BOUNDARY REGULARITY ESTIMATES FOR NONLOCAL ELLIPTIC EQUATIONS IN C^1 AND $C^{1,\alpha}$ DOMAINS

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Abstract. We establish sharp boundary regularity estimates in C^1 and $C^{1,\alpha}$ domains for nonlocal problems of the form $Lu = f$ in Ω, $u = 0$ in Ω^c. Here, L is a nonlocal elliptic operator of order $2s$, with $s \in (0, 1)$.

First, in $C^{1,\alpha}$ domains we show that all solutions u are C^s up to the boundary and that $u/d^s \in C^\alpha(\overline{\Omega})$, where d is the distance to $\partial\Omega$.

In C^1 domains, solutions are in general not comparable to d^s, and we prove a boundary Harnack principle in such domains. Namely, we show that if u_1 and u_2 are positive solutions, then u_1/u_2 is bounded and Hölder continuous up to the boundary.

Finally, we establish analogous results for nonlocal equations with bounded measurable coefficients in non-divergence form. All these regularity results will be essential tools in a forthcoming work on free boundary problems for nonlocal elliptic operators [CRS15].

1. Introduction and results

In this paper we study the boundary regularity of solutions to nonlocal elliptic equations in C^1 and $C^{1,\alpha}$ domains. The operators we consider are of the form

$$Lu(x) = \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) \frac{a(y/|y|)}{|y|^{n+2s}}\, dy, \tag{1.1}$$

with

$$0 < \lambda \le a(\theta) \le \Lambda, \qquad \theta \in S^{n-1}. \tag{1.2}$$

When $a \equiv ctt$, then L is a multiple of the fractional Laplacian $-(-\Delta)^s$.

We consider solutions $u \in L^\infty(\mathbb{R}^n)$ to

$$\begin{cases} Lu = f & \text{in } B_1 \cap \Omega \\ u = 0 & \text{in } B_1 \setminus \Omega, \end{cases} \tag{1.3}$$

with $f \in L^\infty(\Omega \cap B_1)$ and $0 \in \partial\Omega$.

When L is the Laplacian Δ, then the following are well known results:

(i) If Ω is $C^{1,\alpha}$, then $u \in C^{1,\alpha}(\overline{\Omega} \cap B_{1/2})$.

(ii) If Ω is C^1, then solutions are in general *not* $C^{0,1}$.

Still, in C^1 domains one has the following boundary Harnack principle:

Key words and phrases. nonlocal equations, boundary regularity, C^1 domains.

(iii) If Ω is C^1, and u_1 and u_2 are positive in Ω, with $f \equiv 0$, then u_1 and u_2 are comparable in $\overline{\Omega} \cap B_{1/2}$, and $u_1/u_2 \in C^{0,\gamma}(\overline{\Omega} \cap B_{1/2})$ for some small $\gamma > 0$.

Actually, (iii) holds in general Lipschitz domains (for γ small enough), or even in less regular domains; see [Dah77, BBB91]. Analogous results hold for more general second order operators in non-divergence form $L = \sum_{i,j} a_{ij}(x)\partial_{ij}u$ with bounded measurable coefficients $a_{ij}(x)$ [BB94].

The aim of the present paper is to establish analogous results to (i) and (iii) for nonlocal elliptic operators L of the form (1.1)-(1.2), and also for non-divergence operators with bounded measurable coefficients.

1.1. $C^{1,\alpha}$ **domains.** When $L = \Delta$ in (1.3) and Ω is $C^{k,\alpha}$, then solutions u are as regular as the domain Ω provided that f is regular enough. In particular, if Ω is C^∞ and $f \in C^\infty$ then $u \in C^\infty(\overline{\Omega})$.

When $L = -(-\Delta)^s$, then the boundary regularity is well understood in $C^{1,1}$ and in C^∞ domains. In both cases, the optimal Hölder regularity of solutions is $u \in C^s(\overline{\Omega})$, and in general one has $u \notin C^{s+\epsilon}(\overline{\Omega})$ for any $\epsilon > 0$. Still, higher order estimates are given in terms of the regularity of u/d^s: if Ω is C^∞ and $f \in C^\infty$ then $u/d^s \in C^\infty(\overline{\Omega})$; see Grubb [Gru15, Gru14]. Here, $d(x) = \mathrm{dist}(x, \mathbb{R}^n \setminus \Omega)$.

We prove here a boundary regularity estimate of order $s + \alpha$ in $C^{1,\alpha}$ domains. Namely, we show that if Ω is $C^{1,\alpha}$ then $u/d^s \in C^\alpha(\overline{\Omega})$, as stated below.

We first establish the optimal Hölder regularity up to the boundary, $u \in C^s(\overline{\Omega})$.

Proposition 1.1. *Let $s \in (0,1)$, L be any operator of the form* (1.1)-(1.2), *and Ω be any bounded $C^{1,\alpha}$ domain. Let u be a solution of* (1.3)*. Then,*

$$\|u\|_{C^s(B_{1/2})} \le C\left(\|f\|_{L^\infty(B_1\cap\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right).$$

The constant C depends only on n, s, Ω, and ellipticity constants.

Our second result gives a finer description of solutions in terms of the function d^s, as explained above.

Theorem 1.2. *Let $s \in (0,1)$ and $\alpha \in (0,s)$. Let L be any operator of the form* (1.1)-(1.2)*, Ω be any $C^{1,\alpha}$ domain, and d be the distance to $\partial\Omega$. Let u be a solution of* (1.3)*. Then,*

$$\|u/d^s\|_{C^\alpha(B_{1/2}\cap\overline{\Omega})} \le C\left(\|f\|_{L^\infty(B_1\cap\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right).$$

The constant C depends only on n, s, α, Ω, and ellipticity constants.

The previous estimate in $C^{1,\alpha}$ domains was only known for the half-Laplacian $(-\Delta)^{1/2}$; see De Silva and Savin [DS14]. For more general nonlocal operators, such estimate was only known in $C^{1,1}$ domains [RS14b].

The proofs of Proposition 1.1 and Theorem 1.2 follow the ideas of [RS14b], where the same estimates were established in $C^{1,1}$ domains. One of the main difficulties in the present proofs is the construction of appropriate barriers. Indeed, while any $C^{1,1}$ domain satisfies the interior and exterior ball condition, this is not true anymore in $C^{1,\alpha}$ domains, and the construction of barriers is more delicate. We will need a careful computation to show that

$$|L(d^s)| \le C d^{\alpha-s} \quad \text{in } \Omega.$$

In fact, since d^s is not regular enough to compute L, we need to define a new function ψ which behaves like d but it is C^2 inside Ω, and will show that $|L(\psi^s)| \le C d^{\alpha-s}$; see Definition 2.1.

Once we have this, and doing some extra computations we will be able to construct sub and supersolutions which are comparable to d^s, and thus we will have

$$|u| \le C d^s.$$

This, combined with interior regularity estimates, will give the C^s estimate of Proposition 1.1.

Then, combining these ingredients with a blow-up and compactness argument in the spirit of [RS14b, RS14], we will find the expansion

$$\big|u(x) - Q(z) d^s(x)\big| \le C|x-z|^{s+\alpha}$$

at any $z \in \partial\Omega$. And this will yield Theorem 1.2.

1.2. C^1 **domains.** In C^1 domains, in general one does not expect solutions to be comparable to d^s. In that case, we establish the following.

Theorem 1.3. *Let $s \in (0,1)$ and $\alpha \in (0,s)$. Let L be any operator of the form (1.1)-(1.2), and Ω be any C^1 domain.*

Then, there exists is $\delta > 0$, depending only on α, n, s, Ω, and ellipticity constants, such that the following statement holds.

Let u_1 and u_2, be viscosity solutions of (1.3) with right hand sides f_1 and f_2, respectively. Assume that $\|f_i\|_{L^\infty(B_1\cap\Omega)} \le C_0$ (with $C_0 \ge \delta$), $\|u_i\|_{L^\infty(\mathbb{R}^n)} \le C_0$,

$$f_i \ge -\delta \quad \text{in} \quad B_1 \cap \Omega,$$

and that

$$u_i \ge 0 \quad \text{in} \quad \mathbb{R}^n, \qquad \sup_{B_{1/2}} u_i \ge 1.$$

Then,

$$\|u_1/u_2\|_{C^\alpha(\Omega\cap B_{1/2})} \le C C_0, \qquad \alpha \in (0,s),$$

where C depends only on α, n, s, Ω, and ellipticity constants.

We expect the range of exponents $\alpha \in (0,s)$ to be optimal.

In particular, the previous result yields a boundary Harnack principle in C^1 domains.

Corollary 1.4. *Let $s \in (0,1)$, L be any operator of the form (1.1)-(1.2), and Ω be any C^1 domain. Let u_1 and u_2, be viscosity solutions of*

$$\begin{cases} Lu_1 = Lu_2 = 0 & \text{in } B_1 \cap \Omega \\ u_1 = u_2 = 0 & \text{in } B_1 \setminus \Omega, \end{cases}$$

Assume that

$$u_1 \geq 0 \qquad \textit{and} \qquad u_2 \geq 0 \qquad \textit{in} \quad \mathbb{R}^n,$$

and that $\sup_{B_{1/2}} u_1 = \sup_{B_{1/2}} u_2 = 1$. *Then,*

$$0 < C^{-1} \leq \frac{u_1}{u_2} \leq C \qquad \textit{in} \quad B_{1/2},$$

where C *depends only on* n, s, Ω, *and ellipticity constants.*

Theorems 1.3 and 1.2 will be important tools in a forthcoming work on free boundary problems for nonlocal elliptic operators [CRS15]. Namely, Theorem 1.3 (applied to the derivatives of the solution to the free boundary problem) will yield that C^1 free boundaries are in fact $C^{1,\alpha}$, and then thanks to Theorem 1.2 we will get a fine description of solutions in terms of d^s.

1.3. **Equations with bounded measurable coefficients.** We also obtain estimates for equations with bounded measurable coefficients,

$$\begin{cases} M^+u \geq -K_0 & \text{in } B_1 \cap \Omega \\ M^-u \leq K_0 & \text{in } B_1 \cap \Omega \\ u = 0 & \text{in } B_1 \setminus \Omega. \end{cases} \tag{1.4}$$

Here, M^+ and M^- are the extremal operators associated to the class $\mathcal{L}_*$, consisting of all operators of the form (1.1)-(1.2), i.e.,

$$M^+ := M^+_{\mathcal{L}_*} u = \sup_{L \in \mathcal{L}_*} Lu, \qquad M^- := M^+_{\mathcal{L}_*} u = \inf_{L \in \mathcal{L}_*} Lu.$$

Notice that the equation (1.4) is an equation with bounded measurable coefficients, and it is the nonlocal analogue of

$$a_{ij}(x)\partial_{ij} u = f(x), \qquad \text{with} \quad \lambda \mathrm{Id} \leq (a_{ij}(x))_{ij} \leq \Lambda \mathrm{Id}, \qquad |f(x)| \leq K_0.$$

For nonlocal equations with bounded measurable coefficients in $C^{1,\alpha}$ domains, we show the following.

Here, and throughout the paper, we denote $\bar{\alpha} = \bar{\alpha}(n, s, \lambda, \Lambda) > 0$ the exponent in [RS14, Proposition 5.1].

Theorem 1.5. *Let* $s \in (0,1)$ *and* $\alpha \in (0, \bar{\alpha})$. *Let* Ω *be any* $C^{1,\alpha}$ *domain, and* d *be the distance to* $\partial\Omega$. *Let* $u \in C(B_1)$ *be any viscosity solution of* (1.4). *Then, we have*

$$\|u/d^s\|_{C^\alpha(B_{1/2} \cap \overline{\Omega})} \leq C \left(K_0 + \|u\|_{L^\infty(\mathbb{R}^n)} \right).$$

The constant C *depends only on* n, s, α, Ω, *and ellipticity constants.*

In C^1 domains we prove:

Theorem 1.6. *Let* $s \in (0,1)$ *and* $\alpha \in (0, \bar{\alpha})$. *Let* Ω *be any* C^1 *domain.*

Then, there exists is $\delta > 0$, *depending only on* α, n, s, Ω, *and ellipticity constants, such that the following statement holds.*

Let u_1 and u_2, be functions satisfying

$$\begin{cases} M^+(au_1 + bu_2) & \geq & -\delta(|a| + |b|) & \text{in } B_1 \cap \Omega \\ u_1 = u_2 & = & 0 & \text{in } B_1 \setminus \Omega, \end{cases}$$

for any $a, b \in \mathbb{R}$. Assume that

$$u_i \geq 0 \quad \text{in} \quad \mathbb{R}^n,$$

$\|u_i\|_{L^\infty(\mathbb{R}^n)} \leq C_0$, *and that* $\sup_{B_{1/2}} u_i \geq 1$. *Then, we have*

$$\|u_1/u_2\|_{C^\alpha(\Omega \cap B_{1/2})} \leq C,$$

where C depends only on α, n, s, Ω, and ellipticity constants.

The Boundary Harnack principle for nonlocal operators has been widely studied, and in some cases it is even known in general open sets; see Bogdan [Bog97], Song-Wu [SW99], Bogdan-Kulczycki-Kwasnicki [BKK08], and Bogdan-Kumagai-Kwasnicki [BKK15]. The main differences between our Theorems 1.3-1.6 and previous known results are the following. On the one hand, our results allow a right hand side on the equation (1.3), and apply also to viscosity solutions of equations with bounded measurable coefficients (1.4). On the other hand, we obtain a higher order estimate, in the sense that for linear equations we prove that u_1/u_2 is C^α for all $\alpha \in (0, s)$. Finally, the proof we present here is perturbative, in the sense the we make a blow-up and use that after the rescaling the domain will be a half-space. This allows us to get a higher order estimate for u_1/u_2, but requires the domain to be at least C^1.

The paper is organized as follows. In Section 2 we construct the barriers in $C^{1,\alpha}$ domains. Then, in Section 3 we prove the regularity of solutions in $C^{1,\alpha}$ domains, that is, Proposition 1.1 and Theorems 1.2 and 1.5. In Section 4 we construct the barriers needed in the analysis on C^1 domains. Finally, in Section 5 we prove Theorems 1.3 and 1.6.

2. Barriers: $C^{1,\alpha}$ domains

Throughout this section, Ω will be any bounded and $C^{1,\alpha}$ domain, and

$$d(x) = \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega).$$

Since d is only $C^{1,\alpha}$ inside Ω, we need to consider the following "regularized version" of d.

Definition 2.1. Given a $C^{1,\alpha}$ domain Ω, we consider a fixed function ψ satisfying

$$C^{-1} d \leq \psi \leq C d, \tag{2.1}$$

$$\|\psi\|_{C^{1,\alpha}(\overline{\Omega})} \leq C \qquad \text{and} \qquad |D^2\psi| \leq C d^{\alpha-1}, \tag{2.2}$$

with C depending only on Ω.

Remark 2.2. Notice that to construct ψ one may take for example the solution to $-\Delta\psi = 1$ in Ω, $\psi = 0$ on $\partial\Omega$, extended by $\psi = 0$ in $\mathbb{R}^n \setminus \Omega$.

Note also that any $C^{1,\alpha}$ domain Ω can be locally represented as the epigraph of a $C^{1,\alpha}$ function. More precisely, there is a $\rho_0 > 0$ such that for all $z \in \partial\Omega$ the set $\partial\Omega \cap B_{\rho_0}(z)$ is, after a rotation, the graph of a $C^{1,\alpha}$ function. Then, the constant C in (2.1)-(2.2) can be taken depending only on ρ_0 and on the $C^{1,\alpha}$ norms of these functions.

We want to show the following.

Proposition 2.3. *Let $s \in (0,1)$ and $\alpha \in (0,s)$, L be given by (1.1)-(1.2), and Ω be any $C^{1,\alpha}$ domain. Let ψ be given by Definition 2.1. Then,*

$$|L(\psi^s)| \le C d^{\alpha-s} \qquad \text{in } \Omega. \tag{2.3}$$

The constant C depends only on s, n, Ω, and ellipticity constants.

For this, we need a couple of technical Lemmas. The first one reads as follows.

Lemma 2.4. *Let Ω be any $C^{1,\alpha}$ domain, and ψ be given by Definition 2.1. Then, for each $x_0 \in \Omega$ we have*

$$\left|\psi(x_0+y) - \big(\psi(x_0) + \nabla\psi(x_0)\cdot y\big)_+\right| \le C|y|^{1+\alpha} \qquad \text{for } y \in \mathbb{R}^n.$$

The constant C depends only on Ω.

Proof. Let us consider $\tilde\psi$, a $C^{1,\alpha}(\mathbb{R}^n)$ extension of $\psi|_\Omega$ satisfying $\tilde\psi \le 0$ in $\mathbb{R}^n \setminus \Omega$. Then, since $\tilde\psi \in C^{1,\alpha}(\mathbb{R}^n)$ we clearly have

$$\left|\tilde\psi(x) - \psi(x_0) - \nabla\psi(x_0)\cdot(x-x_0)\right| \le C|x-x_0|^{1+\alpha}$$

in all of $\mathbb{R}^n$. Here we used $\tilde\psi(x_0) = \psi(x_0)$ and $\nabla\tilde\psi(x_0) = \nabla\psi(x_0)$.

Now, using that $|a_+ - b_+| \le |a-b|$, combined with $(\tilde\psi)_+ = \psi$, we find

$$\left|\psi(x) - \big(\psi(x_0) + \nabla\psi(x_0)\cdot(x-x_0)\big)_+\right| \le C|x-x_0|^{1+\alpha}$$

for all $x \in \mathbb{R}^n$. Thus, the lemma follows. $\square$

The second one reads as follows.

Lemma 2.5. *Let Ω be any $C^{1,\alpha}$ domain, $p \in \Omega$, and $\rho = d(p)/2$. Let $\gamma > -1$ and $\beta \ne \gamma$. Then,*

$$\int_{B_1\setminus B_{\rho/2}} d^\gamma(p+y)\frac{dy}{|y|^{n+\beta}} \le C\big(1 + \rho^{\gamma-\beta}\big).$$

The constant C depends only on γ, β, and Ω.

Proof. The proof is similar to that of [RV15, Lemma 4.2].

First, we may assume $p = 0$.

Notice that, since Ω is $C^{1,\alpha}$, then there is $\kappa_* > 0$ such that for any $t \in (0, \kappa_*]$ the level set $\{d = t\}$ is $C^{1,\alpha}$. Since

$$\int_{(B_1 \setminus B_\rho)\cap\{d\geq\kappa_*\}} d^\gamma(y)\frac{dy}{|y|^{n+\beta}} \leq C, \tag{2.4}$$

then we just have to bound the same integral in the set $\{d < \kappa_*\}$. Here we used that $B_r \cap \{d \geq \kappa_*\} = \emptyset$ if $r \leq \kappa_* - 2\rho$, which follows from the fact that $d(0) = 2\rho$.

We will use the following estimate for $t \in (0, \kappa_*)$

$$\mathcal{H}^{n-1}\big(\{d = t\} \cap (B_{2^{-k+1}} \setminus B_{2^{-k}})\big) \leq C(2^{-k})^{n-1},$$

which follows for example from the fact that $\{d = t\}$ is $C^{1,\alpha}$ (see the Appendix in [RV15]). Note also that $\{d = t\} \cap B_r = \emptyset$ if $t > r + 2\rho$.

Let $M \geq 0$ be such that $2^{-M} \leq \rho \leq 2^{-M+1}$. Then, using the coarea formula,

$$\begin{aligned}\int_{(B_1\setminus B_\rho)\cap\{d<\kappa_*\}} d^\gamma(y)\frac{dy}{|y|^{n+\beta}} \leq \\ &\leq \sum_{k=0}^{M}\frac{1}{2^{-k(n+\beta)}}\int_{(B_{2^{-k+1}}\setminus B_{2^{-k}})\cap\{d<C2^{-k}\}} d^\gamma(y)|\nabla d(y)|\,dy \\ &\leq \sum_{k=0}^{M}\frac{1}{2^{-k(n+\beta)}}\int_0^{C2^{-k}} t^\gamma dt \int_{(B_{2^{-k+1}}\setminus B_{2^{-k}})\cap\{d=t\}} d\mathcal{H}^{n-1}(y) \\ &\leq C\sum_{k=0}^{M}\frac{(2^{-k})^{\gamma+1}2^{-k(n-1)}}{2^{-k(n+\beta)}} = C\sum_{k=0}^{M}2^{k(\beta-\gamma)} = C(1+\rho^{\gamma-\beta}).\end{aligned} \tag{2.5}$$

Here we used that $\gamma \neq \beta$ —in case $\gamma = \beta$ we would get $C(1 + |\log \rho|)$.

Combining (2.4) and (2.5), the lemma follows. □

We now give the:

Proof of Proposition 2.3. Let $x_0 \in \Omega$ and $\rho = d(x)$.

Notice that when $\rho \geq \rho_0 > 0$ then ψ^s is smooth in a neighborhood of x_0, and thus $L(\psi^s)(x_0)$ is bounded by a constant depending only on ρ_0. Thus, we may assume that $\rho \in (0, \rho_0)$, for some small ρ_0 depending only on Ω.

Let us denote

$$\ell(x) = \big(\psi(x_0) + \nabla\psi(x_0)\cdot(x - x_0)\big)_+,$$

which satisfies

$$L(\ell^s) = 0 \qquad \text{in} \quad \{\ell > 0\};$$

see [RS14, Section 2].

Now, notice that

$$\psi(x_0) = \ell(x_0) \qquad \text{and} \qquad \nabla\psi(x_0) = \nabla\ell(x_0).$$

Moreover, by Lemma 2.4 we have

$$|\psi(x_0 + y) - \ell(x_0 + y)| \leq C|y|^{1+\alpha},$$

and using $|a^s - b^s| \leq C|a-b|(a^{s-1} + b^{s-1})$ for $a, b \geq 0$, we find

$$|\psi^s(x_0+y) - \ell^s(x_0+y)| \leq C|y|^{1+\alpha}\left(d^{s-1}(x_0+y) + \ell^{s-1}(x_0+y)\right). \tag{2.6}$$

Here, we used that $\psi \leq Cd$.

On the other hand, since $\psi \in C^{1,\alpha}(\overline{\Omega})$ and $\psi \geq cd$ in $\overline{\Omega}$, then it is not difficult to check that

$$\ell > 0 \quad \text{in} \quad B_{\rho/2}(x_0),$$

provided that ρ_0 is small (depending only on Ω). Thanks to this, one may estimate

$$\left|D^2(\psi^s - \ell^s)\right| \leq C\rho^{s+\alpha-2} \quad \text{in } B_{\rho/2},$$

and thus

$$\left|\psi^s - \ell^s\right|(x_0+y) \leq \|D^2(\psi^s - \ell^s)\|_{L^\infty(B_{\rho/2}(x_0))}|y|^2 \leq C\rho^{s+\alpha-2}|y|^2 \tag{2.7}$$

for $y \in B_{\rho/2}$.

Therefore, it follows from (2.6) and (2.7) that

$$\left|\psi^s - \ell^s\right|(x_0+y) \leq \begin{cases} C\rho^{s+\alpha-2}|y|^2 & \text{for } y \in B_{\rho/2} \\ C|y|^{1+\alpha}\left(d^{s-1}(x_0+y) + \ell^{s-1}(x_0+y)\right) & \text{for } y \in B_1 \setminus B_{\rho/2} \\ C|y|^s & \text{for } y \in \mathbb{R}^n \setminus B_1. \end{cases}$$

Hence, recalling that $L(\ell^s)(x_0) = 0$, we find

$$\begin{aligned} |L(\psi^s)(x_0)| &= |L\big(\psi^s - \ell^s\big)(x_0)| \\ &= \int_{\mathbb{R}^n} \left|\psi^s - \ell^s\right|(x_0+y)\frac{a(y/|y|)}{|y|^{n+2s}}\,dy \\ &\leq \int_{B_{\rho/2}} C\rho^{s+\alpha-2}|y|^2\frac{dy}{|y|^{n+2s}} + \int_{\mathbb{R}^n\setminus B_1} C|y|^s\frac{dy}{|y|^{n+2s}} + \\ &\quad + \int_{B_1\setminus B_{\rho/2}} C|y|^{1+\alpha}\left(d^{s-1}(x_0+y) + \ell^{s-1}(x_0+y)\right)\frac{dy}{|y|^{n+2s}} \\ &\leq C(\rho^{\alpha-s}+1) + C\int_{B_1\setminus B_{\rho/2}}\left(d^{s-1}(x_0+y) + \ell^{s-1}(x_0+y)\right)\frac{dy}{|y|^{n+2s-1-\alpha}}. \end{aligned}$$

Thus, using Lemma 2.5 twice, we find

$$|L(\psi^s)(x_0)| \leq C\rho^{\alpha-s},$$

and (2.3) follows. □

When $\alpha > s$ the previous proof gives the following result, which states that for any operator (1.1)-(1.2) one has $L(d^s) \in L^\infty(\Omega)$. Here, as in [Gru15, RS14b, RS14], d denotes a fixed function that coincides with $\mathrm{dist}(x, \mathbb{R}^n \setminus \Omega)$ in a neighborhood of $\partial\Omega$, satisfies $d \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, and it is $C^{1,\alpha}$ in Ω.

Proposition 2.6. *Let $s\in(0,1)$, L be given by (1.1)-(1.2), and Ω be any bounded $C^{1,\alpha}$ domain, with $\alpha>s$. Then,*

$$|L(d^s)|\le C \quad in\ \Omega.$$

The constant C depends only on n, s, Ω, and ellipticity constants.

To our best knowledge, this result was only known in case that L is the fractional Laplacian and Ω is $C^{1,1}$, or in case that $a\in C^\infty(S^{n-1})$ in (1.1) and Ω is C^∞ (in this case $L(d^s)$ is $C^\infty(\overline{\Omega})$; see [Gru15]).

Also, recall that for a general stable operator (1.1) (with $a\in L^1(S^{n-1})$ and without the assumption (1.2)) the result is false, since we constructed in [RS14b] an operator L and a C^∞ domain Ω for which $L(d^s)\notin L^\infty(\Omega)$. Hence, the assumption (1.2) is somewhat necessary for Proposition 2.6 to be true.

Proof of Proposition 2.6. Let $x_0\in\Omega$, and $\rho=d(x)$.

Notice that when $\rho\ge\rho_0>0$ then d^s is C^{1+s} in a neighborhood of x_0, and thus $L(d^s)(x_0)$ is bounded by a constant depending only on ρ_0. Thus, we may assume that $\rho\in(0,\rho_0)$, for some small ρ_0 depending only on Ω.

Let us denote

$$\ell(x)=\big(d(x_0)+\nabla d(x_0)\cdot(x-x_0)\big)_+,$$

which satisfies

$$L(\ell^s)=0 \qquad \text{in} \quad \{\ell>0\}.$$

Moreover, as in Proposition 2.3, we have

$$|d^s(x_0+y)-\ell^s(x_0+y)|\le C|y|^{1+\alpha}\left(d^{s-1}(x_0+y)+\ell^{s-1}(x_0+y)\right). \tag{2.8}$$

In particular,

$$|d^s(x_0+y)-\ell^s(x_0+y)|\le C\rho^{s-1}|y|^{1+\alpha} \qquad \text{for } y\in B_{\rho/2}.$$

Hence, recalling that $L(\ell^s)(x_0)=0$, we find

$$\begin{aligned}|L(\psi^s)(x_0)| &= |L\big(\psi^s-\ell^s\big)(x_0)|\\ &= \int_{\mathbb{R}^n}\big|\psi^s-\ell^s\big|(x_0+y)\frac{a(y/|y|)}{|y|^{n+2s}}\,dy\\ &\le \int_{B_{\rho/2}} C\rho^{s-1}|y|^{1+\alpha}\frac{dy}{|y|^{n+2s}}+\int_{\mathbb{R}^n\setminus B_1} C|y|^s\frac{dy}{|y|^{n+2s}}+\\ &\quad+\int_{B_1\setminus B_{\rho/2}} C|y|^{1+\alpha}\left(d^{s-1}(x_0+y)+\ell^{s-1}(x_0+y)\right)\frac{dy}{|y|^{n+2s}}\\ &\le C(1+\rho^{\alpha-s}).\end{aligned}$$

Here we used Lemma 2.5. Since $\alpha>s$, the result follows. □

We next show the following.

Lemma 2.7. *Let $s \in (0,1)$, L be given by (1.1)-(1.2), and Ω be any $C^{1,\alpha}$ domain. Let ψ be given by Definition 2.1. Then, for any $\epsilon \in (0,\alpha)$, we have*

$$L(\psi^{s+\epsilon}) \geq cd^{\epsilon-s} - C \qquad \text{in } \Omega \cap B_{1/2}, \tag{2.9}$$

with $c > 0$. The constants c and C depend only on ϵ, s, n, Ω, and ellipticity constants.

Proof. Exactly as in Proposition 2.3, one finds that

$$\left|\psi^{s+\epsilon}(x_0+y) - \ell^{s+\epsilon}(x_0+y)\right| \leq C|y|^{1+\alpha}\left(d^{s+\epsilon-1}(x_0+y) + \ell^{s+\epsilon-1}(x_0+y)\right), \tag{2.10}$$

and

$$\left|\psi^{s+\epsilon} - \ell^{s+\epsilon}\right|(x_0+y) \leq C\rho^{s+\epsilon+\alpha-2}|y|^2 \tag{2.11}$$

for $y \in B_{\rho/2}$. Therefore, as in Proposition 2.3,

$$|L(\psi^{s+\epsilon} - \ell^{s+\epsilon})(x_0)| \leq C(1+\rho^{\alpha+\epsilon-s}).$$

We now use that, by homogeneity, we have

$$L(\ell^{s+\epsilon})(x_0) = \kappa\rho^{\epsilon-s},$$

with $\kappa > 0$ (see [RS14]). Thus, combining the previous two inequalities we find

$$L(\psi^{s+\epsilon})(x_0) \geq \kappa\rho^{\epsilon-s} - C(1+\rho^{\alpha+\epsilon-s}) \geq \frac{\kappa}{2}\rho^{s-\epsilon} - C,$$

as desired. □

We now construct sub and supersolutions.

Lemma 2.8 (Supersolution)**.** *Let $s \in (0,1)$, L be given by (1.1)-(1.2), and Ω be any bounded $C^{1,\alpha}$ domain. Then, there exists $\rho_0 > 0$ and a function ϕ_1 satisfying*

$$\begin{cases} \qquad L\phi_1 \leq -1 & \text{in } \Omega \cap \{d \leq \rho_0\} \\ C^{-1}d^s \leq \phi_1 \leq Cd^s & \text{in } \Omega \\ \qquad \phi_1 = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

The constants C and ρ_0 depend only on n, s, Ω, and ellipticity constants.

Proof. Let ψ be given by Definition 2.1, and let $\epsilon = \frac{\alpha}{2}$. Then, by Proposition 2.3 we have

$$-C_0 d^{\alpha-s} \leq L(\psi^s) \leq C_0 d^{\alpha-s},$$

and by Lemma 2.7

$$L(\psi^{s+\epsilon}) \geq c_0 d^{\epsilon-s} - C_0.$$

Next, we consider the function

$$\phi_1 = \psi^s - c\psi^{s+\epsilon},$$

with c small enough. Then, ϕ_1 satisfies

$$L\phi_1 \leq C_0 d^{\alpha-s} + C_0 - cc_1 d^{\epsilon-s} \leq -1 \qquad \text{in} \quad \Omega \cap \{d \leq \rho_0\}, \tag{2.12}$$

for some $\rho_0 > 0$. Finally, by construction we clearly have

$$C^{-1}d^s \leq \phi_1 \leq Cd^s \quad \text{in} \quad \Omega,$$

and thus the Lemma is proved. □

Notice that the previous proof gives in fact the following.

Lemma 2.9. *Let $s \in (0,1)$, L be given by (1.1)-(1.2), and Ω be any bounded $C^{1,\alpha}$ domain. Then, there exist $\rho_0 > 0$ and a function ϕ_1 satisfying*

$$\begin{cases} \qquad\quad L\phi_1 \le -d^{\epsilon-s} & \text{in } \Omega \cap \{d \le \rho_0\} \\ C^{-1}d^s \le \phi_1 \le Cd^s & \text{in } \Omega \\ \qquad\quad \phi_1 = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

The constants C and ρ_0 depend only on n, s, Ω, and ellipticity constants.

Proof. The proof is the same as Lemma 2.8; see (2.12). □

We finally construct a subsolution.

Lemma 2.10 (Subsolution)**.** *Let $s \in (0,1)$, L be given by (1.1)-(1.2), and Ω be any bounded $C^{1,\alpha}$ domain. Then, for each $K \subset\subset \Omega$ there exists a function ϕ_2 satisfying*

$$\begin{cases} \qquad\quad L\phi_2 \ge 1 & \text{in } \Omega \setminus K \\ C^{-1}d^s \le \phi_2 \le Cd^s & \text{in } \Omega \\ \qquad\quad \phi_2 = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

The constants c and C depend only on n, s, Ω, K, and ellipticity constants.

Proof. First, notice that if $\eta \in C_c^\infty(K)$ then $L\eta \ge c_1 > 0$ in $\Omega \setminus K$. Hence,

$$\phi_2 = \psi^s + \psi^{s+\epsilon} + C\eta$$

satisfies

$$L\phi_2 \ge -C_0 d^{\alpha-s} + c_0 d^{\epsilon-s} - C_0 + Cc_1 \ge 1 \qquad \text{in} \quad \Omega \setminus K,$$

provided that C is chosen large enough. □

3. Regularity in $C^{1,\alpha}$ domains

The aim of this section is to prove Proposition 1.1 and Theorem 1.2.

3.1. **Hölder regularity up to the boundary.** We will prove first the following result, which is similar to Proposition 1.1 but allows u to grow at infinity and f to be singular near $\partial\Omega$.

Proposition 3.1. *Let $s \in (0,1)$, L be any operator of the form (1.1)-(1.2), and Ω be any bounded $C^{1,\alpha}$ domain. Let u be a solution to (1.3), and assume that*

$$|f| \le Cd^{\epsilon-s} \quad \text{in} \quad \Omega.$$

Then,

$$\|u\|_{C^s(B_{1/2})} \le C \left(\|d^{s-\epsilon} f\|_{L^\infty(B_1\cap\Omega)} + \sup_{R\ge 1} R^{\delta-2s} \|u\|_{L^\infty(B_R)} \right).$$

The constant C depends only on n, s, ϵ, δ, Ω, and ellipticity constants.

Proof. Dividing by a constant, we may assume that

$$\|d^{s-\epsilon}f\|_{L^\infty(B_1\cap\Omega)}+\sup_{R\geq1}R^{\delta-2s}\|u\|_{L^\infty(B_R)}\leq 1.$$

Then, the truncated function $w=u\chi_{B_1}$ satisfies

$$|Lw|\leq Cd^{\epsilon-s}\quad\text{in}\quad \Omega\cap B_{3/4},$$

$w\leq 1$ in B_1, and $w\equiv 0$ in $\mathbb{R}^n\setminus B_1$.

Let $\widetilde{\Omega}$ be a bounded $C^{1,\alpha}$ domain satisfying: $B_1\cap\Omega\subset\widetilde{\Omega}$; $B_{1/2}\cap\partial\Omega\subset\partial\widetilde{\Omega}$; and $\mathrm{dist}(x,\partial\widetilde{\Omega})\geq c>0$ in $\Omega\cap(B_1\setminus B_{3/4})$. Let ϕ_1 be the function given by Lemma 2.8, satisfying

$$\begin{cases} \qquad L\phi_1 \;\leq\; -\tilde d^{\epsilon-s} & \text{in } \widetilde{\Omega}\cap\{\tilde d\leq\rho_0\}\\ c\tilde d^s \;\leq\; \phi_1 \;\leq\; C\tilde d^s & \text{in } \widetilde{\Omega}\\ \qquad \phi_1 \;=\; 0 & \text{in } \mathbb{R}^n\setminus\Omega,\end{cases}$$

where we denoted $\tilde d(x)=\mathrm{dist}(x,\mathbb{R}^n\setminus\widetilde{\Omega})$.

Then, the function $\varphi=C\phi_1$ satisfies

$$\begin{cases} L\varphi \leq -Cd^{\epsilon-s} & \text{in } \Omega\cap B_{1/2}\cap\{d\leq\rho_0\}\\ \varphi \leq Cd^s & \text{in } \Omega\cap B_{1/2}\\ \varphi\geq 1 & \text{in } \Omega\cap(B_1\setminus B_{3/4}) \quad\text{and in}\quad \Omega\cap B_{1/2}\cap\{d\geq\rho_0\}\\ \varphi\geq 0 & \text{in } \mathbb{R}^n.\end{cases}$$

In particular, if C is large enough then we have $L(\varphi-w)\leq 0$ in $\Omega\cap B_{1/2}\cap\{d\leq\rho_0\}$, and $\varphi-w\geq 0$ in $\mathbb{R}^n\setminus(\Omega\cap B_{1/2}\cap\{d\leq\rho_0\})$.

Therefore, the maximum principle yields $w\leq\varphi$, and thus $w\leq Cd^s$ in $B_{1/2}$. Replacing w by $-w$, we find

$$|w|\leq Cd^s\qquad\text{in}\quad B_{1/2}. \tag{3.1}$$

Now, it follows from the interior estimates of [RS14b, Theorem 1.1] that

$$r^s[w]_{C^s(B_r(x_0))}\leq C\big(r^{2s}\|Lw\|_{L^\infty(B_{2r}(x_0))}+\sup_{R\geq1}R^{\delta-2s}\|w\|_{L^\infty(B_{rR}(x_0))}\big)$$

for any ball $B_r(x_0)\subset\Omega\cap B_{1/2}$ with $2r=d(x_0)$. Now, taking $\delta=s$ and using (3.1), we find

$$R^{-s}\|w\|_{L^\infty(B_{rR}(x_0))}\leq Cr^s\quad\text{for all } R\geq 1.$$

Thus, we have

$$[w]_{C^s(B_r(x_0))}\leq C$$

for all balls $B_r(x_0)\subset\Omega\cap B_{1/2}$ with $2r=d(x_0)$. This yields

$$\|w\|_{C^s(B_{1/2})}\leq C.$$

Indeed, take $x,y\in B_{1/2}$, let $r=|x-y|$ and $\rho=\min\{d(x),d(y)\}$. If $2\rho\geq r$, then using $|u|\leq Cd^s$

$$|u(x)-u(y)|\leq|u(x)|+|u(y)|\leq Cr^s+C(r+\rho)^s\leq\bar C\rho^s.$$

If $2\rho < r$ then $B_{2\rho}(x) \subset \Omega$, and hence

$$|u(x) - u(y)| \le \rho^s [u]_{C^s(B_\rho(x))} \le C\rho^s.$$

Thus, the proposition is proved. □

The proof of Proposition is now immediate.

Proof of Proposition 1.1. The result is a particular case of Proposition 3.1. □

3.2. **Regularity for u/d^s.** Let us now prove Theorem 1.2. For this, we first show the following.

Proposition 3.2. *Let $s \in (0,1)$ and $\alpha \in (0,s)$. Let L be any operator of the form (1.1)-(1.2), Ω be any $C^{1,\alpha}$ domain, and ψ be given by Definition 2.1.*

Assume that $0 \in \partial\Omega$, and that $\partial\Omega \cap B_1$ can be represented as the graph of a $C^{1,\alpha}$ function with norm less or equal than 1.

Let u be any solution to (1.3), and let

$$K_0 = \|d^{s-\alpha} f\|_{L^\infty(B_1\cap\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}.$$

Then, there exists a constant Q satisfying $|Q| \le CK_0$ and

$$\left|u(x) - Q\psi^s(x)\right| \le CK_0|x|^{s+\alpha}.$$

The constant C depends only on n, s, and ellipticity constants.

We will need the following technical lemma.

Lemma 3.3. *Let Ω, ψ, and u be as in Proposition 3.2, and define*

$$\phi_r(x) := Q_*(r)\psi^s(x), \tag{3.2}$$

where

$$Q_*(r) := \arg\min_{Q\in\mathbb{R}} \int_{B_r} \left(u - Q\psi^2\right)^2 dx = \frac{\int_{B_r} u\psi^s}{\int_{B_r} \psi^{2s} dx}.$$

Assume that for all $r \in (0,1)$ we have

$$\|u - \phi_r\|_{L^\infty(B_r)} \le C_0 r^{s+\alpha}. \tag{3.3}$$

Then, there is $Q \in \mathbb{R}$ satisfying $|Q| \le C(C_0 + \|u\|_{L^\infty(B_1)})$ such that

$$\|u - Q\psi^s\|_{L^\infty(B_r)} \le CC_0 r^{s+\alpha},$$

for some constant C depending only on s and α.

Proof. The proof is analogue to that of [RS14b, Lemma 5.3].

First, we may assume $C_0 + \|u\|_{L^\infty(B_1)} = 1$. Then, by (3.3), for all $x \in B_r$ we have

$$|\phi_{2r}(x) - \phi_r(x)| \le |u(x) - \phi_{2r}(x)| + |u(x) - \phi_r(x)| \le Cr^{s+\alpha}.$$

This, combined with $\sup_{B_r} \psi^s = cr^s$, gives

$$|Q_*(2r) - Q_*(r)| \le Cr^\alpha.$$

Moreover, we have $|Q_*(1)| \leq C$, and thus there exists the limit $Q = \lim_{r\downarrow 0} Q_*(r)$. Furthermore,

$$|Q - Q_*(r)| \leq \sum_{k\geq 0} |Q_*(2^{-k}r) - Q_*(2^{-k-1}r)| \leq \sum_{k\geq 0} C2^{-m\alpha}r^\alpha \leq Cr^\alpha.$$

In particular, $|Q| \leq C$.

Therefore, we finally find

$$\|u - Q\psi^s\|_{L^\infty(B_r)} \leq \|u - Q_*(r)\psi^s\|_{L^\infty(B_r)} + Cr^s|Q_*(r) - Q| \leq Cr^{s+\alpha},$$

and the lemma is proved. □

We now give the:

Proof of Proposition 3.2. The proof is by contradiction, and uses several ideas from [RS14b, Section 5].

First, dividing by a constant we may assume $K_0 = 1$. Also, after a rotation we may assume that the unit (outward) normal vector to $\partial\Omega$ at 0 is $\nu = -e_n$.

Assume the estimate is not true, i.e., there are sequences Ω_k, L_k, f_k, u_k, for which:

- Ω_k is a $C^{1,\alpha}$ domain that can be represented as the graph of a $C^{1,\alpha}$ function with norm is less or equal than 1;
- $0 \in \partial\Omega_k$ and the unit normal vector to $\partial\Omega_k$ at 0 is $-e_n$;
- L_k is of the form (1.1)-(1.2);
- $\|d^{s-\alpha} f_k\|_{L^\infty(B_1\cap\Omega)} + \|u_k\|_{L^\infty(\mathbb{R}^n)} \leq 1$;
- For any constant Q, $\sup_{r>0} \sup_{B_r} r^{-s-\alpha}|u_k - Q\psi_k^s| = \infty$.

Then, by Lemma 3.3 we will have

$$\sup_k \sup_{r>0} \|u_k - \phi_{k,r}\|_{L^\infty(B_r)} = \infty,$$

where

$$\phi_{k,r}(x) = Q_k(r)\psi_k^s, \qquad Q_k(r) = \frac{\int_{B_r} u_k\psi_k^s}{\int_{B_r} \psi_k^{2s}}.$$

We now define the monotone quantity

$$\theta(r) := \sup_k \sup_{r'>r} (r')^{-s-\alpha}\|u_k - \phi_{k,r'}\|_{L^\infty(B_{r'})},$$

which satisfies $\theta(r) \to \infty$ as $r \to 0$. Hence, there are sequences $r_m \to 0$ and k_m, such that

$$(r_m)^{-s-\alpha}\|u_{k_m} - \phi_{k_m,r_m}\|_{L^\infty(B_{r_m})} \geq \frac{1}{2}\theta(r_m). \tag{3.4}$$

Let us now denote $\phi_m = \phi_{k_m,r_m}$ and define

$$v_m(x) := \frac{u_{k_m}(r_m x) - \phi_m(r_m x)}{(r_m)^{s+\alpha}\theta(r_m)}.$$

Note that

$$\int_{B_1} v_m(x)\psi_k^s(r_m x)dx = 0, \tag{3.5}$$

and also

$$\|v_m\|_{L^\infty(B_1)} \geq \frac{1}{2}, \tag{3.6}$$

which follows from (3.4).

With the same argument as in the proof of Lemma 3.3, one finds

$$|Q_{k_m}(2r) - Q_{k_m}(r)| \leq C r^\alpha \theta(r).$$

Then, by summing a geometric series this yields

$$|Q_{k_m}(rR) - Q_{k_m}(r)| \leq C r^\alpha \theta(r) R^\alpha$$

for all $R \geq 1$ (see [RS14b]).

The previous inequality, combined with

$$\|u_m - Q_{k_m}(r_m R)\psi^s_{k_m}\|_{L^\infty(B_{r_m R})} \leq (r_m R)^{s+\alpha}\theta(r_m R)$$

(which follows from the definition of θ), gives

$$\begin{aligned} \|v_m\|_{L^\infty(B_R)} &= \frac{1}{(r_m)^{s+\alpha}\theta(r_m)} \|u_m - Q_{k_m}(r_m)\psi^s_{k_m}\|_{L^\infty(B_{r_m R})} \\ &\leq \frac{(r_m R)^{s+\alpha}\theta(r_m R)}{(r_m)^{s+\alpha}\theta(r_m)} + \frac{C(r_m R)^s}{(r_m)^{s+\alpha}\theta(r_m)} |Q_{k_m}(r_m R) - Q_{k_m}(r_m)| \\ &\leq C R^{s+\alpha} \end{aligned} \tag{3.7}$$

for all $R \geq 1$. Here we used that $\theta(r_m R) \leq \theta(r_m)$ if $R \geq 1$.

Now, the functions v_m satisfy

$$L_m v_m(x) = \frac{(r_m)^{2s}}{(r_m)^{s+\alpha}\theta(r_m)} f_{k_m}(r_m x) - \frac{(r_m)^{2s}}{(r_m)^{s+\alpha}\theta(r_m)} (L\psi_{k_m})(r_m x)$$

in $(r_m^{-1}\Omega_{k_m}) \cap B_{r_m^{-1}}$. Since $\alpha < s$, and using Proposition 2.3, we find

$$|L_m v_m| \leq \frac{C}{\theta(r_m)} (r_m)^{s-\alpha} d^{\alpha-s}_{k_m}(r_m x) \qquad \text{in} \quad (r_m^{-1}\Omega_{k_m}) \cap B_{r_m^{-1}}.$$

Thus, denoting $\Omega_m = r_m^{-1}\Omega_{k_m}$ and $d_m(x) = \mathrm{dist}(x, r_m^{-1}\Omega_{k_m})$, we have

$$|L_m v_m| \leq \frac{C}{\theta(r_m)} d_m^{\alpha-s}(x) \qquad \text{in} \quad \Omega_m \cap B_{r_m^{-1}}. \tag{3.8}$$

Notice that the domains Ω_m converge locally uniformly to $\{x_n > 0\}$ as $m \to \infty$.

Next, by Proposition 3.1, we find that for each fixed $M \geq 1$

$$\|v_m\|_{C^s(B_M)} \leq C(M) \qquad \text{for all } m \text{ with } r_m^{-1} > 2M.$$

The constant $C(M)$ does not depend on m. Hence, by Arzelà-Ascoli theorem, a subsequence of v_m converges locally uniformly to a function $v \in C(\mathbb{R}^n)$.

In addition, there is a subsequence of operators L_{k_m} which converges weakly to some operator L of the form (1.1)-(1.2) (see Lemma 3.1 in [RS14b]). Hence, for any fixed $K \subset\subset \{x_n > 0\}$, thanks to the growth condition (3.7) and since $v_m \to v$ locally uniformly, we can pass to the limit the equation (3.8) to get

$$Lv = 0 \quad \text{in } K.$$

Here we used that the domains Ω_m converge uniformly to $\{x_n > 0\}$, so that for m large enough we will have $K \subset \Omega_m \cap B_{r_m^{-1}}$. We also used that, in K, the right hand side in (3.8) converges uniformly to 0.

Since this can be done for any $K \subset\subset \{x_n > 0\}$, we find

$$Lv = 0 \quad \text{in } \{x_n > 0\}.$$

Moreover, we also have $v = 0$ in $\{x_n \leq 0\}$, and $v \in C(\mathbb{R}^n)$.

Thus, by the classification result [RS14b, Theorem 4.1], we find

$$v(x) = \kappa(x_n)_+^s \tag{3.9}$$

for some $\kappa \in \mathbb{R}$.

Now, notice that, up to a subsequence, $r_m^{-1}\psi_{k_m}(r_m x) \to c_1(x_n)_+$ uniformly, with $c_1 > 0$. This follows from the fact that ψ_{k_m} are $C^{1,\alpha}(\overline{\Omega}_{k_m})$ (uniformly in m) and that $0 < c_0 d_{k_m} \leq \psi_{k_m} \leq C_0 d_{k_m}$.

Then, multiplying (3.5) by $(r_m)^{-s}$ and passing to the limit, we find

$$\int_{B_1} v(x)(x_n)_+^s dx = 0.$$

This means that $\kappa = 0$ in (3.9), and therefore $v \equiv 0$. Finally, passing to the limit (3.6) we find a contradiction, and thus the proposition is proved. □

We finally give the:

Proof of Theorem 1.2. First, dividing by a constant if necessary, we may assume

$$\|f\|_{L^\infty(B_1\cap\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)} \leq 1.$$

Second, by definition of ψ we have $\psi/d \in C^\alpha(\overline{\Omega} \cap B_{1/2})$ and

$$\|\psi^s/d^s\|_{C^\alpha(\overline{\Omega}\cap B_{1/2})} \leq C.$$

Thus, it suffices to show that

$$\|u/\psi^s\|_{C^\alpha(\overline{\Omega}\cap B_{1/2})} \leq C. \tag{3.10}$$

To prove (3.10), let $x_0 \in \Omega \cap B_{1/2}$ and $2r = d(x_0)$. Then, by Proposition 3.2 there is $Q = Q(x_0)$ such that

$$\|u - Q\psi^s\|_{L^\infty(B_r(x_0))} \leq Cr^{s+\alpha}. \tag{3.11}$$

Moreover, by rescaling and using interior estimates, we get

$$\|u - Q\psi^s\|_{C^\alpha(B_r(x_0))} \leq Cr^s. \tag{3.12}$$

Finally, (3.11)-(3.12) yield (3.10), exactly as in the proof of Theorem 1.2 in [RS14b]. □

Remark 3.4. Notice that, thanks to Proposition 3.2, we have that Theorem 1.2 holds for all right hand sides satisfying $|f(x)| \leq Cd^{\alpha-s}$ in Ω.

3.3. **Equations with bounded measurable coefficients.** We prove now Theorem 1.5.

First, we show the following C^α estimate up to the boundary.

Proposition 3.5. *Let $s \in (0,1)$, and Ω be any bounded $C^{1,\alpha}$ domain.*

Let u be a solution to

$$\begin{cases} M^+u & \geq & -K_0 d^{\epsilon-s} & \text{in } B_1\cap\Omega \\ M^-u & \leq & K_0 d^{\epsilon-s} & \text{in } B_1\cap\Omega \\ u & = & 0 & \text{in } B_1\setminus\Omega. \end{cases} \tag{3.13}$$

Then,

$$\|u\|_{C^{\bar\alpha}(B_{1/2})} \leq C\left(K_0 + \sup_{R\geq 1} R^{\delta-2s}\|u\|_{L^\infty(B_R)}\right).$$

The constant C depends only on n, s, ϵ, δ, Ω, and ellipticity constants.

Proof. The proof is very similar to that of Proposition 3.5.

First, using the supersolution given by Lemma 2.8, and by the exact same argument of Proposition 3.5, we find

$$|w| \leq Cd^s \qquad \text{in} \quad B_{1/2}.$$

Now, using the interior estimates of [CS09] one finds

$$[w]_{C^{\bar\alpha}(B_r(x_0))} \leq C$$

for all balls $B_r(x_0) \subset \Omega\cap B_{1/2}$ with $2r = d(x_0)$, and this yields

$$\|w\|_{C^{\bar\alpha}(B_{1/2})} \leq C,$$

as desired. □

We next show:

Proposition 3.6. *Let $s \in (0,1)$ and $\alpha \in (0,\bar\alpha)$. Let L be any operator of the form (1.1)-(1.2), Ω be any $C^{1,\alpha}$ domain, and ψ be given by Definition 2.1.*

Assume that $0 \in \partial\Omega$, and that $\partial\Omega\cap B_1$ can be represented as the graph of a $C^{1,\alpha}$ function with norm less or equal than 1.

Let u be any solution to (1.4), and let

$$K_0 = \|f\|_{L^\infty(B_1\cap\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)}.$$

Then, there exists a constant Q satisfying $|Q| \leq CK_0$ and

$$\left|u(x) - Q\psi^s(x)\right| \leq CK_0|x|^{s+\alpha}.$$

The constant C depends only on n, s, and ellipticity constants.

Proof. The proof is very similar to that of Proposition 3.2.

Assume by contradiction that we have Ω_k and u_k such that:

- Ω_k is a $C^{1,\alpha}$ domain that can be represented as the graph of a $C^{1,\alpha}$ function with norm is less or equal than 1;
- $0 \in \partial\Omega_k$ and the unit normal vector to $\partial\Omega_k$ at 0 is $-e_n$;
- u_k satisfies (1.4) with $K_0 = 1$;
- For any constant Q, $\sup_{r>0} \sup_{B_r} r^{-s-\alpha}|u_k - Q\psi_k^s| = \infty$.

Then, by Lemma 3.3 we will have

$$\sup_k \sup_{r>0} \|u_k - \phi_{k,r}\|_{L^\infty(B_r)} = \infty,$$

where

$$\phi_{k,r}(x) = Q_k(r)\psi_k^s, \qquad Q_k(r) = \frac{\int_{B_r} u_k\psi_k^s}{\int_{B_r} \psi_k^{2s}}.$$

We now define $\theta(r)$, $r_m \to 0$, and v_m as in the proof of Proposition 3.2. Then, we have

$$\int_{B_1} v_m(x)\psi_k^s(r_m x)dx = 0, \tag{3.14}$$

$$\|v_m\|_{L^\infty(B_1)} \geq \frac{1}{2}, \tag{3.15}$$

and

$$\|v_m\|_{L^\infty(B_R)} \leq CR^{s+\alpha} \qquad \text{for all} \quad R \geq 1. \tag{3.16}$$

Moreover, the functions v_m satisfy

$$M^- v_m(x) \leq \frac{(r_m)^{2s}}{(r_m)^{s+\alpha}\theta(r_m)} + \frac{(r_m)^{2s}}{(r_m)^{s+\alpha}\theta(r_m)}\,(M^+\psi_{k_m})(r_m x)$$

in $(r_m^{-1}\Omega_{k_m}) \cap B_{r_m^{-1}}$. Using Lemma 2.3, and denoting $\Omega_m = r_m^{-1}\Omega_{k_m}$ and $d_m(x) = \mathrm{dist}(x, r_m^{-1}\Omega_{k_m})$, we find

$$M^- v_m \leq \frac{C}{\theta(r_m)}\, d_m^{\alpha-s}(x) \qquad \text{in} \quad \Omega_m \cap B_{r_m^{-1}}. \tag{3.17}$$

Similarly, we find

$$M^+ v_m \geq -\frac{C}{\theta(r_m)}\, d_m^{\alpha-s}(x) \qquad \text{in} \quad \Omega_m \cap B_{r_m^{-1}}.$$

Notice that the domains Ω_m converge locally uniformly to $\{x_n > 0\}$ as $m \to \infty$.

Next, by Proposition 3.5, we find that for each fixed $M \geq 1$

$$\|v_m\|_{C^{\bar\alpha}(B_M)} \leq C(M) \qquad \text{for all } m \text{ with } r_m^{-1} > 2M.$$

The constant $C(M)$ does not depend on m. Hence, by Arzelà-Ascoli theorem, a subsequence of v_m converges locally uniformly to a function $v \in C(\mathbb{R}^n)$.

Hence, passing to the limit the equation (3.17) we get

$$M^- v \leq 0 \leq M^+ v \quad \text{in } \{x_n > 0\}.$$

Moreover, we also have $v = 0$ in $\{x_n \leq 0\}$, and $v \in C(\mathbb{R}^n)$.

Thus, by the classification result [RS14, Proposition 5.1], we find

$$v(x) = \kappa(x_n)_+^s \tag{3.18}$$

for some $\kappa \in \mathbb{R}$. But passing (3.14) —multiplied by $(r_m)^{-s}$— to the limit, we find

$$\int_{B_1} v(x)(x_n)_+^s dx = 0.$$

This means that $v \equiv 0$, a contradiction with (3.15). □

Finally, we give the:

Proof of Theorem 1.5. The result follows from Proposition 3.6; see the proof of Theorem 1.2. □

4. Barriers: C^1 domains

We construct now sub and supersolutions that will be needed in the proof of Theorem 1.3. Recall that in C^1 domains one does not expect solutions to be comparable to d^s, and this is why the sub and supersolutions we construct have slightly different behaviors near the boundary. Namely, they will be comparable to $d^{s+\epsilon}$ and $d^{s-\epsilon}$, respectively.

Lemma 4.1. *Let $s \in (0,1)$, and $e \in S^{n-1}$. Define*

$$\Phi_{\rm sub}(x) := \left(e \cdot x - \eta|x|\left(1 - \frac{(e\cdot x)^2}{|x|^2}\right)\right)_+^{s+\epsilon}$$

and

$$\Phi_{\rm super}(x) := \left(e \cdot x + \eta|x|\left(1 - \frac{(e\cdot x)^2}{|x|^2}\right)\right)_+^{s-\epsilon}$$

For every $\epsilon > 0$ there is $\eta > 0$ such that two functions $\Phi_{\rm sub}$ and $\Phi_{\rm super}$ satisfy, for all $L \in \mathcal{L}_*$,

$$\begin{cases} L\Phi_{\rm sub} \geq c_\epsilon d^{\epsilon - s} > 0 & \text{in } \mathcal{C}_\eta \\ \Phi_{\rm sub} = 0 & \text{in } \mathbb{R}^n \setminus \mathcal{C}_\eta \end{cases}$$

and

$$\begin{cases} L\Phi_{\rm super} \leq -c_\epsilon d^{-\epsilon - s} < 0 & \text{in } \mathcal{C}_{-\eta} \\ \Phi_{\rm super} = 0 & \text{in } \mathbb{R}^n \setminus \mathcal{C}_{-\eta} \end{cases}$$

where $\mathcal{C}_{\pm\eta}$ is the cone

$$\mathcal{C}_{\pm\eta} := \left\{ x \in \mathbb{R}^n \ : e \cdot \frac{x}{|x|} > \pm\eta\left(1 - \left(e \cdot \frac{x}{|x|}\right)^2\right)\right\}.$$

The constant η depends only on ϵ, s, and ellipticity constants.

Proof. We prove the statement for $\Phi_{\rm sub}$. The statement for $\Phi_{\rm super}$ is proved similarly.

Let us denote $\Phi := \Phi_{\rm sub}$. By homogeneity it is enough to prove that $L\Phi \ge c_\epsilon > 0$ on points belonging to $e+\partial\mathcal{C}_\eta$, since all the positive dilations of this set with respect to the origin cover the interior of $\tilde{\mathcal{C}}_\eta$.

Let thus $P \in \partial\mathcal{C}_\eta$, that is,

$$e\cdot P - \eta\left(|P| - \frac{(e\cdot P)^2}{|P|}\right) = 0.$$

Consider

$$\begin{aligned}\Phi_{P,\eta}(x) &:= \Phi(P+e+x)\\ &= \left(e\cdot(P+e+x) - \eta\left(|P+e+x| - \frac{(e\cdot(P+e+x))^2}{|P+e+x|}\right)\right)_+^{s+\epsilon}\\ &= \left(1+e\cdot x - \eta\left(|P+e+x| - |P| - \frac{(e\cdot(P+e+x))^2}{|P+e+x|} + \frac{(e\cdot P)^2}{|P|}\right)\right)_+^{s+\epsilon}\\ &= \bigl(1+e\cdot x - \eta\psi_P(x)\bigr)_+^{s+\epsilon},\end{aligned}$$

where we define

$$\psi_P(x) := |P+e+x| - |P| - \frac{(e\cdot(P+e+x))^2}{|P+e+x|} + \frac{(e\cdot P)^2}{|P|}.$$

Note that the functions ψ_P satisfy

$$\psi_P(0) = 0,$$

$$|\nabla\psi_P(x)| \le C \quad \text{in } \mathbb{R}^n\setminus\{-P-e\},$$

and

$$|D^2\psi_P(x)| \le C \quad \text{for } x\in B_{1/2},$$

where C does not depend on P (recall that $|e|=1$).

Then, the family $\Phi_{P,\eta}$ satisfies

$$\Phi_{P,\eta} \to (1+e\cdot x)_+^{s+\epsilon} \quad \text{in } C^2(\overline{B_{1/2}})$$

as $\eta\searrow 0$, uniformly in P and moreover

$$\int_{\mathbb{R}^n} \frac{\bigl|\Phi_{P,\eta} - (1+e\cdot x)_+^{s+\epsilon}\bigr|}{1+|x|^{n+2s}}\,dx \le \int_{\mathbb{R}^n}\frac{C(C\eta|x|)^{s+\epsilon}}{1+|x|^{n+2s}}\,dx \le C\eta^{s+\epsilon}.$$

Thus,

$$L\Phi_{P,\eta}(0) \to L\bigl((1+e\cdot\quad)_+^{s+\epsilon}\bigr)(0) \ge c(s,\epsilon,\lambda) > 0 \quad \text{as } \eta\searrow 0$$

uniformly in P.

In particular one can chose $\eta = \eta(s,\epsilon,\lambda,\Lambda)$ so that $L\Phi_{P,\eta}(0) \ge c_\epsilon > 0$ for all $P\in\partial\tilde{\mathcal{C}}_\eta$ and for all $L\in\mathcal{L}_*$, and the lemma is proved. □

5. Regularity in C^1 domains

We prove here Theorems 1.3 and 1.6.

Definition 5.1. Let $r_0 > 0$ and let $\rho : (0, r_0] \to 0$ be a nonincreasing function with $\lim_{t\downarrow 0} \rho(t) = 0$. We say that a domain Ω is *improving Lipschitz* at 0 with inwards unit normal vector $e_n = (0, \ldots, 0, 1)$ and modulus ρ if

$$\Omega \cap B_r = \{(x', x_n) : x_n > g(x')\} \cap B_r \qquad \text{for} \quad r \in (0, r_0],$$

where $g : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfies

$$\|g\|_{\mathrm{Lip}(B_r)} \le \rho(r) \quad \text{for } 0 < r \le r_0.$$

We say that Ω is *improving Lipschitz* at $x_0 \in \partial\Omega$ with inwards unit normal $e \in S^{n-1}$ if the normal vector to $\partial\Omega$ at x_0 is e and, after a rotation, the domain $\Omega - x_0$ satisfies the previous definition.

We first prove the following C^α estimate up to the boundary.

Lemma 5.2. *Let $s \in (0, 1)$, and let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain in B_1 with Lipschitz constant less than ℓ. Namely, assume that after a rotation we have*

$$\Omega \cap B_1 = \{(x', x_n) : x_n > g(x')\} \cap B_1,$$

with $\|g\|_{\mathrm{Lip}(B_1)} \le \ell$. Let $u \in C(B_1)$ be a viscosity solution of

$$M^+ u \ge -K_0 d^{\epsilon - s} \quad \textit{and} \quad M^- u \le K_0 d^{\epsilon - s} \quad \textit{in } \Omega \cap B_1,$$
$$u = 0 \quad \textit{in } B_1 \setminus \Omega.$$

Assume that

$$\|u\|_{L^\infty(B_R)} \le K_0 R^{2s-\epsilon} \quad \textit{for all } R \ge 1.$$

Then, if $\ell \le \ell_0$, where $\ell_0 = \ell_0(n, s, \lambda, \Lambda)$, we have

$$\|u\|_{C^{\bar\alpha}(B_{1/2})} \le C K_0.$$

The constants C and $\bar\alpha$ depend only on n, s, ϵ and ellipticity constants.

Proof. By truncating u in B_2 and dividing it by CK_0 we may assume that

$$\|u\|_{L^\infty(\mathbb{R}^n)} = 1$$

and that

$$M^+ u \ge -d^{\epsilon - s} \quad \text{and} \quad M^- u \le d^{\epsilon - s} \quad \text{in } \Omega \cap B_1.$$

Now, we divide the proof into two steps.

Step 1. We first prove that

$$\big|u(x)\big| \le C|x - x_0|^\alpha \quad \text{in } \Omega \cap B_{3/4}, \tag{5.1}$$

where $x_0 \in \partial\Omega$ is the closest point to x on $\partial\Omega$. We will prove (5.1) by using a supersolution. Indeed, given $\epsilon \in (0, s)$, let Φ_{super} and $\mathcal{C}_\eta$ be the homogeneous supersolution and the cone from Lemma 4.1, where $e = e_n$. Note that Φ_{super} is a positive function satisfying $M^- \Phi_{\mathrm{super}} \le -c d^{-\epsilon - s} < 0$ outside the convex cone $\mathbb{R}^n \setminus \mathcal{C}_\eta$, and it is homogeneous of degree $s - \epsilon$.

Then, we easily check that the function $\psi = C\Phi_{\rm super} - \chi_{B_1(z_0)}$, with C large and $|z_0| \ge 2$ such that $\Phi_{\rm super} \ge 1$ in $B_1(z_0)$, satisfies $M^+\psi \le -d^{\epsilon-s}$ in $B_{1/4}\cap\mathcal{C}_\eta$ and $\psi \ge \frac14$ in $\mathcal{C}_\eta \setminus B_{1/4}$. Indeed, we simply use that $M^-\chi_{B_1(z_0)} \ge c_0 > 0$ in $B_{1/4}$. Note that this argument exploits the nonlocal character of the operator and a slightly more complicated one would be needed in order to obtain a result that is stable as $s \uparrow 1$.

Note that the supersolution ψ vanishes in $B_{1/4}\setminus\mathcal{C}_\eta$. Then, if ℓ_0 is small enough, for every point in $x_0 \in \partial\Omega\cap B_{3/4}$ we will have

$$x_0 + (B_{1/4}\setminus\mathcal{C}_\eta) \subset B_1\setminus\Omega.$$

Then, using translates of ψ (and $-\psi$) upper (lower) barriers we get $\big|u(x)\big| \le \psi(x_0+x) \le C|x-x_0|^{s-\epsilon}$, as desired.

Step 2. To obtain a C^α estimate up to the boundary, we use the following interior estimate from [CS09]: Let $r\in(0,1)$,

$$M^+u \ge -r^{\alpha-2s} \quad \text{and } M^-u \le r^{\alpha-2s} \quad \text{in } B_r(x)$$

and

$$|u(z)| \le r^\alpha\left(1+\frac{(z-x)^\alpha}{r^\alpha}\right) \quad \text{in all of } \mathbb{R}^n.$$

Then,

$$[u]_{C^\alpha(B_{r/2}(x))} \le C,$$

with C and $\alpha>0$ depending only s, ellipticity constants and dimension.

Combining this estimate with (5.1), it follows that

$$\|u\|_{C^\alpha(B_{1/2})} \le C.$$

Thus, the lemma is proved. □

We will also need the following.

Lemma 5.3. *Let $s\in(0,1)$, $\alpha\in(0,\bar\alpha)$, and $C_0\ge1$. Given $\epsilon\in(0,\alpha]$, there exist $\delta>0$ depending only on ϵ, n, s, and ellipticity constants, such that the following statement holds.*

Assume that $\Omega\subset\mathbb{R}^n$ is a Lipchitz domain such that $\partial\Omega\cap B_{1/\delta}$ is a Lipchitz graph of the form $x_n = g(x')$, in $|x'|<1/\delta$ with

$$[g]_{{\rm Lip}(B_{1/\delta})} \le \delta,$$

and $0\in\partial\Omega$.

Let $\varphi\in C(\mathbb{R}^n)$ be a viscosity solution of

$$M^+\varphi \ge -\delta\, d^{\epsilon-s} \quad \textit{and} \quad M^-\varphi \le \delta\, d^{\epsilon-s} \quad \textit{in } \Omega\cap B_{1/\delta},$$

$$\varphi = 0 \quad \textit{in } B_{1/\delta}\setminus\Omega,$$

satisfying

$$\varphi \ge 0 \quad \textit{in } B_1.$$

Assume that φ satisfies

$$\sup_{B_1} \varphi = 1 \quad \textit{and} \quad \|\varphi\|_{L^\infty(B_{2^l})} \le C_0(2^l)^{s+\alpha} \quad \textit{for all } l \ge 0.$$

Then, we have

$$\int_{B_1} \varphi^2\, dx \ge \frac{1}{2} \int_{B_1} (x_n)_+^{2s}\, dx \tag{5.2}$$

and

$$\left(\frac{1}{2}\right)^{s+\epsilon} \le \frac{\sup_{B_{2^{l-1}}} \varphi}{\sup_{B_{2^l}} \varphi} \le \left(\frac{1}{2}\right)^{s-\epsilon} \quad \textit{for all } l \le 0. \tag{5.3}$$

Proof. Step 1. We first prove that, for δ small enough, we have (5.2) and

$$\left(\frac{1}{2}\right)^{s+\epsilon} \le \frac{\sup_{B_{1/2}} \varphi}{\sup_{B_1} \varphi} \le \left(\frac{1}{2}\right)^{s-\epsilon} \tag{5.4}$$

In a second step we will iterate (5.4) to show (5.3).

The proof of (5.4) is by compactness. Suppose that there is a sequence φ_k of functions satisfying the assumptions with $\delta = \delta_k \downarrow 0$ for which one of the three possibilities

$$\left(\frac{1}{2}\right)^{s+\epsilon} > \frac{\sup_{B_{1/2}} \varphi_k}{\sup_{B_1} \varphi_k}, \tag{5.5}$$

$$\frac{\sup_{B_{1/2}} \varphi_k}{\sup_{B_1} \varphi_k} > \left(\frac{1}{2}\right)^{s-\epsilon} \tag{5.6}$$

or

$$\int_{B_1} \varphi_k^2\, dx < \frac{1}{2} \int_{B_1} (x_n)_+^{2s}\, dx \tag{5.7}$$

holds for all $k \ge 1$.

Let us show that a subsequence of φ_k converges locally uniformly $\mathbb{R}^n$ to the function $(x_n)_+^s$. Indeed, thanks to Lemma 5.2 and the Arzela-Ascoli theorem a subsequence of φ_k converges to a function $\varphi \in C(\mathbb{R}^n)$, which satisfies $M^+\varphi \ge 0$ and $M^-\varphi \le 0$ in $\mathbb{R}^n_+$, and $\varphi = 0$ in $\mathbb{R}^n_-$. Here we used that $\delta_k \to 0$. Moreover, by the growth control $\|\varphi\|_{L^\infty(B_R)} \le CR^{s+\alpha}$ and the classification theorem [RS14, Proposition 5.1], we find $\varphi(x) = K(x_n)_+^s$. But since $\sup_{B_1} \varphi_k = 1$, then $K = 1$.

Therefore, we have proved that a subsequence of φ_k converges uniformly in B_1 to $(x_n)_+^s$. Passing to the limit (5.5), (5.6) or (5.7), we reach a contradiction.

Step 2. We next show that we can iterate (5.4) to obtain (5.3) by induction. Assume that for some $m \le 0$ we have

$$\left(\frac{1}{2}\right)^{s+\epsilon} \le \frac{\sup_{B_{2^{l-1}}} \varphi}{\sup_{B_{2^l}} \varphi} \le \left(\frac{1}{2}\right)^{s-\epsilon} \quad \text{for } m \le l \le 0. \tag{5.8}$$

We then consider the function

$$\bar\varphi = \frac{\varphi(2^{-m}x)}{\sup_{B_{2^m}} \varphi},$$

and notice that

$$2^{(s+\epsilon)l} \le \sup_{B_{2^l}} \varphi \le 2^{(s-\epsilon)l} \quad \text{for } m \le l \le 0.$$

Thus,

$$M^+\bar\varphi \ge -\frac{\delta 2^{2sm}}{2^{(s+\epsilon)m}} \ge -\delta \quad \text{in } (2^{-m}\Omega) \cap B_{2^{-m}/\delta}$$

and similarly

$$M^-\bar\varphi \le \delta \quad \text{in } (2^{-m}\Omega) \cap B_{2^{-m}/\delta}.$$

Clearly

$$\bar\varphi = 0 \quad \text{in } (2^{-m}\mathcal{C}\Omega) \cap B_{2^{-m}/\delta}$$

and

$$\varphi \ge 0 \quad \text{in } B_{2^{-m}} \supset B_1.$$

Since $\partial\Omega$ is Lipchitz with constant δ in $B_{1/\delta}$ and $2^{-m} \ge 1$ ($m \le 0$) we have that the rescaled domain $(2^{-m}\Omega) \cap B_{2^{-m}/\delta}$ is also Lipchitz with the same constant $1/\delta$ in a larger ball.

Finally, using (5.8) again we find

$$\sup_{B_{2^l}} \bar\varphi = \frac{\sup_{B_{2^{l+m}}} \varphi}{\sup_{B_{2^m}} \varphi} \le 2^{(s+\epsilon)l} \le 2^{(s+\alpha)l} \qquad \text{for } l \ge 0 \text{ with } l+m \le 0,$$

For $l+m > 0$ we have

$$\sup_{B_{2^l}} \bar\varphi = \frac{\sup_{B_{2^{l+m}}} \varphi}{2^{(s+\epsilon)m}\varphi} \le \frac{C_0 2^{(s+\alpha)(l+m)}}{2^{(s+\epsilon)m}} = C_0 2^{(s+\alpha)l} 2^{(\alpha-\epsilon)m} \le C_0 2^{(s+\alpha)l}.$$

Hence, using Step 1, we obtain

$$\left(\frac{1}{2}\right)^{s+\epsilon} \le \frac{\sup_{B_{1/2}} \bar\varphi}{\sup_{B_1} \bar\varphi} \le \left(\frac{1}{2}\right)^{s-\epsilon}.$$

Thus (5.8) holds for $l = m-1$, and the lemma is proved. □

We next prove the following.

Proposition 5.4. *Let $s \in (0,1)$, $\alpha \in (0,\bar\alpha)$, and $C_0 \ge 1$.*

Let $\Omega \subset \mathbb{R}^n$ be a domain that is improving Lipschitz at 0 with unit outward normal $e \in S^{n-1}$ and with modulus of continuity ρ (see Definition 5.1). Then, there exists $\delta > 0$, depending only on α, s, C_0, ellipticity constants, and dimension such that the following statement holds.

Assume that $r_0 = 1/\delta$ and $\rho(1/\delta) < \delta$. Suppose that $u, \varphi \in C(\mathbb{R}^n)$ are viscosity solutions of

$$\begin{cases} M^+(au+b\varphi) \ge -\delta(|a|+|b|)d^{\alpha-s} & \text{in } B_{1/\delta} \cap \Omega \\ u = \varphi = 0 & \text{in } B_{1/\delta} \setminus \Omega, \end{cases} \tag{5.9}$$

for all $a, b \in \mathbb{R}$. Moreover, assume that

$$\|au+b\varphi\|_{L^\infty(\mathbb{R}^n)} \le C_0(|a|+|b|)R^{s+\alpha} \quad \text{for all } R \ge 1, \tag{5.10}$$

$$\varphi \geq 0 \quad in\ B_1, \quad and \quad \sup_{B_1} \varphi = 1.$$

Then, there is $K \in \mathbb{R}$ with $|K| \leq C$ such that

$$\big|u(x) - K\varphi(x)\big| \leq C|x|^{s+\alpha} \quad in\ B_1,$$

where C depends only on ρ, C_0, α, s, ellipticity constants, and dimension.

Proof. Step 1 (preliminary results). Fix $\epsilon \in (0, \alpha)$. Using Lemma 5.3, if δ is small enough we have

$$\int_{B_1} \varphi^2\,dx \geq \frac{1}{2}\int_{B_1} (x_n)_+^{2s}\,dx \geq c(n,s) > 0 \tag{5.11}$$

and

$$\left(\frac{1}{2}\right)^{s+\epsilon} \leq \frac{\sup_{B_{2^{l-1}}} \varphi}{\sup_{B_{2^l}} \varphi} \leq \left(\frac{1}{2}\right)^{s-\epsilon} \qquad \text{for all } l \leq 0. \tag{5.12}$$

In particular, since $\sup_{B_1} \varphi = 1$ then

$$(r/2)^{s+\epsilon} \leq \sup_{B_r} \varphi \leq (2r)^{s-\epsilon} \quad \text{for all } r \in (0,1). \tag{5.13}$$

Step 2. We prove now, with a blow-up argument, that

$$\big\|u(x) - K_r\varphi(x)\big\|_{L^\infty(B_r)} \leq Cr^{s+\alpha} \tag{5.14}$$

for all $r \in (0,1]$, where

$$K_r := \frac{\int_{B_r} u\,\varphi\,dx}{\int_{B_r} \varphi^2\,dx}. \tag{5.15}$$

Notice that (5.14) implies the estimate of the proposition with $K = \lim_{r\searrow 0} K_r$. Indeed, we have $|K_1| \leq C$ —which is immediate using (5.10) with $a = 1$ and $b = 0$ and (5.11)— and

$$\begin{aligned}
|K_r - K_{r/2}|(r/2)^{s+\epsilon} &\leq \big\|K_r\varphi - K_{r/2}\varphi\big\|_{L^\infty(B_r)} \\
&\leq \big\|u - K_r\varphi\big\|_{L^\infty(B_r)} + \big\|u - K_{r/2}\varphi\big\|_{L^\infty(B_r)} \\
&\leq Cr^{s+\alpha}.
\end{aligned}$$

Thus,

$$|K| \leq |K_1| + \sum_{j=0}^{\infty} |K_{2^{-j}} - K_{2^{-j-1}}| \leq C + C\sum_{j=0}^{\infty} 2^{-j(\alpha-\epsilon)} \leq C,$$

provided that ϵ is taken smaller that α.

Let us prove (5.14) by contradiction. Assume that we have a sequences Ω_j, e_j,u_j, φ_j satisfying the assumptions of the Proposition, but not (5.14). That is,

$$\lim_{j\to\infty} \sup_{r>0} r^{-s-\alpha}\big\|u_j(x) - K_{r,j}\varphi_j\big\|_{L^\infty(B_r)} = \infty,$$

were $K_{r,j}$ is defined as in (5.15) with u replaced by u_j and φ replace by φ_j.

Define, for $r \in (0,1]$ the nonincreasing quantity

$$\theta(r) = \sup_{r' \in (r,1)} (r')^{-s-\alpha} \big\| u_j(x) - K_{r',j}\varphi_j \big\|_{L^\infty(B_{r'})}.$$

Note that $\theta(r) < \infty$ for $r > 0$ since $\|u_j\|_{L^\infty(\mathbb{R}^n)} \le 1$ and that $\lim_{r \searrow 0} \theta(r) = \infty$.

For every $m \in \mathbb{N}$, by definition of θ there exist $r'_m \ge 1/m$, j_m, $\Omega_m = \Omega_{j_m}$, and $e_m = e_{j_m}$ such that

$$(r'_m)^{-s-\alpha} \big\| u_{j_m}(x) - K_{r'_m,j_m}\varphi_{j_m} \big\|_{L^\infty(B_{r'_m})} \ge \frac{1}{2}\theta(1/m) \ge \frac{1}{2}\theta(r'_m).$$

Note that $r'_m \to 0$. Taking a subsequence we may assume that $e_m \to e \in S^{n-1}$. Denote

$$u_m = u_{j_m}, \quad K_m = K_{r'_m,j_m} \quad \text{and} \quad \varphi_m = \varphi_{j_m}.$$

We now consider the blow-up sequence

$$v_m(x) = \frac{u_m(r'_m x) - K_m \varphi_m(r'_m x)}{(r'_m)^{s+\alpha}\theta(r'_m)}.$$

By definition of θ and r'_m we will have

$$\|v_m\|_{L^\infty(B_1)} \ge \frac{1}{2}. \tag{5.16}$$

In addition, by definition of $K_m = K_{r'_m,j_m}$ we have

$$\int_{B_1} v_m(x)\varphi_m(r'_m x)\, dx = 0 \tag{5.17}$$

for all $m \ge 1$.

Let us prove that

$$\|v_m\|_{L^\infty(B_R)} \le C R^{s+\alpha} \quad \text{for all } R \ge 1. \tag{5.18}$$

Indeed, first, by definition of $\theta(2r)$ and $\theta(r)$,

$$\begin{aligned}
\frac{\big\| K_{2r,j}\varphi_j - K_{r,j}\varphi_j \big\|_{L^\infty(B_r)}}{r^{s+\alpha}\theta(r)} &\le \frac{2^{s+\alpha}\theta(2r)}{\theta(r)} \frac{\big\| u_j - K_{r,j}\varphi_j \big\|_{L^\infty(B_{2r})}}{(2r)^{s+\alpha}\theta(2r)} + \frac{\big\| u_j - K_{r/2,j}\varphi_j \big\|_{L^\infty(B_r)}}{r^{s+\alpha}\theta(r)} \\
&\le 2^{s+\alpha} + 1 \le 5.
\end{aligned}$$

On the one hand, using Step 1 we have

$$\begin{aligned}
\frac{|K_{2r,j} - K_{r,j}|(r/2)^{s+\epsilon}}{r^{s+\alpha}\theta(r)} &\le \frac{|K_{2r,j} - K_{r,j}|\, \|\varphi_j\|_{L^\infty(B_r)}}{r^{s+\alpha}\theta(r)} \\
&= \frac{\big\| K_{2r,j}\varphi_j - K_{r,j}\varphi_j \big\|_{L^\infty(B_r)}}{r^{s+\alpha}\theta(r)} \\
&\le 5,
\end{aligned}$$

and therefore

$$|K_{2r,j} - K_{r,j}| \le 10\, r^{\alpha-\epsilon}\theta(r), \tag{5.19}$$

which we will use later on in this proof.

On the other hand, by (5.12) in Step 1 we have, whenever $0 < 2^l r \le 2^N r \le 1$,

$$\big\|\varphi_j\big\|_{L^\infty(B_{2^N r})} \le 2^{(s+\epsilon)(N-l)}\big\|\varphi_j\big\|_{L^\infty(B_{2^l r})}$$

and therefore

$$\begin{aligned}\frac{\big\|K_{2^{l+1}r,j}\varphi_j - K_{2^l r,j}\varphi_j\big\|_{L^\infty(B_{2^N r})}}{r^{s+\alpha}\theta(r)} &= \frac{\big|K_{2^{l+1}r,j} - K_{2^l r,j}\big|\,\big\|\varphi_j\big\|_{L^\infty(B_{2^N r})}}{r^{s+\alpha}\theta(r)}\\ &\le \frac{\big|K_{2^{l+1}r,j} - K_{2^l r,j}\big|\,2^{(s+\epsilon)(N-l)}\big\|\varphi_j\big\|_{L^\infty(B_{2^l r})}}{r^{s+\alpha}\theta(r)}\\ &= \frac{2^{(s+\epsilon)(N-l)}\big\|K_{2^{l+1}r,j}\varphi_j - K_{2^l r,j}\varphi_j\big\|_{L^\infty(B_{2^l r})}}{r^{s+\alpha}\theta(r)}\\ &= \frac{2^{l(s+\alpha)}\theta(2^l r)}{\theta(r)}\,\frac{2^{(s+\epsilon)(N-l)}\big\|K_{2^{l+1}r,j}\varphi_j - K_{2^l r,j}\varphi_j\big\|_{L^\infty(B_{2^l r})}}{(2^l r)^{s+\alpha}\theta(2^l r)}\\ &\le 10\,2^{(s+\epsilon)N}\,2^{l(\alpha-\epsilon)}.\end{aligned}$$

Thus,

$$\frac{\big\|K_{2^N r,j}\varphi_j - K_{r,j}\varphi_j\big\|_{L^\infty(B_{2^N r})}}{r^{s+\alpha}\theta(r)} \le 2^{(s+\epsilon)N}\sum_{l=0}^{N-1}2^{l(\alpha-\epsilon)} \le C2^{(s+\alpha)N},$$

where we have used that $\epsilon \in (0,\alpha)$.

Form the previous equation we deduce

$$\frac{\big\|K_{Rr,j}\varphi_j - K_{r,j}\varphi_j\big\|_{L^\infty(B_{Rr})}}{r^{s+\alpha}\theta(r)} \le CR^{s+\alpha}$$

whenever $0 < r \le Rr \le 1$.

Hence,

$$\begin{aligned}\|v_m\|_{L^\infty(B_R)} &= \frac{1}{\theta(r'_m)(r'_m)^{s+\alpha}}\big\|u_m - K_m\varphi_m\big\|_{L^\infty(B_{Rr'm})}\\ &\le \frac{R^{s+\alpha}\big\|u_{j_m} - K_{Rr'_m,j_m}\varphi_{j_m}\big\|_{L^\infty(B_{Rr'm})}}{\theta(r'_m)(Rr'_m)^{s+\alpha}} + \frac{\big\|K_{Rr'_m,j_m}\varphi_{j_m} - K_{r'_m,j_m}\varphi_{j_m}\big\|_{L^\infty(B_{Rr'_m})}}{(r'_m)^{s+\alpha}\theta(r'_m)}\\ &\le \frac{R^{s+\alpha}\theta(Rr'_m)}{\theta(r'_m)} + CR^{s+\alpha}\\ &\le CR^{s+\alpha},\end{aligned}$$

whenever $Rr'_m \le 1$.

When $Rr'_m \ge 1$ we simply use the assumption (5.10), namely,

$$\|au_m + b\varphi_m\|_{L^\infty(\mathbb{R}^n)} \le C_0(|a|+|b|)R^{s+\alpha} \quad \text{for all } R \ge 1,$$

twice, with $a = 1$, $b = -K_{1,j_m}$ and with $a = 0$, $b = 1$ to estimate

$$\begin{aligned}
\|v_m\|_{L^\infty(B_R)} &= \frac{1}{\theta(r'_m)(r'_m)^{s+\alpha}}\big\|u_m - K_m\varphi_m\big\|_{L^\infty(B_{Rr'm})} \\
&\le \frac{R^{s+\alpha}\big\|u_{j_m} - K_{1,j_m}\varphi_{j_m}\big\|_{L^\infty(B_{Rr'm})}}{\theta(r'_m)(Rr'_m)^{s+\alpha}} + \frac{\big\|K_{1,j_m}\varphi_{j_m} - K_{r'_m,j_m}\varphi_{j_m}\|_{L^\infty(B_{Rr'_m})}}{(r'_m)^{s+\alpha}\theta(r'_m)} \\
&\le C_0(1 + |K_{1,j_m}|)R^{s+\alpha} + \frac{\big\|K_{1,j_m}\varphi_{j_m} - K_{r'_m,j_m}\varphi_{j_m}\|_{L^\infty(B_1)}}{(r'_m)^{s+\alpha}\theta(r'_m)}\frac{\|\varphi_{j_m}\|_{L^\infty(B_{Rr'_m})}}{\|\varphi_{j_m}\|_{L^\infty(B_1)}} \\
&\le CR^{s+\alpha} + C\left(\frac{1}{r'_m}\right)^{s+\alpha}(Rr'_m)^{s+\epsilon} \\
&\le CR^{s+\alpha} + C(r'_m)^{-s-\alpha}(Rr'_m)^{s+\alpha} \le CR^{s+\alpha},
\end{aligned}$$

where we have used $|K_{1,j_m}| \le C$ (that we will prove in detail in Step 3).

Step 3. We prove that a subsequence of v_m converges locally uniformly to a entire solution v_∞ of the problem

$$\begin{cases} M^+v_\infty \ge 0 \ge M^-v_\infty & \text{in } \{e\cdot x > 0\} \\ v_\infty = 0 & \text{in } \{e \cdot x < 0\}. \end{cases} \tag{5.20}$$

By assumption, the function $w = au_m + b\varphi_m$ satisfies

$$\begin{cases} M^+(au_m + b\varphi_m) \ge -\delta(|a| + |b|)d^{\alpha-s} & \text{in } B_1 \cap \Omega_m \\ u_m = \varphi_m = 0 & \text{in } B_1 \setminus \Omega_m, \end{cases} \tag{5.21}$$

for all $a, b \in \mathbb{R}$.

Now, using (5.19) we obtain

$$\begin{aligned}
\frac{|K_{1,j} - K_{2^{-N},j}|}{\theta(2^{-N})} &\le \sum_{l=0}^{N-1}\frac{|K_{2^{-N+l+1},j} - K_{2^{-N+l},j}|}{\theta(2^{-N})} \\
&= \sum_{l=0}^{N-1} 10\,\frac{\theta(2^{-N+l})}{\theta(2^{-N})}2^{(-N+l)(\alpha-\epsilon)} \\
&\le 10\sum_{l=0}^{N-1}2^{(-N+l)(\alpha-\epsilon)} \le C,
\end{aligned}$$

since $\alpha - \epsilon > 0$.

Next, using (5.11) —that holds with φ replaced by φ_j—,the definition $K_{r,j}$, and that $\|\varphi_j\|_{L^\infty(B_1)} = 1$ while $\|u_j\|_{L^\infty(B_1)} \le C_0$, we obtain

$$\big|K_{1,j}\big| = \left|\frac{\int_{B_1} u_j\,\varphi_j\,dx}{\int_{B_1}\varphi_j^2\,dx}\right| \le C. \tag{5.22}$$

Thus

$$\frac{\left|K_{2^{-N},j}\right|}{\theta(2^{-N})} \le \frac{\left|K_{1,j}\right|}{\theta(2^{-N})} + \frac{\left|K_{1,j} - K_{2^{-N},j}\right|}{\theta(2^{-N})} \le C$$

Using this control for $K_{r,j}$ and setting in (5.21)

$$a = \frac{1}{\theta(r'_m)} \quad \text{and} \quad b = \frac{-K_{r'_m,j_m}}{\theta(r'_m)}$$

we obtain

$$\begin{aligned} M^+ v_m &= \frac{(r'_m)^{2s}}{(r'_m)^{s+\alpha}\theta(r'_m)} M^+ \left(\frac{1}{\theta(r'_m)}\, u_m - \frac{K_{r'_m,j_m}}{\theta(r'_m)} \varphi_m \right) (r'_m\,\cdot) \\ &\ge -C\delta\, \frac{d_m^{\alpha-s}}{\theta(r'_m)} \quad \text{in } B_{(r'_m)^{-1}} \cap (r'_m)^{-1}\Omega_m, \end{aligned}$$

where $d_m(x) = \mathrm{dist}(x, r_m^{-1}\Omega_{k_m})$. Similarly, changing sign in the previous choices of a and b we obtain

$$-M^-(v_m) = M^+(-v_m) \ge -C\delta\, \frac{d_m^{\alpha-s}}{\theta(r'_m)} \quad \text{in } B_{(r'_m)^{-1}} \cap (r'_m)^{-1}\Omega_m$$

As complement datum we clearly have

$$v_m = 0 \quad \text{in } B_{(r'_m)^{-1}} \setminus (r'_m)^{-1}\Omega_m.$$

Then, by Lemma 5.2 we have

$$\|v_m\|_{C^\gamma(B_R)} \le C(R) \quad \text{for all } m \text{ large enough.}$$

The constant $C(R)$ depends on R, but not on m.

Then, by Arzelà-Ascoli and the stability lemma in [CS11b, Lemma 4.3] we obtain that

$$v_m \to v_\infty \in C(\mathbb{R}^n),$$

locally uniformly, where v_∞ satisfies the growth control

$$\|v_\infty\|_{L^\infty(B_R)} \le CR^{s+\alpha} \quad \text{for all } R \ge 1$$

and solves (5.20) in the viscosity sense. Thus, by the Liouville-type result [RS14, Proposition 5.1], we find $v_\infty(x) = K(x\cdot e)_+^s$ for some $K \in \mathbb{R}$.

Step 4. We prove that as subsequence of $\tilde\varphi_m$, where

$$\tilde\varphi_m(x) = \frac{\varphi_m(r'_m x)}{\sup_{B_{r'_m}} \varphi_m},$$

converges locally uniformly to $(x\cdot e)_+^s$.

This is similar to Step 3 and we only need to use the estimates in Step 1, and the growth control (5.10), to obtain a uniform control of the type

$$\|\tilde\varphi_m\|_{L^\infty(B_R)} \le C_0 R^{s+\alpha} \quad \text{for all } R \ge 1.$$

Using the estimates in Step 1 we easily show that

$$\frac{(r'_m)^{2s}}{\sup_{B_{r'_m}} \varphi_m} \downarrow 0.$$

Thus, we use (5.21) with $a = 0$ and $b = \left(\sup_{B_{r'_m}} \varphi_m\right)^{-1}$ to prove that $\tilde{\varphi}_m$ converges locally uniformly to a solution $\tilde{\varphi}_\infty$ of

$$\begin{cases} M^+\tilde{\varphi}_\infty \geq 0 \geq M^-\tilde{\varphi}_\infty & \text{in } \{e \cdot x > 0\} \\ \tilde{\varphi}_\infty = 0 & \text{in } \{e \cdot x < 0\}, \end{cases}$$

Then, using the Liouville-type result [RS14, Proposition 5.1] and since

$$\|\tilde{\varphi}_\infty\|_{L^\infty(B_1)} = \lim_{m\to\infty} \|\tilde{\varphi}_m\|_{L^\infty(B_1)} = \lim_{m\to\infty} 1 = 1$$

we get

$$\tilde{\varphi}_\infty \equiv (x \cdot e)_+^s.$$

Hence, $\tilde{\varphi}_m(x) \to (x \cdot e)_+^s$ locally uniformly in $\mathbb{R}^n$.

Step 5. We have $v_m \to K(x \cdot e)_+^s$ and $\tilde{\varphi}_m \to (x \cdot e)_+^s$ locally uniformly. Now, by (5.17),

$$\int_{B_1} v_m(x)\tilde{\varphi}_m(x)\, dx = 0.$$

Thus, passing this equation to the limits,

$$\int_{B_1} v_\infty(x)(x \cdot e)_+^s\, dx = 0.$$

This implies $K = 0$ and $v_\infty \equiv 0$.

But then passing to the limit (5.16) we get

$$\|v_\infty\|_{L^\infty(B_1)} \geq \frac{1}{2},$$

a contradiction. □

We next prove Theorems 1.3 and 1.6.

Proof of Theorem 1.6. Step 1. We first show, by a barrier argument, that for any given $\epsilon > 0$ we have

$$cd^{s+\epsilon} \leq u_i \leq Cd^{s-\epsilon} \quad \text{in } B_{1/2},$$

where $d = \operatorname{dist}(\,\cdot\,, B_1 \setminus \Omega)$, and $c > 0$ is a constant depending only on Ω, n, s, ellipticity constants.

First, notice that by assumption we have $M^-u_i = -M^+(-u_i) \leq \delta$ and $M^+u_i \geq -\delta$ in $B_1 \cap \Omega$. Therefore, since $\sup_{B_{1/2}} u_i \geq 1$, for any small $\rho > 0$ by the interior Harnack inequality we find

$$\inf_{B_{3/4}\cap\{d\geq\rho\}} u_i \geq C^{-1} - C\delta \geq c > 0,$$

provided that δ is small enough (depending on ρ).

Now, let $x_0 \in B_{1/2} \cap \partial\Omega$, and let $e \in S^{n-1}$ be the normal vector to $\partial\Omega$ at x_0. By the previous inequality,

$$\inf_{B_\rho(x_0+2\rho e)} u_i \geq c.$$

Since Ω is C^1, then for any $\eta > 0$ there is $\rho > 0$ for which

$$(x_0 + \mathcal{C}_\eta) \cap B_{4\rho} \subset \Omega,$$

where $\mathcal{C}_\eta$ is the cone in Lemma 4.1.

Therefore, using the function $\Phi_{\rm sub}$ given by Lemma 4.1, we may build the subsolution

$$\psi = \Phi_{\rm sub}\chi_{B_{4\rho}(x_0)} + C_1\chi_{B_{\rho/2}(x_0+2\rho e)}.$$

Indeed, if C_1 is large enough then ψ satisfies

$$M^-\psi \geq 1 \quad \text{in} \quad (x_0 + \mathcal{C}_\eta) \cap \big(B_{3\rho}(x_0) \setminus B_\rho(x_0 + 2\rho e)\big)$$

and $\psi \equiv 0$ outside $x_0 + \mathcal{C}_\eta$.

Hence, we may use $c_2\psi$ as a barrier, with c_2 small enough so that $u_i \geq c_2\psi$ in $B_\rho(x_0 + 2\rho e)$. Then, by the comparison principle we find

$$u_i \geq c_2\psi,$$

and in particular

$$u_i(x_0 + te) \geq c_3\, t^{s+\epsilon}$$

for $t \in (0, \rho)$. Since this can be done for all $x_0 \in B_{1/2} \cap \partial\Omega$, we find

$$u_i \geq c\, d^{s+\epsilon} \quad \text{in } B_{1/2}. \tag{5.23}$$

Similarly, using the supersolution $\Phi_{\rm sup}$ from Lemma 4.1, we find

$$u_i \leq C\, d^{s-\epsilon} \quad \text{in } B_{1/2}, \tag{5.24}$$

for $i = 1, 2$.

Step 2. Let us prove now that

$$u_1 \leq Cu_2 \quad \text{in} \quad B_{1/2}. \tag{5.25}$$

To prove 5.25, we rescale the functions u_1 and u_2 and use Proposition 5.4.

Let $x_0 \in B_{1/2} \cap \partial\Omega$, and let

$$\theta(r) = \sup_{r'>r} \frac{\|u_1\|_{L^\infty(B_{r'}(x_0))} + \|u_2\|_{L^\infty(B_{r'}(x_0))}}{(r')^{s+\epsilon}}.$$

Notice that $\theta(r)$ is monotone nonincreasing and that $\theta(r) \to \infty$ by (5.23). Let $r_k \to 0$ be such that

$$\|u_1\|_{L^\infty(B_{r_k}(x_0))} + \|u_2\|_{L^\infty(B_{r_k}(x_0))} \geq \frac{1}{2}\,(r_k)^{s+\epsilon}\theta(r_k),$$

with $c_0 > 0$, and define

$$v_k(x) = \frac{u_1(x_0 + r_k x)}{(r_k)^{s+\epsilon}\theta(r_k)}, \qquad w_k(x) = \frac{u_2(x_0 + r_k x)}{(r_k)^{s+\epsilon}\theta(r_k)}.$$

Note that

$$\|v_k\|_{L^\infty(B_1)} + \|w_k\|_{L^\infty(B_1)} \geq \frac{1}{2}.$$

Moreover,

$$\|v_k\|_{L^\infty(B_R)} = \frac{\|u_1\|_{L^\infty(B_{r_kR})}}{(r_k)^{s+\epsilon}\theta(r_k)} \leq \frac{\theta(r_kR)(r_kR)^{s+\epsilon}}{(r_k)^{s+\epsilon}\theta(r_k)} \leq R^{s+\epsilon},$$

for all $R \geq 1$, and analogously

$$\|w_k\|_{L^\infty(B_R)} \leq R^{s+\epsilon}$$

for all $R \geq 1$.

Now, the functions v_k, w_k satisfy the equation

$$M^+(av_k + bw_k)(x) = \frac{(r_k)^{2s}}{(r_k)^{s+\epsilon}\theta(r_k)}\, M^+(au_1 + bu_2)(x_0 + r_kx) \geq -C_0(r_k)^{s-\epsilon}\delta(|a| + |b|)$$

in $\Omega_k \cap B_{r_k^{-1}}$, where $\Omega_k = r_k^{-1}(\Omega - x_0)$.

Taking k large enough, we will have that Ω_k satisfies the hypotheses of Proposition 5.4 in $B_{1/\delta}$, and

$$M^+(av_k + bw_k) \geq -\delta(|a| + |b|) \qquad \text{in} \quad \Omega_k \cap B_{1/\delta}.$$

Moreover, since $\sup_{B_1} v_k + \sup_{B_1} w_k \geq 1/2$, then either $\sup_{B_1} v_k \geq 1/4$ or $\sup_{B_1} w_k \geq 1/4$. Therefore, by Proposition 5.4 we find that either

$$|v_k(x) - K_1w_k(x)| \leq C|x|^{s+\alpha}$$

or

$$|w_k(x) - K_2v_k(x)| \leq C|x|^{s+\alpha}$$

for some $|K| \leq C$. This yields that either

$$|u_1(x) - K_1u_2(x)| \leq C|x - x_0|^{s+\alpha} \tag{5.26}$$

or

$$|u_2(x) - K_2u_1(x)| \leq C|x - x_0|^{s+\alpha}, \tag{5.27}$$

with a bigger constant C.

Now, we may choose $\epsilon > 0$ so that $\epsilon < \alpha/2$, and then (5.27) combined with (5.23)-(5.24) gives $K_2 \geq c > 0$, which in turn implies (5.26) for $K_1 = K_2^{-1}$, $|K_1| \leq C$. Thus, in any case (5.26) is proved.

In particular, for all $x_0 \in B_{1/2} \cap \partial\Omega$ and all $x \in B_{1/2} \cap \Omega$ we have

$$u_1(x)/u_2(x) \leq K_1 + \left|\frac{u_1(x)}{u_2(x)} - K_1\right| \leq K_1 + C|x - x_0|^{s+\alpha}/u_2(x).$$

Choosing x_0 such that $|x - x_0| \leq Cd(x)$ and using (5.24), we deduce

$$u_1(x)/u_2(x) \leq K_1 + Cd^{s+\alpha}/d^{s-\epsilon} \leq C,$$

and thus (5.25) is proved.

Step 3. We finally show that $u_1/u_2 \in C^\alpha(\overline{\Omega}\cap B_{1/2})$ for all $\alpha\in(0,\bar\alpha)$. Since this last step is somewhat similar to the proof of Theorem 1.2 in [RS14b], we will omit some details.

We use that, for all $\alpha\in(0,\bar\alpha)$ and all $x\in B_{1/2}\cap\Omega$, we have

$$\left|\frac{u_1(x)}{u_2(x)} - K(x_0)\right| \leq C|x-x_0|^{\alpha-\epsilon}, \tag{5.28}$$

where $x_0\in B_{1/2}\cap\partial\Omega$ is now the closest point to x on $B_{1/2}\cap\partial\Omega$. This follows from (5.26), as shown in Step 2.

We also need interior estimates for u_1/u_2. Indeed, for any ball $B_{2r}(x)\subset\Omega\cap B_{1/2}$, with $2r=d(x)$, there is a constant K such that $\|u_1-Ku_2\|_{L^\infty(B_r(x))}\leq Cr^{s+\alpha}$. Thus, by interior estimates we find that $[u_1-Ku_2]_{C^{\alpha-\epsilon}(B_r(x))}\leq Cr^{s+\epsilon}$. This, combined with (5.23)-(5.24) yields

$$[u_1/u_2]_{C^{\alpha-\epsilon}(B_r(x))}\leq C. \tag{5.29}$$

Let now $x,y\in B_{1/2}\cap\Omega$, and let us show that

$$\left|\frac{u_1(x)}{u_2(x)} - \frac{u_1(y)}{u_2(y)}\right| \leq C|x-y|^{\alpha-\epsilon}. \tag{5.30}$$

If $y\in B_r(x)$, $2r=d(x)$, or if $x\in B_r(y)$, $2r=d(y)$, then this follows from (5.29). Otherwise, we have $|x-y|\geq\frac12\max\{d(x),d(y)\}$, and then (5.30) follows from (5.28).

In any case, (5.30) is proved, and therefore we have

$$\|u_1/u_2\|_{C^{\alpha-\epsilon}(\overline{\Omega}\cap B_{1/2})}\leq C.$$

Since this can be done for any $\alpha\in(0,\bar\alpha)$ and any $\epsilon>0$, the result follows. □

Proof of Theorem 1.3. The proof is the same as Theorem 1.6, replacing the Liouville-type result [RS14, Proposition 5.1] by [RS14b, Theorem 4.1], and replacing $\bar\alpha$ by s. □

Remark 5.5. Notice that in Proposition 5.4 we only require the right hand side of the equation to be bounded by $d^{\alpha-s}$. Thanks to this, Theorem 1.3 holds as well for

$$-\,\delta\, d^{\alpha-s}\leq f_i(x)\leq C_0 d^{\alpha-s}, \qquad \alpha\in(0,s). \tag{5.31}$$

In that case, we get

$$\|u_1/u_2\|_{C^\alpha(\Omega\cap B_{1/2})}\leq CC_0,$$

with the exponent α in (5.31).

Proof of Corollary 1.4. The result follows from Theorem 1.3. □

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