

THE BOUNDARY HARNACK PRINCIPLE FOR NONLOCAL ELLIPTIC OPERATORS IN NON-DIVERGENCE FORM

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ABSTRACT. We prove a boundary Harnack inequality for nonlocal elliptic operators L in non-divergence form with bounded measurable coefficients. Namely, our main result establishes that if $Lu_1 = Lu_2 = 0$ in $\Omega \cap B_1$, $u_1 = u_2 = 0$ in $B_1 \setminus \Omega$, and $u_1, u_2 \geq 0$ in \mathbb{R}^n , then u_1 and u_2 are comparable in $B_{1/2}$. The result applies to arbitrary open sets Ω .

When Ω is Lipschitz, we show that the quotient u_1/u_2 is Hölder continuous up to the boundary in $B_{1/2}$.

These results will be used in forthcoming works on obstacle-type problems for nonlocal operators.

1. INTRODUCTION AND RESULTS

The aim of this note is to establish new boundary Harnack inequalities for nonlocal elliptic operators in non-divergence form in general open sets.

To our knowledge, the first boundary Harnack principle for nonlocal elliptic operators was established by Bogdan [Bog97], who proved it for the fractional Laplacian in Lipschitz domains. Later, his result was extended to arbitrary open sets by Song and Wu in [SW99]; see also Bogdan-Kulczycki-Kwasnicki [BKK08]. More recently, Bogdan-Kumagai-Kwasnicki [BKK15] established the Boundary Harnack principle in general open sets for a wide class of Markov processes with jumps. In particular, their results apply to all linear operators of the form

$$Lu(x) = \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) K(y) dy, \quad (1.1)$$

with kernels $K(y) = K(-y)$ satisfying

$$0 < \frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}}, \quad y \in \mathbb{R}^n; \quad (1.2)$$

see [BKK15, Example 5.6].

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Here, we consider *non-divergence* form operators

$$Lu(x) = \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) K(x, y) dy, \quad (1.3)$$

with kernels $K(x, y) = K(x, -y)$ satisfying

$$0 < \frac{\lambda}{|y|^{n+2s}} \leq K(x, y) \leq \frac{\Lambda}{|y|^{n+2s}}, \quad x, y \in \mathbb{R}^n. \quad (1.4)$$

No regularity in x is assumed. These are the nonlocal analogues of second order uniformly elliptic operators $L = \sum_{i,j} a_{ij}(x) \partial_{ij}$ with bounded measurable coefficients; see [BL02, Sil06, CS09].

To our knowledge, our results are the first ones that establish boundary Harnack inequalities for such class of nonlocal operators in non-divergence form. Quite recently, we established in [RS15] a boundary Harnack estimate for operators of the form (1.3)-(1.4) under the important extra assumption that $K(x, y)$ is *homogeneous* in y . The results of [RS15] are for C^1 domains, and all the proofs are by blow-up and perturbative arguments. The techniques of the present paper are of very different nature, and completely independent from those in [RS15].

Our first result establishes the boundary Harnack principle in general open sets Ω , and reads as follows.

Theorem 1.1. *Let $s \in (0, 1)$, and L be any operator of the form (1.3)-(1.4). Let $\Omega \subset \mathbb{R}^n$ be any open set, with $0 \in \partial\Omega$, and $u_1, u_2 \in C(B_1)$ be two viscosity solutions of*

$$\begin{cases} Lu_1 = Lu_2 = 0 & \text{in } B_1 \cap \Omega \\ u_1 = u_2 = 0 & \text{in } B_1 \setminus \Omega, \end{cases} \quad (1.5)$$

satisfying $u_i \geq 0$ in \mathbb{R}^n and

$$\int_{\mathbb{R}^n} \frac{u_i(x)}{1 + |x|^{n+2s}} dx = 1.$$

Then,

$$C^{-1}u_2 \leq u_1 \leq C u_2 \quad \text{in } B_{1/2}.$$

The constant C depends only on n, s, Ω , and ellipticity constants.

Here, the equation $Lu = 0$ should be understood in the viscosity sense as $M^+u \geq 0 \geq M^-u$, where

$$M^+u = M_{\mathcal{L}_0}^+u = \sup_{L \in \mathcal{L}_0} Lu, \quad M^-u = M_{\mathcal{L}_0}^-u = \inf_{L \in \mathcal{L}_0} Lu,$$

and \mathcal{L}_0 is the class of operators of the form (1.1)-(1.2); see [CS09] for more details. The fact that both u_1 and u_2 solve the *same* equation $Lu_1 = Lu_2 = 0$ can be stated as $M^+(au_1 + bu_2) \geq 0$ for all $a, b \in \mathbb{R}$. Notice that taking $a = \pm 1$ and $b = 0$, or $a = 0$ and $b = \pm 1$, we get that $M^+u_i \geq 0 \geq M^-u_i$.

We will in fact prove a more general version of Theorem 1.1, in which we allow a right hand side in the equation, $Lu_1 = f_1$ and $Lu_2 = f_2$ in $\Omega \cap B_1$, with $\|f_i\|_{L^\infty} \leq \delta$,

and $\delta > 0$ small enough. In terms of the extremal operators M^+ and M^- , it reads as follows.

Theorem 1.2. *Let $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$ be any open set. Assume that there is $x_0 \in B_{1/2}$ and $\varrho > 0$ such that $B_{2\varrho}(x_0) \subset \Omega \cap B_{1/2}$.*

Then, there exists $\delta > 0$, depending only on n, s, ϱ , and ellipticity constants, such that the following statement holds.

Let $u_1, u_2 \in C(B_1)$ be viscosity solutions of

$$\begin{cases} M^+(au_1 + bu_2) & \geq -\delta(|a| + |b|) & \text{in } B_1 \cap \Omega \\ u_1 = u_2 & = 0 & \text{in } B_1 \setminus \Omega \end{cases} \quad (1.6)$$

for all $a, b \in \mathbb{R}$, and such that

$$u_i \geq 0 \quad \text{in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} \frac{u_i(x)}{1 + |x|^{n+2s}} dx = 1. \quad (1.7)$$

Then,

$$C^{-1}u_2 \leq u_1 \leq C u_2 \quad \text{in } B_{1/2}.$$

The constant C depends only on n, s, ϱ , and ellipticity constants.

One of the advantages of Theorem 1.2 is that it allows us to establish the following result.

Theorem 1.3. *Let $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$ be any Lipschitz domain, with $0 \in \partial\Omega$. Then, there is $\delta > 0$, depending only on n, s, Ω , and ellipticity constants, such that the following statement holds.*

Let $u_1, u_2 \in C(B_1)$ be viscosity solutions of (1.6) satisfying (1.7). Then, there is $\alpha \in (0, 1)$ such that

$$\left\| \frac{u_1}{u_2} \right\|_{C^{0,\alpha}(\bar{\Omega} \cap B_{1/2})} \leq C.$$

The constants α and C depend only on n, s, Ω , and ellipticity constants.

Theorems 1.2 and 1.3 will be used in forthcoming works on obstacle-type problems for nonlocal operators. In such context, boundary Harnack estimates of this type can be used to show that Lipschitz free boundaries are in fact $C^{1,\alpha}$; see for example [CSS08, CRS16].

The proof of Theorems 1.1 and 1.2 that we give here is quite short and simple, and uses very strongly the nonlocal character of the operator (recall that the boundary Harnack principle is in general false for second order (local) operators in Hölder domains [BB94]). Then, we prove Theorem 1.3 by iterating appropriately Theorem 1.2.

The paper is organized as follows. In Section 2 we give some preliminaries. In Section 3 we establish Theorems 1.2 and 1.1. In Section 4 we prove Theorem 1.3. Finally, in Section 5 we extend those results to non-symmetric operators and to operators with drift.

2. PRELIMINARIES

In this section we recall some results that will be used in our proofs.

An important ingredient to prove our boundary Harnack inequality is the interior Harnack inequality for nonlocal equations in non-divergence form, which states that if u solves

$$M^+u \geq -C_0 \quad \text{and} \quad M^-u \leq C_0 \quad \text{in } B_1,$$

and $u \geq 0$ in \mathbb{R}^n , then

$$\sup_{B_{1/2}} u \leq C \left(\inf_{B_{1/2}} u + C_0 \right);$$

see [CS09] and also [BL02].

In our proof, in fact, we will need the following two results, which imply the Harnack inequality. The first one is a half Harnack inequality for subsolutions.

Theorem 2.1 ([CS11]). *Assume that $u \in C(B_1)$ satisfies*

$$M^+u \geq -C_0 \quad \text{in } B_1$$

in the viscosity sense. Then,

$$\sup_{B_{1/2}} u \leq C \left(\int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} dx + C_0 \right).$$

The constant C depends only on n , s , and ellipticity constants.

The second one is the other half Harnack inequality, for supersolutions.

Theorem 2.2. *Assume that $u \in C(B_1)$ satisfies*

$$M^-u \leq C_0 \quad \text{in } B_1$$

in the viscosity sense. Assume in addition that $u \geq 0$ in \mathbb{R}^n . Then,

$$\int_{\mathbb{R}^n} \frac{u(x)}{1 + |x|^{n+2s}} dx \leq C \left(\inf_{B_{1/2}} u + C_0 \right).$$

The constant C depends only on n , s , and ellipticity constants.

When $s \geq \frac{1}{2}$, the result can be found in [CD16, Corollary 6.2], where it is proved in the more general setting of parabolic and nonsymmetric operators with drift. For completeness, we give a short proof of Theorem 2.2 here.

Proof of Theorem 2.2. Let $b \in C_c^\infty(B_{3/4})$ be such that $0 \leq b \leq 1$ and $b \equiv 1$ in $B_{1/2}$. Let $t > 0$ be the maximum value for which $u \geq tb$. Notice that $t \leq \inf_{B_{1/2}} u$. Since u and b are continuous in B_1 , then there is $x_0 \in B_{3/4}$ such that $u(x_0) = tb(x_0)$.

Now, on the one hand, we have

$$M^-(u - tb)(x_0) \leq M^-u(x_0) - tM^-b \leq C_0 + Ct.$$

On the other hand, since $u - tb \geq 0$ in \mathbb{R}^n and $(u - tb)(x_0) = 0$ then

$$M^-(u - tb)(x_0) = \lambda \int_{\mathbb{R}^n} \frac{u(z) - tb(z)}{|x_0 - z|^{n+2s}} dz \geq c \int_{\mathbb{R}^n} \frac{u(z)}{1 + |z|^{n+2s}} dz - Ct.$$

Combining the previous identities, we get

$$\inf_{B_{1/2}} u \geq t \geq -c_1 C_0 + c_2 \int_{\mathbb{R}^n} \frac{u(z)}{1 + |z|^{n+2s}} dz,$$

and the result follows. \square

3. PROOF OF THEOREM 1.2

Theorem 1.1 is a particular case of Theorem 1.2. We give below the proof of Theorem 1.2. Before that, we need a Lemma.

Lemma 3.1. *Let $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$ be any open set. Assume that there is $x_0 \in B_{1/2}$ and $\varrho > 0$ such that $B_{2\varrho}(x_0) \subset \Omega \cap B_{1/2}$. Denote $D = B_\varrho(x_0)$.*

Let $u \in C(B_1)$ be a viscosity solution of

$$\begin{cases} M^+u \geq -C_0 & \text{and} & M^-u \leq C_0 & \text{in } B_1 \cap \Omega \\ u = 0 & & & \text{in } B_1 \setminus \Omega \end{cases}$$

Assume in addition that $u \geq 0$ in \mathbb{R}^n . Then,

$$\sup_{B_{3/4}} u \leq C \left(\inf_D u + C_0 \right),$$

with C depending only on n, s, ϱ , and ellipticity constants.

Proof. Since $u \geq 0$ in B_1 and $M^+u \geq -C_0$ in $B_1 \cap \{u > 0\}$, then $M^+u \geq -C_0$ in all of B_1 . Thus, by Theorem 2.1 we have

$$\sup_{B_{3/4}} u \leq C \left(\int_{\mathbb{R}^n} \frac{u(x)}{1 + |x|^{n+2s}} dx + C_0 \right).$$

(Notice that Theorem 2.1 gives a the bound in $B_{1/2}$, but by a standard covering argument we get the same in $B_{3/4}$.) Now, using Theorem 2.2 in the ball $B_{2\varrho}(x_0)$, we find

$$\int_{\mathbb{R}^n} \frac{u(x)}{1 + |x|^{n+2s}} dx \leq C \left(\inf_D u + C_0 \right),$$

where $D = B_\varrho(x_0)$. Combining the previous estimates, the Lemma follows. \square

We next give the:

Proof of Theorem 1.2. First, as in Lemma 3.1, by (1.7) we have

$$u_i \leq C \quad \text{in } B_{3/4} \tag{3.1}$$

and

$$u_i \geq c > 0 \quad \text{in } B_\varrho(x_0), \tag{3.2}$$

provided that $\delta > 0$ is small enough. Notice that c depends on n , s , ellipticity constants, and ϱ , but not on Ω .

Let now $b \in C_c^\infty(B_{1/2})$ be such that $0 \leq b \leq 1$ and $b \equiv 1$ in $B_{1/4}$, and let $\eta \in C_c^\infty(B_\varrho(x_0))$ such that $0 \leq \eta \leq 1$ in $B_\varrho(x_0)$ and $\eta = 1$ in $B_{\varrho/2}(x_0)$. Let

$$w := u_1 \chi_{B_{3/4}} + C_1(b - 1) + C_2 \eta.$$

Then, thanks to (3.1), if C_1 is chosen large enough we will have

$$w \leq 0 \quad \text{in } \mathbb{R}^n \setminus B_{1/2}.$$

Moreover, taking now C_2 large enough,

$$\begin{aligned} M^+ w &\geq M^+ u_1 + M^-(u_1 \chi_{\mathbb{R}^n \setminus B_{3/4}}) + C_1 M^- b + C_2 M^- \eta \\ &\geq -\delta - C - C C_1 + c C_2 \geq 1 \quad \text{in } \Omega \cap B_{1/2} \setminus B_\varrho(x_0). \end{aligned}$$

Here we used that $M^+ u_1 \geq -\delta$ in $\Omega \cap B_1$, that $M^-(u_1 \chi_{\mathbb{R}^n \setminus B_{3/4}}) \geq -C \int_{\mathbb{R}^n} u_1(x)/(1 + |x|^{n+2s}) dx \leq C$ in $B_{1/2}$, that $M^- b \geq -C$, and that $M^- \eta \geq c > 0$ in $B_1 \setminus B_\varrho(x_0)$. Analogously, for any $C_3 \leq \delta^{-1}$ we get that

$$M^+(w - C_3 u_2) \geq 1 - C_3 \delta \geq 0 \quad \text{in } \Omega \cap B_{1/2} \setminus B_\varrho(x_0),$$

Finally, since $w \leq C$ in $B_\varrho(x_0)$ and $u_2 \geq c > 0$ in $B_\varrho(x_0)$, we clearly have

$$w \leq C_3 u_2 \quad \text{in } B_\varrho(x_0)$$

for some big constant C_3 . Taking δ small enough so that $\delta^{-1} \geq C_3$, by the comparison principle we find $w \leq C_3 u_2$ in all of \mathbb{R}^n .

In particular, since $w \equiv u_1$ in $B_{1/4} \setminus B_\varrho(x_0)$, this yields

$$u_1 \leq C_3 u_2 \quad \text{in } B_{1/4} \setminus B_\varrho(x_0).$$

Since u_1 and u_2 are comparable in $B_\varrho(x_0)$, we deduce

$$u_1 \leq C u_2 \quad \text{in } B_{1/4},$$

maybe with a bigger constant C . Finally, a standard covering argument yields the same result in $B_{1/2}$, and thus the theorem is proved. \square

4. PROOF OF THEOREM 1.3

We prove here Theorem 1.3. Throughout this section, Ω will be a Lipschitz domain with $0 \in \partial\Omega$. In particular, there is $\varrho > 0$ such that for every $r \in (0, 1)$ there is $x_r \in B_{r/2}$ for which

$$B_{2\varrho r}(x_r) \subset \Omega \cap B_{r/2}. \quad (4.1)$$

Throughout this section, we denote $D_r = B_{\varrho r}(x_r)$.

We will divide the proof of Theorem 1.3 in several steps. First, we have the following boundary Harnack type estimate, which is an immediate consequence of Theorem 1.2.

Lemma 4.1. *Let $s \in (0,1)$ and $\Omega \subset \mathbb{R}^n$ be any open set. Assume that there is $x_0 \in B_{1/2}$ and $\varrho > 0$ such that $B_{2\varrho}(x_0) \subset \Omega \cap B_{1/2}$. Denote $D = B_\varrho(x_0)$.*

Then, there exists $\delta > 0$, depending only on n, s, ϱ , and ellipticity constants, such that the following statement holds.

Let u_1 and u_2 be two functions satisfying, for all $a, b \in \mathbb{R}$,

$$\begin{cases} M^+(au_1 + bu_2) \geq -|a|C_0 - |b|\delta & \text{in } B_1 \cap \Omega \\ u_1 = u_2 = 0 & \text{in } B_1 \setminus \Omega, \end{cases} \quad (4.2)$$

with $u_1, u_2 \geq 0$ in \mathbb{R}^n and $\inf_D u_2 = 1$. Then,

$$\inf_D \frac{u_1}{u_2} \leq C \left(\inf_{B_{1/2}} \frac{u_1}{u_2} + C_0 \right). \quad (4.3)$$

The constant C depends only on n, s, ϱ , and ellipticity constants.

Proof. Dividing by $\inf_D u_1$ if necessary, we may assume $\inf_D u_1 = 1$.

By the interior Harnack inequality, $1 = \inf_D u_2 \leq \sup_D u_2 \leq C$ (provided that δ is small enough). Thus,

$$\inf_D \frac{u_1}{u_2} \leq C_1,$$

with C_1 independent of C_0 .

Now, if $C_0 \leq \delta$, then by Theorem 1.2 we have $u_2 \leq C_2 u_1$ in $B_{1/2}$, and therefore

$$\inf_D \frac{u_1}{u_2} \leq C_1 \leq C_1 C_2 \left(\inf_{B_{1/2}} \frac{u_1}{u_2} \right).$$

If $C_0 \geq \delta$, then we simply have

$$\inf_D \frac{u_1}{u_2} \leq C_1 \leq \frac{C_1}{\delta} C_0 = CC_0.$$

In any case, (4.3) is proved. \square

Second, we need the following consequence of the interior Harnack.

Lemma 4.2. *Let $s \in (0,1)$ and $\Omega \subset \mathbb{R}^n$ be any open set. Assume that there is $x_0 \in B_{1/2}$ and $\varrho > 0$ such that $B_{2\varrho}(x_0) \subset \Omega \cap B_{1/2}$. Denote $D = B_\varrho(x_0)$.*

Then, there exists $\delta > 0$, depending only on n, s, ϱ , and ellipticity constants, such that the following statement holds.

Let u_1 and u_2 be two functions satisfying $u_1, u_2 \geq 0$ in \mathbb{R}^n , (4.2), and $\inf_D u_2 = 1$. Then,

$$\sup_D \frac{u_1}{u_2} \leq C \left(\inf_D \frac{u_1}{u_2} + C_0 \right). \quad (4.4)$$

The constant C depends only on n, s, ϱ , and ellipticity constants.

Proof. Notice that $M^+u_1 \geq -C_0$ and $M^-u_1 \leq C_0$ in $\Omega \cap B_1$, while $M^+u_2 \geq -\delta$ and $M^-u_2 \leq \delta$ in $\Omega \cap B_1$.

By interior Harnack inequality, we have $1 = \inf_D u_2 \leq \sup_D u_2 \leq C$ (provided that δ is small enough). Moreover, for u_1 we have $\sup_D u_1 \leq C(\inf_D u_1 + C_0)$, and thus

$$\sup_D \frac{u_1}{u_2} \leq C \sup_D u_1 \leq C \left(\inf_D u_1 + C_0 \right) \leq C \left(\inf_D \frac{u_1}{u_2} + C_0 \right),$$

as desired. \square

We will also need the following rescaled versions of the previous Lemmas.

Corollary 4.3. *Let $s \in (0, 1)$, $r \in (0, 1)$, and $\Omega \subset \mathbb{R}^n$ be any Lipschitz domain, with $0 \in \partial\Omega$. Then, there exists $\delta > 0$, depending only on n, s, ϱ in (4.1), ellipticity constants, such that the following statement holds.*

Let u_1 and u_2 be two functions satisfying, for all $a, b \in \mathbb{R}$,

$$\begin{cases} M^+(au_1 + bu_2) & \geq -|a|K - |b|\delta/C_1 & \text{in } B_r \cap \Omega \\ u_1 = u_2 & = 0 & \text{in } B_r \setminus \Omega, \end{cases} \quad (4.5)$$

with $C_1 > 0$ and $u_1, u_2 \geq 0$ in \mathbb{R}^n . Assume in addition that

$$\frac{r^{2s}}{\inf_{D_r} u_2} \leq C_1. \quad (4.6)$$

Then,

$$\inf_{D_r} \frac{u_1}{u_2} \leq C \left(\inf_{B_{r/2}} \frac{u_1}{u_2} + K \frac{r^{2s}}{\inf_{D_r} u_2} \right). \quad (4.7)$$

The constant C depends only on n, s, ϱ , and ellipticity constants.

Proof. The functions $v_1(x) := u_1(rx)/\inf_{D_r} u_2$ and $v_2(x) := C_1 u_2(rx)/\inf_{D_r} u_2$ satisfy

$$\begin{cases} M^+(av_1 + bv_2) & \geq -|a|K \frac{r^{2s}}{\inf_{D_r} u_2} - |b|\delta & \text{in } B_1 \cap \Omega \\ v_1 = v_2 & = 0 & \text{in } B_1 \setminus \Omega. \end{cases}$$

Thus, the result follows from Lemma 4.1. \square

Corollary 4.4. *Let $s \in (0, 1)$, $r \in (0, 1)$, and $\Omega \subset \mathbb{R}^n$ be any Lipschitz domain, with $0 \in \partial\Omega$. Then, there exists $\delta > 0$, depending only on n, s, ϱ in (4.1), and ellipticity constants, such that the following statement holds.*

Let u_1 and u_2 be two functions satisfying $u_1, u_2 \geq 0$ in \mathbb{R}^n , and (4.5). Assume in addition (4.6). Then,

$$\sup_{D_r} \frac{u_1}{u_2} \leq C \left(\inf_{D_r} \frac{u_1}{u_2} + K \frac{r^{2s}}{\inf_{D_r} u_2} \right). \quad (4.8)$$

The constant C depends only on n, s, ϱ , and ellipticity constants.

Proof. Setting $v_1(x) := u_1(rx)/\inf_{D_r} u_2$ and $v_2(x) := C_1 u_2(rx)/\inf_{D_r} u_2$, the result follows from Lemma 4.2. \square

We will also need the following.

Lemma 4.5. *Let $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$ be any Lipschitz domain, with $0 \in \partial\Omega$. There exists $\delta > 0$, $\gamma \in (0, 1)$, and $c_0 > 0$ depending only on n , s , Ω , and ellipticity constants, such that the following statement holds.*

Let u be a viscosity solution of $M^+u \geq -\delta$ and $M^-u \leq \delta$ in $B_1 \cap \Omega$, with $u = 0$ in $B_1 \setminus \Omega$. Assume in addition that $u \geq 0$ in \mathbb{R}^n and $\inf_{D_1} u = 1$.

Then, $u \geq c_0 d^{2s-\gamma}$ in $B_{1/2}$, where $d(x) = \text{dist}(x, B_1 \setminus \Omega)$. In particular,

$$\inf_{D_r} u \geq c_0 r^{2s-\gamma} \quad \text{for all } r \in (0, 1).$$

The constants γ and c_0 depend only on n , s , Ω , and ellipticity constants.

Proof. We defer the proof to the Appendix. \square

As a consequence, we find the following.

Corollary 4.6. *Let $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$ be any Lipschitz domain, with $0 \in \partial\Omega$. There exists $\delta > 0$, depending only on n , s , Ω , and ellipticity constants, such that the following statement holds.*

Let u_2 be a viscosity solution of $M^+u_2 \geq -\delta$ and $M^-u_2 \leq \delta$ in $B_1 \cap \Omega$, with $u_2 = 0$ in $B_1 \setminus \Omega$. Assume in addition that $u_2 \geq 0$ in \mathbb{R}^n .

Then, there is $\gamma \in (0, 1)$ such that

$$\sup_{B_{2r}|z|} u_2 \leq C |z|^{2s-\gamma} \inf_{D_r} u_2 \quad \text{whenever } |z| \geq \frac{1}{2} \quad \text{and } r|z| \leq \frac{1}{4}.$$

The constants γ and C depend only on n , s , Ω , and ellipticity constants.

Proof. We use the previous Lemma with

$$v(x) := \frac{u_2(4r|z|x)}{\inf_{D_{4r|z|}} u_2},$$

to find

$$c|z|^{\gamma-2s} = t^{2s-\gamma} \leq C \inf_{D_t} v = C \frac{\inf_{D_r} u_2}{\inf_{D_{4r|z|}} u_2},$$

where $t = \frac{1}{4}|z|^{-1}$. Thus,

$$\inf_{D_{4r|z|}} u_2 \leq C |z|^{2s-\gamma} \inf_{D_r} u_2.$$

Moreover, by Lemma 3.1 we have

$$\sup_{B_{2r}|z|} u_2 \leq C \inf_{D_{4r|z|}} u_2,$$

then

$$\sup_{B_{2r}|z|} u_2 \leq C |z|^{2s-\gamma} \inf_{D_r} u_2,$$

and we are done. \square

Using the previous results, we now prove the following oscillation reduction Lemma. Similar oscillation reduction Lemmas for nonlocal operators can be found in [Bog99], [CS98], or [RS16].

Lemma 4.7. *Let $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$ be any Lipschitz domain, with $0 \in \partial\Omega$. Then, there exists $\delta > 0$, depending only on n, s, ϱ in (4.1), and ellipticity constants, such that the following statement holds.*

Let $u_1, u_2 \in C(B_1)$ be viscosity solutions (1.6) satisfying (1.7). Then,

$$\sup_{\Omega \cap B_r} \frac{u_1}{u_2} - \inf_{\Omega \cap B_r} \frac{u_1}{u_2} \leq Cr^\alpha \quad (4.9)$$

for all $r \leq 3/4$. The constants C and $\alpha \in (0, 1)$ depend only on n, s, ϱ , and ellipticity constants.

Proof. We will prove that there exist constants $C_1 > 0$ and $\alpha > 0$, and monotone sequences $\{m_k\}_{k \geq 1}$ and $\{\bar{m}_k\}_{k \geq 1}$, such that

$$\bar{m}_k - m_k = 4^{-\alpha k}, \quad 0 \leq m_k \leq m_{k+1} < \bar{m}_{k+1} \leq \bar{m}_k \leq 1,$$

and

$$m_k u_2 \leq C_1^{-1} u_1 \leq \bar{m}_k u_2 \quad \text{in } B_{r_k}, \quad r_k = 4^{-k}. \quad (4.10)$$

Clearly, if such sequences exist, then (4.9) holds for all $r \leq \frac{1}{4}$. We will construct such sequences inductively.

First notice that, by Theorem 1.2 (and a covering argument), we have

$$0 \leq u_1 \leq \tilde{C}_1 u_2 \quad \text{in } B_{3/4}, \quad (4.11)$$

for some constant \tilde{C}_1 . Thus, it follows that (4.9) holds for $\frac{1}{4} \leq r \leq \frac{3}{4}$, and that we may take $m_1 = 0, \bar{m}_1 = 1$. Furthermore, by taking $C_1 \geq \tilde{C}_1 4^{\alpha k_0}$ we see that (4.10) holds with for all $k \leq k_0$, with $m_k = 0$ and $\bar{m}_k = 4^{-\alpha k}$ for $1 \leq k \leq k_0$, and k_0 is to be chosen later.

Assume now that we have sequences up to m_k and \bar{m}_k (with $k \geq k_0$), and let

$$v_k := C_1^{-1} u_1 - m_k u_2.$$

Notice that by induction hypothesis we have $v_k \geq 0$ in B_{r_k} (but not in all of \mathbb{R}^n). Moreover, since $C_1^{-1} u_1 \geq m_j u_2$ in B_{r_j} for $j \leq k$, then

$$v_k \geq (m_j - m_k) u_2 \geq (m_j - \bar{m}_j + \bar{m}_k - m_k) u_2 = -(4^{-\alpha j} - 4^{-\alpha k}) u_2 \quad \text{in } B_{r_j},$$

for every $j \leq k$. Using now that for every $x \in B_1 \setminus B_{r_k}$ there is $j < k$ such that $|x| < r_j = 4^{-j} \leq 4|x|$, we find

$$v_k(x) \geq -u_2(x) (|4x|^\alpha - r_k^\alpha) \quad \text{in } B_{1/4} \setminus B_{r_k}.$$

Thanks to this, and since $v_k \geq 0$ in B_{r_k} , for every $x \in B_{r_k/2}$ we have that the negative part of v_k satisfies

$$\begin{aligned}
 0 &\leq M^- v_k^-(x) \leq M^+ v_k^-(x) = \Lambda \int_{x+y \notin B_{r_k}} v_k^-(x+y) \frac{dy}{|y|^{n+2s}} \\
 &\leq C \int_{\frac{r_k}{2} \leq |y| \leq \frac{1}{4}} u_2(x+y) (|4y|^\alpha - r_k^\alpha) \frac{dy}{|y|^{n+2s}} + \int_{\mathbb{R}^n \setminus B_{1/4}} C_1^{-1} u_1(x+y) \frac{dy}{|y|^{n+2s}} \\
 &= C r_k^{\alpha-2s} \int_{\frac{1}{2} \leq |z| \leq \frac{1}{4r_k}} \frac{(|4z|^\alpha - 1) u_2(x+r_k z)}{|z|^{n+2s}} dz + C C_1^{-1} \int_{\mathbb{R}^n} u_1(y) \frac{dy}{1+|y|^{n+2s}} \\
 &\leq C r_k^{\alpha-2s} \int_{\frac{1}{2} \leq |z| \leq \frac{1}{4r_k}} \frac{(|4z|^\alpha - 1) \sup_{B_{2r_k|z|}} u_2}{|z|^{n+2s}} dz + C C_1^{-1}.
 \end{aligned}$$

Now, by Corollary 4.6 there is $\gamma > 0$ such that

$$\sup_{B_{2r_k|z|}} u_2 \leq C |z|^{2s-\gamma} \left(\inf_{D_{r_k}} u_2 \right)$$

for every $|z| \geq \frac{1}{2}$ and $r|z| \leq \frac{1}{4}$, and thus

$$\begin{aligned}
 C r_k^{\alpha-2s} \int_{\frac{1}{2} \leq |z| \leq \frac{1}{4r_k}} \frac{(|4z|^\alpha - 1) \sup_{B_{2r_k|z|}} u_2}{|z|^{n+2s}} dz &\leq C r_k^{\alpha-2s} \left(\inf_{D_{r_k}} u_2 \right) \int_{\frac{1}{2} \leq |z| \leq \frac{1}{4r_k}} \frac{(|4z|^\alpha - 1) |z|^{2s-\gamma}}{|z|^{n+2s}} dz \\
 &\leq \varepsilon_0 r_k^{\alpha-2s} \left(\inf_{D_{r_k}} u_2 \right),
 \end{aligned}$$

with

$$\varepsilon_0 := C \int_{|z| \geq \frac{1}{2}} \frac{(|4z|^\alpha - 1) |z|^{2s-\gamma}}{|z|^{n+2s}} dz \longrightarrow 0 \quad \text{as } \alpha \rightarrow 0.$$

This means that

$$0 \leq M^- v_k^- \leq M^+ v_k^- \leq \varepsilon_0 r_k^{\alpha-2s} \left(\inf_{D_{r_k}} u_2 \right) + C C_1^{-1} \quad \text{in } B_{r_k/2}.$$

Therefore, since $v_k^+ = C_1^{-1} u_1 - m_k u_2 + v_k^-$, we have

$$\begin{aligned}
 M^- v_k^+ &\leq C_1^{-1} M^-(u_1 - m_k u_2) + M^+ v_k^- \leq C_1^{-1} (1 + m_k) \delta + \varepsilon_0 r_k^{\alpha-2s} \left(\inf_{D_{r_k}} u_2 \right) + C C_1^{-1} \\
 &\leq \delta + \varepsilon_0 r_k^{\alpha-2s} \left(\inf_{D_{r_k}} u_2 \right) + C C_1^{-1}
 \end{aligned}$$

in $\Omega \cap B_{r_k/2}$. Also,

$$M^+ v_k^+ \geq M^+ v_k \geq -(C_1^{-1} + m_k) \delta \geq -\delta \quad \text{in } \Omega \cap B_{r_k/2}.$$

Similarly, we have

$$M^+(a v_k^+ + b u_2) \geq -|a| \left(\delta + \varepsilon_0 r_k^{\alpha-2s} \left(\inf_{D_{r_k}} u_2 \right) + C C_1^{-1} \right) - |b| \delta \quad \text{in } \Omega \cap B_{r_k/2}.$$

Now, recall that by Corollary 4.6 we have

$$\frac{r_k^{2s}}{\inf_{D_{r_k}} u_2} \leq Cr_k^\gamma \leq C_1.$$

Thus, we can apply Corollaries 4.3 and 4.4 to the functions v_k^+ and u_2 , to obtain

$$\begin{aligned} \inf_{D_{r_k}} \frac{v_k^+}{u_2} &\leq C \inf_{B_{r_k/2}} \frac{v_k^+}{u_2} + C \left(\delta + \varepsilon_0 r_k^{\alpha-2s} \left(\inf_{D_{r_k}} u_2 \right) + CC_1^{-1} \right) \frac{r_k^{2s}}{\inf_{D_{r_k}} u_2} \\ &\leq C \inf_{B_{r_k/2}} \frac{v_k^+}{u_2} + C(\delta + C_1^{-1})r_k^\gamma + C\varepsilon_0 r_k^\alpha, \end{aligned}$$

and

$$\sup_{D_{r_k/2}} \frac{v_k^+}{u_2} \leq C \inf_{D_{r_k/2}} \frac{v_k^+}{u_2} + C(\delta + C_1^{-1})r_k^\gamma + C\varepsilon_0 r_k^\alpha.$$

Recalling that $v_k^+ = v_k = C_1^{-1}u_1 - m_k u_2$ in $B_{r_k/2}$, we find

$$\inf_{D_{r_k/2}} (C_1^{-1}u_1/u_2 - m_k) \leq C \inf_{B_{r_k/4}} (C_1^{-1}u_1/u_2 - m_k) + C(\delta + C_1^{-1})r_k^\gamma + C\varepsilon_0 r_k^\alpha,$$

and

$$\sup_{D_{r_k/2}} (C_1^{-1}u_1/u_2 - m_k) \leq C \inf_{D_{r_k/2}} (C_1^{-1}u_1/u_2 - m_k) + C(\delta + C_1^{-1})r_k^\gamma + C\varepsilon_0 r_k^\alpha.$$

Therefore, we deduce

$$\sup_{D_{r_k/2}} (C_1^{-1}u_1/u_2 - m_k) \leq C \inf_{B_{r_k/4}} (C_1^{-1}u_1/u_2 - m_k) + C(\delta + C_1^{-1})r_k^\gamma + C\varepsilon_0 r_k^\alpha.$$

Repeating the same argument with $\bar{v}_k := \bar{m}_k - C_1^{-1}u_1$ instead of v_k , we find

$$\sup_{D_{r_k/2}} (\bar{m}_k - C_1^{-1}u_1/u_2) \leq C \inf_{B_{r_k/4}} (\bar{m}_k - C_1^{-1}u_1/u_2) + C(\delta + C_1^{-1})r_k^\gamma + C\varepsilon_0 r_k^\alpha.$$

Thus, combining the previous estimates, we get

$$\begin{aligned} \bar{m}_k - m_k &\leq C \inf_{B_{r_k/4}} (C_1^{-1}u_1/u_2 - m_k) + C \inf_{B_{r_k/4}} (\bar{m}_k - C_1^{-1}u_1/u_2) + C(\delta + C_1^{-1})r_k^\gamma + C\varepsilon_0 r_k^\alpha \\ &= C \left(\inf_{B_{r_k/4}} (C_1^{-1}u_1/u_2) - \sup_{B_{r_k/4}} (C_1^{-1}u_1/u_2) + \bar{m}_k - m_k + (\delta + C_1^{-1})r_k^\gamma + \varepsilon_0 r_k^\alpha \right). \end{aligned}$$

Using that $\bar{m}_k - m_k = 4^{-\alpha k}$, $r_k = 4^{-k}$, and $k \geq k_0$, we obtain

$$\sup_{B_{r_{k+1}}} (C_1^{-1}u_1/u_2) - \inf_{B_{r_{k+1}}} (C_1^{-1}u_1/u_2) \leq \left(\frac{C-1}{C} + (\delta + C_1^{-1})4^{-(\gamma-\alpha)k_0} + \varepsilon_0 \right) 4^{-\alpha k}.$$

Taking α small enough and k_0 large enough, we get

$$\sup_{B_{r_{k+1}}} (C_1^{-1}u_1/u_2) - \inf_{B_{r_{k+1}}} (C_1^{-1}u_1/u_2) \leq 4^{-\alpha(k+1)}.$$

This means that we can choose m_{k+1} and \bar{m}_{k+1} , and thus we are done. \square

We finally give the:

Proof of Theorem 1.3. We will combine Lemma 4.7 with interior estimates in order to get the desired result.

Let $x, y \in \Omega \cap B_{1/2}$, let

$$r = |x - y| \quad \text{and} \quad d = \min\{d(x), d(y)\},$$

where $d(x) = \text{dist}(x, \partial\Omega)$. Let $x_* \in \partial\Omega$ be such that $d(x) = |x - x_*|$. We need to show that $|(u_1/u_2)(x) - (u_1/u_2)(y)| \leq Cr^{\alpha'}$, with $\alpha' > 0$. Since u_1/u_2 is bounded in $B_{3/4}$, we may assume that $0 < r \leq r_0$, with r_0 small enough.

If $r \leq d/2$, then by interior estimates [CS09] we have

$$\|u_i\|_{C^\alpha(B_{d/2}(x))} \leq Cd^{-\alpha}.$$

Since $\inf_{B_{d/2}(x)} u_2 \geq c_0 d^{2s-\gamma}$, then

$$\|u_2^{-1}\|_{C^\alpha(B_{d/2}(x))} \leq Cd^{\gamma-\alpha-2s}.$$

Therefore, for $r \leq d/2$ we have

$$|(u_1/u_2)(x) - (u_1/u_2)(y)| \leq Cr^\alpha d^{\gamma-2\alpha-2s} \leq Cr^\alpha d^{-2s}.$$

provided that $\alpha \leq \gamma/2$. In particular, if $r \leq d^\theta/2$, with $\theta > 2s/\alpha > 1$, then

$$|(u_1/u_2)(x) - (u_1/u_2)(y)| \leq Cr^{\alpha-2s/\theta}. \quad (4.12)$$

On the other hand, for all $r \in (0, r_0)$ we have $x, y \in B_{d+r}(x_*)$, and thus by Lemma 4.7 we have

$$|(u_1/u_2)(x) - (u_1/u_2)(y)| \leq \sup_{B_{d+r}(x_*) \cap \Omega} \frac{u_1}{u_2} - \inf_{B_{d+r}(x_*) \cap \Omega} \frac{u_1}{u_2} \leq C(d+r)^\alpha.$$

In particular, if $r \geq d^\theta/2$ then

$$|(u_1/u_2)(x) - (u_1/u_2)(y)| \leq Cr^{\theta\alpha}. \quad (4.13)$$

Combining (4.12) and (4.13), we find

$$|(u_1/u_2)(x) - (u_1/u_2)(y)| \leq Cr^{\alpha'} \quad \text{for all } r \in (0, 1),$$

with $\alpha' = \min\{\alpha - 2s/\theta, \theta\alpha\} > 0$. Thus, the Theorem is proved. \square

5. NON-SYMMETRIC OPERATORS WITH DRIFT

The above proofs of Theorems 1.2 and Theorem 1.3 work as well for operators of the form

$$\tilde{L}u(x) = \int_{\mathbb{R}^n} (u(x+y) - u(x) - \nabla u(x) \cdot y\chi_{B_1}(y))K(x, y)dy + b(x) \cdot \nabla u,$$

provided that $s \geq \frac{1}{2}$. Namely, consider the class of nonlocal and non-symmetric operators

$$\tilde{L}u(x) = \int_{\mathbb{R}^n} (u(x+y) - u(x) - \nabla u(x) \cdot y\chi_{B_1}(y))K(y)dy + b \cdot \nabla u, \quad (5.1)$$

with K satisfying (1.2) and

$$|b| + \left| r^{2s-1} \int_{B_1 \setminus B_r} y K(y) dy \right| \leq \beta. \quad (5.2)$$

Given λ , Λ , and β , we define the class $\mathcal{L}(\lambda, \Lambda, \beta)$ as the set of all linear operators (5.1) satisfying (1.2) and (5.2). Then, we may define \widetilde{M}^\pm as

$$\widetilde{M}^+ u = \widetilde{M}_{\mathcal{L}(\lambda, \Lambda, \beta)}^+ u = \sup_{\tilde{L} \in \mathcal{L}(\lambda, \Lambda, \beta)} \tilde{L} u, \quad \widetilde{M}^- u = \widetilde{M}_{\mathcal{L}(\lambda, \Lambda, \beta)}^- u = \inf_{\tilde{L} \in \mathcal{L}(\lambda, \Lambda, \beta)} \tilde{L} u.$$

For such operators, Theorems 2.1 and 2.2 were established in [CD16]; see Corollaries 4.3 and 6.2 therein. Using such results, and with the exact same proofs given in the previous Sections, we find the following.

Theorem 5.1. *Let $s \in [\frac{1}{2}, 1)$ and $\Omega \subset \mathbb{R}^n$ be any open set. Assume that there is $x_0 \in B_{1/2}$ and $\varrho > 0$ such that $B_{2\varrho}(x_0) \subset \Omega \cap B_{1/2}$.*

Then, there exists $\delta > 0$, depending only on n , s , ϱ , λ , Λ , and β , such that the following statement holds.

Let $u_1, u_2 \in C(B_1)$ be viscosity solutions of

$$\begin{cases} \widetilde{M}^+(au_1 + bu_2) \geq -\delta(|a| + |b|) & \text{in } B_1 \cap \Omega \\ u_1 = u_2 = 0 & \text{in } B_1 \setminus \Omega \end{cases} \quad (5.3)$$

for all $a, b \in \mathbb{R}$, and such that

$$u_i \geq 0 \quad \text{in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} \frac{u_i(x)}{1 + |x|^{n+2s}} dx = 1. \quad (5.4)$$

Then,

$$C^{-1}u_2 \leq u_1 \leq C u_2 \quad \text{in } B_{1/2}.$$

The constant C depends only on n , s , ϱ , λ , Λ , and β .

Moreover, we also have the following.

Theorem 5.2. *Let $s \in [\frac{1}{2}, 1)$ and $\Omega \subset \mathbb{R}^n$ be any Lipschitz domain, with $0 \in \partial\Omega$. Then, there is $\delta > 0$, depending only on n , s , Ω , λ , Λ , and β , such that the following statement holds.*

Let $u_1, u_2 \in C(B_1)$ be viscosity solutions of (5.3) satisfying (5.4). Then, there is $\alpha \in (0, 1)$ such that

$$\left\| \frac{u_1}{u_2} \right\|_{C^{0,\alpha}(\overline{\Omega} \cap B_{1/2})} \leq C.$$

The constants α and C depend only on n , s , Ω , λ , Λ , and β .

Those results will be used in the forthcoming paper [FR16].

6. APPENDIX: SUBSOLUTION IN LIPSCHITZ DOMAINS

We prove here a lower bound for positive solutions u in Lipschitz domains, namely $u \geq cd^{2s-\gamma}$ in Ω for some small $\gamma > 0$. This is stated in Lemma 4.5, which we prove below.

For this, we need to construct the following subsolution.

Lemma 6.1. *Let $s \in (0, 1)$, and $e \in S^{n-1}$. Given $\eta > 0$, there is $\epsilon > 0$ depending only on n, s, η and ellipticity constants such that the following holds.*

Define

$$\Phi(x) := \left(e \cdot x - \eta|x| \left(1 - \frac{(e \cdot x)^2}{|x|^2} \right) \right)_+^{2s-\epsilon}$$

Then,

$$\begin{cases} M^- \Phi \geq 0 & \text{in } \mathcal{C}_\eta \\ \Phi = 0 & \text{in } \mathbb{R}^n \setminus \mathcal{C}_\eta \end{cases}$$

where \mathcal{C}_η is the cone defined by

$$\mathcal{C}_\eta := \left\{ x \in \mathbb{R}^n : e \cdot \frac{x}{|x|} > \eta \left(1 - \left(e \cdot \frac{x}{|x|} \right)^2 \right) \right\}.$$

The constant ϵ depends only on η, s , and ellipticity constants.

In particular Φ satisfies $M^- \Phi \geq 0$ in all of \mathbb{R}^n .

Proof. By homogeneity it is enough to prove that, for ϵ small enough, we have $M^- \Phi \geq 1$ on points belonging to $e + \partial\mathcal{C}_\eta$, since all the positive dilations of this set with respect to the origin cover the interior of $\tilde{\mathcal{C}}_\eta$.

Let thus $P \in \partial\mathcal{C}_\eta$, that is,

$$e \cdot P - \eta \left(|P| - \frac{(e \cdot P)^2}{|P|} \right) = 0.$$

Consider

$$\begin{aligned} \Phi_P(x) &:= \Phi(P + e + x) \\ &= \left(e \cdot (P + e + x) - \eta \left(|P + e + x| - \frac{(e \cdot (P + e + x))^2}{|P + e + x|} \right) \right)_+^{2s-\epsilon} \\ &= \left(1 + e \cdot x - \eta \left(|P + e + x| - |P| - \frac{(e \cdot (P + e + x))^2}{|P + e + x|} + \frac{(e \cdot P)^2}{|P|} \right) \right)_+^{2s-\epsilon} \\ &= (1 + e \cdot x - \eta\psi_P(x))_+^{s+\epsilon}, \end{aligned}$$

where we define

$$\psi_P(x) := |P + e + x| - |P| - \frac{(e \cdot (P + e + x))^2}{|P + e + x|} + \frac{(e \cdot P)^2}{|P|}.$$

Note that the functions ψ_P satisfy

$$|\nabla\psi_P(x)| \leq C \quad \text{in } \mathbb{R}^n \setminus \{-P - e\},$$

and

$$|D^2\psi_P(x)| \leq C \quad \text{for } x \in B_{1/2}, \quad (6.1)$$

where C does not depend on P (recall that $|e| = 1$).

Now for fixed $\tilde{e} \in \partial\mathcal{C}_\eta \cap \partial B_1$ let us compute

$$\lim_{t \uparrow +\infty} \psi_{t\tilde{e}}(x) = \lim_{t \uparrow +\infty} (|t\tilde{e} + e + x| - |t\tilde{e}|) - \lim_{t \uparrow +\infty} \left(\frac{(e \cdot (t\tilde{e} + e + x))^2}{|t\tilde{e} + e + x|} - \frac{(e \cdot t\tilde{e})^2}{|t\tilde{e}|} \right).$$

On the one hand, we have

$$\lim_{t \uparrow +\infty} (|t\tilde{e} + e + x| - |t\tilde{e}|) = \tilde{e} \cdot (e + x).$$

On the other hand to compute for $f_t(y) := \frac{(e \cdot (t\tilde{e} + y))^2}{|t\tilde{e} + y|}$ we have

$$\partial_{y_i} f_t(y) = \frac{2(e \cdot (t\tilde{e} + y))e_i}{|t\tilde{e} + y|} - \frac{(e \cdot (t\tilde{e} + y))^2}{|t\tilde{e} + y|^3} (t\tilde{e} + y)_i$$

and hence

$$\lim_{t \uparrow +\infty} \partial_{y_i} f_t(y) = (2(e \cdot \tilde{e})e_i - (e \cdot \tilde{e})^2 \tilde{e}_i).$$

Therefore,

$$\lim_{t \uparrow +\infty} \left(\frac{(e \cdot (t\tilde{e} + e + x))^2}{|t\tilde{e} + e + x|} - \frac{(e \cdot t\tilde{e})^2}{|t\tilde{e}|} \right) = (2(e \cdot \tilde{e})e - (e \cdot \tilde{e})^2 \tilde{e}) \cdot (e + x).$$

We have thus found

$$\lim_{t \uparrow +\infty} \psi_P(x) = (\tilde{e} - 2(e \cdot \tilde{e})e + (e \cdot \tilde{e})^2 \tilde{e}) \cdot (e + x)$$

and

$$\lim_{t \uparrow +\infty} (1 + e \cdot x - \eta\psi_P(x)) = (e - \eta\tilde{e} + 2\eta(e \cdot \tilde{e})e - \eta(e \cdot \tilde{e})^2 \tilde{e}) \cdot (e + x)$$

Note that for δ small enough (depending only on η), if we define

$$\mathcal{C}_{\tilde{e}} := \left\{ x \in \mathbb{R}^n : \frac{x + e}{|x + e|} \cdot \frac{e - (e \cdot \tilde{e})\tilde{e}}{|e - (e \cdot \tilde{e})\tilde{e}|} \geq (1 - \delta) \right\}$$

satisfies

$$\lim_{t \uparrow +\infty} (1 + e \cdot x - \eta\psi_P(x)) \geq c|x| \quad \text{for all } x \in \mathcal{C}_{\tilde{e}} \quad (6.2)$$

where $c > 0$. Indeed, the vector $e' := e - (e \cdot \tilde{e})\tilde{e}$ is perpendicular to \tilde{e} and has positive scalar product with e . Thus, we have

$$(e - \eta\tilde{e} + 2\eta(e \cdot \tilde{e})e - \eta(e \cdot \tilde{e})^2 \tilde{e}) \cdot e' > 0$$

Let us show now that for $\varepsilon > 0$ small enough the function Φ_P satisfies

$$M^- \Phi_P(0) \geq 1. \quad (6.3)$$

We first prove (6.3) in the case $|P| \geq R$ with R large enough. Indeed let $P = t\tilde{e}$ for $t \uparrow +\infty$ and $\tilde{e} \in \partial\mathcal{C}_\eta \cap \partial B_1$. Let us denote

$$\delta^2 u(x, y) = \frac{u(x+y) + u(x-y)}{2} - u(x).$$

Using (6.1), and (6.2), and $\Phi_P \geq 0$ we obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} M^- \Phi_P(0) &\geq \int_{\mathbb{R}^n} \left((\delta^2 u)_+ \frac{\lambda}{|y|^{n+2s}} - (\delta^2 u)_- \frac{\Lambda}{|y|^{n+2s}} \right) dy \\ &\geq \int_{\mathcal{C}_{\tilde{e}}} (c|y| - C)_+^{2s-\epsilon} \frac{dy}{|y|^{n+2s}} - C \int_{\mathbb{R}^n} \min\{1, |y|^2\} \frac{dy}{|y|^{n+2s}} \\ &\geq \frac{c}{\epsilon} - C. \end{aligned}$$

Thus (6.3) follows for $|P| \geq R$ with R large, provided that ϵ is taken small enough.

We now concentrate in the case $|P| < R$. In this case we use that, taking $\delta > 0$ small enough (depending on η) and defining the cone

$$\mathcal{C}_e := \left\{ x \in \mathbb{R}^n : \frac{x}{|x|} \cdot e \geq (1 - \delta) \right\}$$

we have

$$e \cdot (P + e + x) - \eta \left(|P + e + x| - \frac{(e \cdot (P + e + x))^2}{|P + e + x|} \geq c|x| \right)$$

for $x \in \mathcal{C}_e$ with $|x| \geq L$ with L large enough (depending on R).

Thus, reasoning similarly as above but now integrating in $\mathcal{C}_e \cap \{|x| > L\}$ instead of on $\mathcal{C}_{\tilde{e}}$ we prove (6.3) also in the case $P \geq R$, provided that ϵ is small enough. Therefore the lemma is proved. \square

Finally, we give the:

Proof of Lemma 4.5. Note that we only need to prove the conclusion of the Lemma for $r > 0$ small enough, since the conclusion for non-small r follows from the interior Harnack inequality.

Recall that $\Omega \subset \mathbb{R}^n$ is assumed to Lipschitz domain, with $0 \in \partial\Omega$. Then, for some $e \in S^{n-1}$, $\eta > 0$ (typically large), and $r_0 > 0$ depending on (the Lipschitz regularity of) Ω we have

$$\tilde{\mathcal{C}}_\eta \cap B_{2r_0} \subset \Omega$$

where $\tilde{\mathcal{C}}_\eta$ is the cone of Lemma 6.1, which is very sharp for η large.

Let Φ and $\epsilon > 0$ be the subsolution and the constant in Lemma 6.1. We now take

$$\tilde{\Phi} = (\Phi - (|x|/r_0)^2) \chi_{2r_0}.$$

By Lemma (6.1) we have

$$M^- \tilde{\Phi} \geq -C \quad \text{in } B_{r_0}$$

while clearly $\tilde{\Phi} \leq 0$ outside B_{r_0} .

Now we take observe that, for $c_1 > 0$ small enough we have

$$M^-(c_1\tilde{\Phi} + \chi_{D_1}) \geq -c_1C + c \geq c/2 > 0$$

in B_{r_0} — not that $B_{r_0} \cap D_1 = \emptyset$ since r_0 is small.

Then, taking $\delta \in (0, c/2)$ we have

$$M^-(u - c_1\tilde{\Phi} + \chi_{D_1}) \leq 0 \quad \text{in } B_{r_0}$$

while

$$u - c_1\tilde{\Phi} + \chi_{D_1} \geq 0 - c_1\tilde{\Phi} + 0 \geq 0 \quad \text{in } (\mathbb{R}^n \setminus B_{r_0}) \setminus D_1$$

and

$$u - c_1\tilde{\Phi} + \chi_{D_1} = (u - 1) - c_1\tilde{\Phi} \geq 0 - c_1\tilde{\Phi} \geq 0 \quad \text{in } (\mathbb{R}^n \setminus B_{r_0}) \cap D_1.$$

Then, by the maximum principle we obtain

$$u - c_1\tilde{\Phi} = u - c_1\tilde{\Phi} + \chi_{D_1} \geq 0 \quad \text{in } B_{r_0}$$

and hence

$$u(x) \geq c_1\Phi(x) - C|x|^2 \quad \text{for } x \in B_{r_0}$$

which clearly implies the Lemma (taking $\gamma = \epsilon$). \square

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