

APPROXIMATE TRACKING OF PERIODIC REFERENCES IN A CLASS OF BILINEAR SYSTEMS VIA STABLE INVERSION

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ABSTRACT. This article deals with the tracking control of periodic references in single-input single-output bilinear systems using a stable inversion-based approach. Assuming solvability of the exact tracking problem and asymptotic stability of the nominal error system, the study focuses on the output behavior when the control scheme uses a periodic approximation of the nominal feedforward input signal u_d . The investigation shows that this results in a periodic, asymptotically stable output; moreover, a sequence of periodic control inputs u_n uniformly convergent to u_d produce a sequence of output responses that, in turn, converge uniformly to the output reference. It is also shown that, for a special class of bilinear systems, the internal dynamics equation can be approximately solved by an iterative procedure that provides closed-form analytic expressions uniformly convergent to its exact solution. Then, robustness in front of bounded parametric disturbances/uncertainties is achievable through dynamic compensation. The theoretical analysis is applied to nonminimum phase switched power converters.

1. Introduction. Bilinear control systems are a class of nonlinear systems modelled by linear differential equations with the control inputs appearing as coefficients. A number of processes arising in engineering and sciences admit a description or an approximation in terms of bilinear models. The interested reader is referred to [1] for a survey on theory and applications of bilinear control theory. It is also worth mentioning the recently published text [2], which contains remarkable up-to-date material on this topic.

Power electronics is an area in which bilinear systems are specially interesting, because the state-space averaged models of basic DC-DC nonlinear switched converters are indeed bilinear [3]. This type of converters possess a very simple structure, and considerable research effort has been directed to study the possibility of

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using them in DC-AC conversion schemes. Step-down AC conversion by means of a linear buck-derived converter is actually solved [4]. However, step-up inversion is a challenging problem because it requires the use of boost or buck-boost converters, which are nonminimum phase systems. This means that when the output voltage undergoes a direct control action the internal variable, i.e. the inductor current, becomes unstable.

The standard proposal to overcome this problem is indirect control of the output voltage through the inductor current [5]. This might be seen as a stable inversion-based procedure [6] in which, given a voltage profile, the key problem consists of finding a stable reference for the nonminimum phase variable that arises as a solution of an Abel ODE in the normal form. Then, the control action forces the current to track such a reference and the internal dynamics yields the output voltage expected behavior. However, the general methods for the computation of stable inverses reported so far [6, 7] involve backwards time numeric integration, which makes exact output tracking controllers extremely sensitive to plant parameter disturbances and/or uncertainties. Additionally, both general and specific techniques such as the one used in [5] need an off-line calculation of the inverse reference profile, thus making robustness a main concern of the approach.

Robust tracking control of periodic output voltage references for nonlinear power converters is usually addressed obtaining bounded, closed-form analytic approximations of the unstable internal variable reference and later using them in dynamic compensation schemes. Nonetheless, the validity of the approaches proposed so far is constrained by different reasons. In [8] the stable inverse is computed from the expression of the equilibrium current in the regulation case, just replacing the set-point voltage reference by the actual time-varying one: this yields a severe trade-off between system parameters and command profile in order to keep the tracking error between acceptable bounds. The proposal in [9] obtains a bounded reference for the inductor current as a solution of the linearized internal dynamics, which reduces its effectiveness to a vicinity of the operating point. The method introduced in [10] exploits the differential flatness of the system to obtain an iterative sequence of bounded approximations of the nonminimum phase variable reference; however, no convergence proof is provided. Finally, [11] obtains a uniformly convergence sequence of Galerkin approximations of the inductor current reference, but two main handicaps appear. On the one hand, only the first Galerkin approximation is available in closed-form and, therefore, useful for dynamic compensation. On the other hand, the effectiveness of the control scheme depends on a number of hypotheses for which sufficient conditions are not provided.

These last issues are partially solutioned in [12] with the introduction of a Banach's fixed-point theorem-based iterative technique that produces an L_∞ -norm convergent sequence of periodic functions that are analytically computable in the closed form. The method is further refined in [13], the result being a procedure which depends on the fulfillment of few and rather easily checkable constraints over the system parameters and the voltage reference profile that provides robustness to piecewise constant load disturbances lying in a known compact set by means of dynamic compensation. However, the use of a state feedback control law rather than a pure feedforward action, as proposed in [6], results in two concerns. The first one deals with possible control saturation during transients, this yielding restrictions on the set of admissible initial conditions. The second one is related to the fact that output voltage dynamics is studied assuming that the controlled variable, i.e. the

input current, has reached the control target and is in the steady state tracking its reference; this approximation may result in unpredicted problems during transients.

This article intends to overcome the lacks of the approach proposed in [13] by using the approximate references obtained therein in pure feedforward control laws, thus following the original solution procedure for the exact tracking problem [6]. Then, the dynamics of the closed-loop system is fully analyzable and, at the same time, unexpected transient behavior and possible saturations of the control action are avoided as well.

The approach proposed in the preceding paragraph is certainly applicable to a wider class of bilinear systems sharing equivalent control targets. Hence, the paper tackles the situation from a generic viewpoint and deals with the tracking control of periodic references in Single-Input Single-Output (SISO) bilinear systems using stable inversion. Then, assuming that the exact tracking problem is solvable and that the nominal error system is asymptotically stable, the study considers the output behavior when the control scheme uses a periodic approximation of the nominal feedforward input signal u_d . It is shown that this results in a periodic, asymptotically stable output; moreover, a sequence of periodic control inputs u_n uniformly convergent to u_d produce a sequence of output responses that, in turn, converge uniformly to the output reference. Furthermore, it is proved that, for the special class of bilinear systems such that its internal dynamics equation can be written as an Abel ODE in the normal form, this may be approximately solved by an iterative procedure that provides closed-form analytic expressions uniformly convergent to its exact solution. Then, robustness in front of bounded parametric disturbances/uncertainties is achievable through dynamic compensation. The theoretical analysis is finally applied to the nonminimum phase switched power converters boost and buck-boost.

The paper is organized as follows. Section 2 poses the exact tracking problem for SISO bilinear systems and summarizes its solution through a classical stable inversion process that uses partial linearization techniques. A study of the approximate tracking of periodic references is carried out in Section 3. Section 4 contains an iterative procedure to solve one-dimensional internal dynamics equations that can be written as Abel ODEs in the normal form. The developed theory is applied to nonminimum phase bilinear switched power converters in Section 5, while simulation results are presented in Section 6.

2. Stable inversion by partial linearization in SISO bilinear systems. The material of this section has been adapted from [6, 14].

Consider the n th-dimensional, SISO bilinear system¹

$$\dot{x} = Ax + \delta + (Bx + \gamma)u, \quad (1)$$

$$y = h(x), \quad (2)$$

where A, B are square real matrices, δ, γ are \mathbb{R}^n vectors and $h : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth scalar map.

The stable inversion problem may be posed as follows: given a smooth output reference trajectory $y_d(t)$, find a bounded state reference trajectory $x_d(t)$ and a

¹Systems matching (1) may also be called inhomogeneous bilinear systems or biaffine systems; see Chapter 1.4 in [2].

bounded nominal control action, $u_d(t)$, such that the triple (x_d, u_d, y_d) satisfies:

$$\dot{x}_d = Ax_d + \delta + (Bx_d + \gamma)u_d, \quad (3)$$

$$y_d = h(x_d). \quad (4)$$

Let us denote by $L_f h(x)$ the Lie derivative of h with respect or along a vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and let us use the following notation:

$$L_f^0 h(x) = h(x), \quad L_f^k h(x) = L_f \left(L_f^{k-1} h(x) \right), \quad k \geq 1.$$

Definition 2.1. System (1)-(2) is said to have relative degree ρ , $1 \leq \rho \leq n$, in a region $D_0 \subseteq D$ iff

$$L_{Bx+\gamma} L_{Ax+\delta}^i h(x) = 0, \quad i = 0, 2, \dots, \rho - 2, \quad \text{and} \quad L_{Bx+\gamma} L_{Ax+\delta}^{\rho-1} h(x) \neq 0.$$

The assumption of a well-defined relative degree ρ for (1)-(2) in D_0 allows a partial linearization of the system therein. This is possible due to the fact that the expressions of $y, \dot{y}, \dots, y^{(\rho-1)}$ calculated along the system trajectories are independent of the control action u , while $y^{(\rho)}$ effectively depends on u . As a consequence, the input-output map can be reduced to a chain of ρ integrators by means of state feedback. Namely, let us set

$$\xi = \psi(x) = \left(y, \dot{y}, \dots, y^{(\rho-1)} \right) = \left(h(x), L_{Ax+\delta} h(x), \dots, L_{Ax+\delta}^{\rho-1} h(x) \right),$$

and select $\eta = \varphi(x)$ as an $n - \rho$ dimensional functional on D in such a way that

$$\begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} = T(x) \quad (5)$$

is a diffeomorphism on D_0 . In these new coordinates, (1)-(2) become

$$\begin{aligned} \dot{\eta} &= L_{Ax+\delta} \varphi(x) + u L_{Bx+\gamma} \varphi(x), \\ \dot{\xi}_i &= \xi_{i+1}, & i = 1, \dots, \rho - 1, \\ \dot{\xi}_\rho &= L_{Ax+\delta}^\rho h(x) + u L_{Bx+\gamma} L_{Ax+\delta}^{\rho-1} h(x), \\ y &= \xi_1. \end{aligned} \quad (6)$$

A further compaction may be achieved defining

$$\begin{aligned} \varphi_1(\eta, \xi) &= [L_{Ax+\delta} \varphi(x)]_{x=T^{-1}(\eta, \xi)}, \\ \varphi_2(\eta, \xi) &= [L_{Bx+\gamma} \varphi(x)]_{x=T^{-1}(\eta, \xi)}, \\ \beta(\eta, \xi) &= [L_{Bx+\gamma} L_{Ax+\delta}^{\rho-1} h(x)]_{x=T^{-1}(\eta, \xi)}, \\ \alpha(\eta, \xi) &= - \left[\frac{L_{Ax+\delta}^\rho h(x)}{L_{Bx+\gamma} L_{Ax+\delta}^{\rho-1} h(x)} \right]_{x=T^{-1}(\eta, \xi)}, \end{aligned}$$

and using the canonical form representation of a chain of ρ integrators:

$$A_c = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B_c = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad C_c = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Thus, (6) may be written as

$$\begin{aligned}\dot{\eta} &= \varphi_1(\eta, \xi) + \varphi_2(\eta, \xi)u, \\ \dot{\xi} &= A_c \xi + B_c \beta(\eta, \xi) [u - \alpha(\eta, \xi)], \\ y &= C_c \xi.\end{aligned}$$

Notice that the control law

$$u = \beta^{-1}(\eta, \xi)v + \alpha(\eta, \xi) \quad (7)$$

linearizes the input-output map. Also, recall that $y = y_d(t)$ denotes the output command profile. Then, the setting $v = y_d^{(\rho)}(t)$ in (7) produces the exact tracking $\xi = \xi_d(t)$. As a consequence, the internal dynamics is governed by the ODE

$$\dot{\eta} = \varphi_1(\eta, \xi_d(t)) + \varphi_2(\eta, \xi_d(t)) \left[\beta^{-1}(\eta, \xi_d(t)) y_d^{(\rho)}(t) + \alpha(\eta, \xi_d(t)) \right]. \quad (8)$$

Remark 1. Systems with unstable internal dynamics are said to be *nonminimum phase*.

Let $\eta = \eta_d(t)$ be a solution of (8). Then, the corresponding state reference $x_d(t)$ and nominal control $u_d(t)$ are given by:

$$x_d = x_d(t) = T^{-1}(\eta_d(t), \xi_d(t)), \quad (9)$$

$$u_d = u_d(t) = \beta^{-1}(\eta_d(t), \xi_d(t)) y_d^{(\rho)}(t) + \alpha(\eta_d(t), \xi_d(t)). \quad (10)$$

Let also

$$\dot{e} = [A + B(u_d(t) + e_u(t))]e + [Bx_d(t) + \gamma]e_u(t), \quad (11)$$

be the error system associated to (1)-(3), with $e = x - x_d$, $e_u = u - u_d$.

Assumption 1. $y_d(t)$ is a smooth, bounded output reference, and there exists a bounded solution $\eta = \eta_d(t)$ for (8) such that, $\forall t \geq 0$,

$$T^{-1}(\eta_d, \xi_d) \in D_0 \quad \text{and} \quad \beta(\eta_d, \xi_d) = \left[L_{Bx+\gamma} L_{Ax+\delta}^{\rho-1} h(x) \right]_{x=T^{-1}(\eta_d, \xi_d)} \neq 0.$$

Assumption 2. The zero-input system associated to (11), i.e. that with $e_u = 0$, admits $e = 0$ as a Globally Asymptotically Stable (GAS) equilibrium point.

Proposition 2.1. *Let Assumption 1 be fulfilled; then, the input-state trajectory (x_d, u_d) given by (9)-(10) is bounded and solves the exact tracking problem. Furthermore, if Assumption 2 is additionally satisfied, the use of the feedforward control law $u = u_d$ in (1) yields asymptotically the desired output response, i.e.*

$$\lim_{t \rightarrow +\infty} [y(t) - y_d(t)] = 0. \quad (12)$$

Proof. The first statement is straightforward. Assumption 2 entails $x - x_d \rightarrow 0$ for $t \rightarrow +\infty$, and (12) follows from the assumed smoothness of $h(\cdot)$. \square

Remark 2. It is worth pointing out the following issues about Assumption 2:

- (i) In case that Assumption 2 is verified not globally but locally, the fulfillment of Proposition 2.1 is restricted to a set of initial conditions for the state sufficiently close to $x_d(0)$. Nonetheless, if Assumption 2 is not verified, (1) might still be stabilized with different techniques: see, for example, [15] and the references therein.
- (ii) Assumption 2 is equivalent to say that (11) satisfies the 0-GAS property or that is 0-GAS.

3. Approximate tracking of periodic references. Consider (1)-(2), and let the control goal be the tracking of a certain T -periodic output reference y_d , i.e. $y_d(t+T) = y_d(t)$. Recall that Assumption 1 is sufficient to guarantee that a bounded input-state trajectory (x_d, u_d) satisfying (3)-(4) may be obtained by means of stable inversion via partial linearization, i.e. following the exposition in Section 2, while Assumption 2 assures that the use of the feedforward control action $u = u_d$ in (1)-(2) may yield the desired output response $y = y_d$ in the steady state.

Next result establishes that if (1) does not receive an exact control action $u = u_d(t)$ but a continuous, T -periodic approximation $u(t) = u_d(t) + e_u(t)$, the system answers with a continuous, T -periodic, GAS output $y = y(t)$. For, let us previously define the set $U \subset \mathcal{C}([0, T])$ as follows:

$$U = \{u \in \mathcal{C}([0, T]); \dot{e} = [A + Bu(t)]e \text{ is GAS}\}. \quad (13)$$

Assumption 3. Assumptions 1 and 2 are satisfied, and both the output command profile y_d and the input-state trajectory (x_d, u_d) are continuous and T -periodic, with $x_d \in \mathcal{C}^1([0, T])$ and $u_d \in \mathcal{C}([0, T])$.

Theorem 3.1. *Let Assumption 3 be fulfilled. Then, when (1) undergoes a continuous, T -periodic feedforward control action $u(t) = u_d(t) + e_u(t) \in U$, there exists one and only one continuous, T -periodic and GAS output response $y(t)$. \square*

Proof. As $h(\cdot)$ is assumed to be Lipschitz continuous, it is sufficient to prove the existence of one and only one \mathcal{C}^1 , T -periodic, GAS error state for (11). The system being linear and taking into account the continuity and T -periodicity of x_d and u_d derived from Assumption 3, a necessary and sufficient condition for this to happen [16] is that $e = 0$ be a GAS equilibrium point of the linear part of (11), which is true by hypothesis. Globality is due to the linear character of (11). \square

Theorem 3.1 ensures that, under Assumption 3, a sequence u_n of continuous, T -periodic approximations of u_d such that $u_n \in U$, $\forall n \in \mathbb{N}$, produces, in turn, a sequence y_n of continuous, T -periodic, GAS outputs. By next result, which is based on Theorem 7.2, the uniform convergence $u_n \rightarrow u_d$ yields the uniform convergence $y_n \rightarrow y_d$.

Theorem 3.2. *Let Assumption 3 be fulfilled. Let also $\{u_n\}$, with $u_n = u_d + e_{u_n}$ be a sequence of continuous, T -periodic feedforward control actions undergone by system (1) and such that $u_n \in U$, $\forall n \in \mathbb{N}$. If $\{u_n\}$ converges uniformly to u_d , then there exists a sequence $\{y_n\}$ of continuous, T -periodic, GAS output responses that converges uniformly to y_d .*

Proof. As discussed above, the existence of the sequence y_n is ensured by Theorem 3.1. Let us now deal with the convergence issue. Identifying (40) with the equivalent error system (11) through the assignment

$$A_n(t) = A + Bu_n(t), \quad b_n(t) = (Bx_d + \gamma)e_{u_n}(t),$$

the uniform convergence $u_n \rightarrow u_d$ ensure that

$$A_n(t) \rightarrow A(t) = A + Bu_d(t), \quad b_n(t) \rightarrow 0,$$

whilst the hypothesis $u_n \in U$, $\forall n \in \mathbb{N}$, and the linear character of the systems involved, guarantees the hyperbolicity of the equilibrium solution $x = 0$ of (41), (42) also demanded in Theorem 7.2. Hence, the sequence of continuous, T -periodic and GAS error states $\{e_n\}$ converge uniformly to 0 as the control error $\{e_{u_n}\} \rightarrow 0$.

Finally, $h(x)$ being Lipschitz-continuous, the output sequence $\{y_n\}$, with $y_n(t) = h(e_n + x_d)$, is such that $\{y_n\} \rightarrow y_d = h(x_d)$. \square

Remark 3. Notice that the 0-GAS hypothesis included in Assumption 3 assures $u_d \in U$. Therefore, by continuity arguments, there exists an open neighborhood U_d of u_d such that $U_d \subseteq U$. In case that $U = U_d$, Theorem 3.1 follows from a standard result on periodic perturbations of periodic systems [16]. In turn, the uniform convergence hypothesis $u_n \rightarrow u_d$ entails the existence of $n_0 \in \mathbb{N}$ such that $u_n \in U_d$, $\forall n \geq n_0$; therefore, Theorem 3.2 may follow for the subsequence $\{u_n\}_{n \geq n_0}$.

Alternatively, Local Input-to-State Stability (LISS) theory [17] also appears to be an appropriate framework to prove Theorems 3.1 and 3.2 for $U = U_d$, because 0-GAS is a sufficient condition for a system to be LISS. Roughly speaking, the LISS approach copes with the stability of the mapping $u(\cdot) \rightarrow x(\cdot)$ and accounts for asymptotic tendency and transient behavior as well. Indeed, the LISS concept encompasses the properties that inputs which are bounded or convergent produce states with the respective property; at the same, it provides estimates of the magnitude of the transient as a function of the initial state. The reader is also referred to [18] for an excellent survey on ISS, the global version of LISS.

4. Iterative solution of a class of internal dynamics equation. Consider a second order, bilinear control system (1)-(2), with $x, \gamma, \delta \in \mathbb{R}^2$, $A, B \in M_2(\mathbb{R})$, the output being one of the state variables. Assume that $y = x_2$ and also that the relative degree of the system with respect to such an output is 1, i.e. there exists $D \subseteq \mathbb{R}^2$ with

$$L_{Bx+\gamma}L_{Ax+\delta}^0 x_2 = (1, 0)^\top \cdot (Bx + \gamma) \neq 0, \quad \forall x \in D.$$

Then, setting $\eta = x_1$, $\xi = x_2$, (8) can be written as an Abel ODE of the second kind:

$$[f_0(t) + f_1(t)\eta] \dot{\eta} = g_0(t) + g_1(t)\eta + g_2(t)\eta^2, \quad (14)$$

with

$$f_k(t), g_l(t) \in \text{span}\{1, \xi_d(t), \xi_d^2(t), \dot{\xi}_d(t), \dot{\xi}_d(t)\xi_d(t)\}.$$

Moreover, using a standard transformation [19] and abusively keeping the notation for the internal variable η , (14) may take the normal form:

$$\eta \dot{\eta} = \eta - g(t), \quad (15)$$

with $g(t) = g(\xi_d(t))$.

Remark 4. The below developed technique is applicable to any generic system

$$\begin{aligned} \dot{x} &= F(x) + G(x)u, \\ y &= h(x), \end{aligned}$$

with relative degree $\rho = n - 1$ and such that its internal dynamics equation (8) can be written as -or transformed into- an Abel ODE in the normal form (15).

Theorem 4.1. [20] *Let $g(t)$ be T -periodic, smooth and such that $g(t) > 0$, $\forall t \geq 0$. Then, equation (15) has one and only one T -periodic solution $\phi(t)$, which is positive and unstable.* \square

Let us obtain an iterative sequence of T -periodic approximations of $\phi(t)$.

Denote as $\mathcal{C}_{per}^n([0, T])$, $n = 0, 1, \dots, \infty$, the subset of elements of $\mathcal{C}^n([0, T])$ that allow a continuous and T -periodic extension in \mathbb{R} , i.e.

$$\mathcal{C}_{per}^n([0, T]) = \{\eta \in \mathcal{C}^n([0, T]); \eta(0) = \eta(T)\},$$

with $\mathcal{C}_{per}([0, T]) = \mathcal{C}_{per}^0([0, T])$. It is well known that $(\mathcal{C}_{per}([0, T]), \|\cdot\|)$, where $\|\cdot\|$ stands for the uniform norm, i.e.

$$\|\eta\| = \sup_{[0, T]} \{|\eta(t)|\}, \quad \eta \in \mathcal{C}([0, T]),$$

is a Banach space.

Let us denote as $P_0 : \mathcal{C}_{per}([0, T]) \longrightarrow \mathbb{R}$ the map that extracts the mean value of periodic functions:

$$P_0(\eta) = \frac{1}{T} \int_0^T \eta(t) dt, \quad \eta \in \mathcal{C}_{per}([0, T]),$$

and let \bar{X} denote the subset of $\mathcal{C}_{per}([0, T])$ that contains the elements with zero mean value:

$$\bar{X} = \{\eta \in \mathcal{C}_{per}([0, T]); P_0(\eta) = 0\}.$$

Then, any $\eta \in \mathcal{C}_{per}([0, T])$ can be uniquely decomposed as

$$\eta = \eta_0 + \bar{\eta}, \quad \text{with } \eta_0 = P_0(\eta) \text{ and } \bar{\eta} \in \bar{X}. \quad (16)$$

Finally, \bar{X} being closed by integration, for all $\bar{\eta} \in \bar{X}$ there exists a unique element $\hat{\eta} \in \bar{X}$ such that $\hat{\eta} = \bar{\eta}$.

Assumption A. Let $\xi_d(t)$ be smooth, T -periodic, positive and such that the corresponding function $g(t) = g(\xi_d(t))$ is positive and verifies:

$$g_0 > \frac{T}{2} + \sqrt{2\|\hat{g}\|}. \quad (17)$$

Lemma 4.2. *Let us define*

$$\alpha = \frac{1}{g_0} \sqrt{\left(g_0 - \frac{T}{2}\right)^2 - 2\|\hat{g}\|}, \quad L_- = g_0(1 - \alpha) - \frac{T}{2}, \quad L_a = ag_0 - \frac{T}{2}.$$

If Assumption A holds, then:

(i) $0 < \alpha \leq 1 - T(2g_0)^{-1}$.

(ii) $\forall a \in (1 - \alpha, 1)$, $0 \leq L_- < L_a$. □

Theorem 4.3. [13] *If Assumption A holds, then $\forall a \in (1 - \alpha, 1)$ and $\forall L \in (L_-, L_a]$, there exist a closed, nonempty subset M_L of $\mathcal{C}_{per}([0, T])$ defined as*

$$M_L = \{\bar{\eta} \in \bar{X}; \|\bar{\eta}\| \leq L\}, \quad (18)$$

such that the sequence $\{\eta_n\} = \{g_0 + \bar{\eta}_n\}$, obtained by means of the iterative procedure

$$\bar{\eta}_{n+1} = \bar{A}(\bar{\eta}_n) = \frac{1}{g_0} \left[\hat{\eta}_n - \hat{g} - \frac{1}{2} (\bar{\eta}_n^2 - P_0(\bar{\eta}_n^2)) \right], \quad \bar{\eta}_0 \in M_L, \quad (19)$$

converges uniformly to the hyperbolic, T -periodic solution $\phi(t)$ of (15). □

Corollary 1. *Assume that*

$$g(t) = g_0 + \bar{g}(t) = g_0 + \sum_{k=1}^r A_k \cos k\omega t + B_k \sin k\omega t, \quad (20)$$

and let $\bar{\eta}_0 \in M_L$ be selected as a

$$\bar{\eta}_0(t) = \sum_{k=1}^s \alpha_{0k} \cos k\omega t + \beta_{0k} \sin k\omega t. \quad (21)$$

Then, $\forall n \geq 1$, the successive approximations $\eta_n = g_0 + \bar{\eta}_n$ obtained from (19) follow the assignment

$$\eta_n(t) = g_0 + \sum_{k=1}^m \alpha_{nk} \cos k\omega t + \beta_{nk} \sin k\omega t, \quad m = \max\{2^{n-1}r, 2^n s\},$$

with $\alpha_{nk} = \alpha_{nk}(\alpha_{0j}, \beta_{0j}, A_j, B_j)$, $\beta_{nk} = \beta_{nk}(\alpha_{0j}, \beta_{0j}, A_j, B_j)$. \square

Remark 5. When the hypotheses of Corollary 1 are satisfied and Fourier expansions of $g(t)$ and $\bar{\eta}_0$ (see (20) and (21)) are finite, the coefficients α_{nk} , β_{nk} are not only numerically but analytically computable in closed form.

5. Application to nonminimum phase switched power converters. The state-space averaged model of the DC-to-DC switched power converters boost and buck-boost is given by the SISO system

$$L \frac{di_L}{d\tau} = -v_C + \mu v_C + V_g [1 + k(\mu - 1)], \quad (22)$$

$$C \frac{dv_C}{d\tau} = i_L - \frac{v_C}{R} - \mu i_L, \quad (23)$$

where the inductor current i_L and the capacitor voltage v_C act as state variables and the control action $\mu = \mu(t)$ takes values in the interval $(0, 1)$. Recall that the control action in the physical converter is actually carried out by means of a switch; hence, $\mu(t)$ is implemented through a PWM signal. The constant voltage source V_g , the inductance L and the capacitance C are considered well known parameters, while perturbations may affect the load resistance R . Boost and buck-boost converters are modelled, respectively, by $k = 0$ and $k = 1$. Finally, the capacitor voltage is considered as the control output.

An appropriate change of state variables and a time re-scaling yield a dimensionless model with a minimum number of parameters that simplify a systematic analysis. Namely, using

$$x_1 = \frac{1}{V_g} \sqrt{\frac{L}{C}} i_L, \quad x_2 = \frac{v_C}{V_g}, \quad t = \frac{1}{\sqrt{LC}} \tau, \quad \lambda = \frac{1}{R} \sqrt{\frac{L}{C}}, \quad u = 1 - \mu,$$

(22), (23) and the output become

$$\dot{x}_1 = 1 - u(x_2 + k), \quad (24)$$

$$\dot{x}_2 = -\lambda x_2 + u x_1, \quad (25)$$

$$y = x_2. \quad (26)$$

Since L , R and C take positive values, λ is always positive. Meanwhile, $u : [0, +\infty) \rightarrow (0, 1)$. Assigning

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -\lambda \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} -k \\ 0 \end{pmatrix} \quad (27)$$

and $h(x) = x_2$, (24)-(25)-(26) may be written as (1)-(2).

Assume that a smooth, T -periodic output reference candidate $y_d = x_{2d}$ is selected. Let us solve the tracking problem by means of the stable inversion procedure detailed in Section 2.

For, let us first notice that $L_{Bx+\gamma} L_{Ax+\delta}^0 h(x) = x_1$. Hence, according to Definition 2.1, the relative degree of (24)-(25)-(26) in $D_0 = \{x \in \mathbb{R}^2; x_1 \neq 0\}$ is $\rho = 1$.

Then, we select $\xi = x_2$, $\eta = x_1$, which means that the transformation (5) is the identity and, thus, a diffeomorphism on D_0 . Afterwards, a straightforward calculation shows that the internal dynamics equation (8) is now:

$$\dot{\eta} = 1 - \frac{[\xi_d(t) + k] [\dot{\xi}_d(t) + \lambda \xi_d(t)]}{\eta}, \quad (28)$$

where $\xi_d(t) = x_{2d}(t)$. As the system is demanded to evolution in D_0 due to relative degree restrictions, (28) may take the form (15) with

$$g(t) = g(\xi_d(t)) = [\xi_d(t) + k] [\dot{\xi}_d(t) + \lambda \xi_d(t)]. \quad (29)$$

Therefore, the results about both exact and iteratively approximated solutions of (28) derived in Section 4 are readily applicable to the present case under appropriate restrictions on the output voltage reference profile $\xi_d(t)$.

Hence, the existence of a continuous, T -periodic and positive solution for (28) is a straightforward consequence of Theorem 4.1.

Proposition 5.1. *If $\xi_d(t)$ is T -periodic, smooth and such that $g(\xi_d(t)) > 0$, $\forall t \geq 0$, then (28) has one and only one T -periodic solution $\eta_d(t)$, which is positive and unstable.* \square

Remark 6. Notice that:

- (i) The hypotheses of Proposition 5.1 ensure the fulfillment of Assumption 1.
- (ii) The nonminimum phase character of (24)-(25)-(26) for $y = x_2$ follows immediately from Proposition 5.1.

Additional restrictions over $\xi_d(t)$ are needed to ensure that the converter operates inside the control unsaturation region, that is, $0 < u_d(t) < 1$. To this end, let

$$u_d(t) = \frac{\dot{\xi}_d(t) + \lambda \xi_d(t)}{\eta_d(t)} = \frac{\dot{x}_{2d}(t) + \lambda x_{2d}(t)}{x_{1d}(t)} \quad (30)$$

be the nominal control input obtained from (10). Next result is proved in [20]:

Proposition 5.2. *If $x_{2d} > 0$ and $\inf \{g(x_{2d}(t))\} > \sup \{\dot{x}_{2d}(t) + \lambda x_{2d}(t)\}$, then $u_d \in \mathcal{C}([0, T])$ and $0 < u_d(t) < 1$.* \square

In order to establish the effective solution of the exact and the approximate tracking problems, let us study the stability of the zero-input system associated to (11):

Proposition 5.3. *System (11) is 0-GAS for all $u_d(t) \in \mathcal{C}([0, T])$, $u_d(t) \neq 0$, $\forall t \geq 0$.*

Proof. The zero-input system associated to (11) reads as

$$\dot{e} = [A + Bu_d(t)]e,$$

with A , B , defined in (27). Let $V(e) = \frac{1}{2}e^\top e$ be a Lyapunov function candidate. Then,

$$\dot{V}(e) = e^\top Ae \leq 0,$$

because A is negative semidefinite and B is skew-symmetric. Nevertheless, the set S where $\dot{V} = 0$ is $S = \{e \in \mathbb{R}^2; e_2 = 0\}$, and the greatest invariant set inside S is $e = 0$:

$$e_2 = 0 \Rightarrow \dot{e}_1 = -0 \cdot u_d(t) = 0 \Rightarrow e_1 = K, \quad K \in \mathbb{R};$$

however, $\dot{e}_2 = 0$ because, otherwise, the system would abandon S immediately, this yielding

$$0 = -\lambda \cdot 0 + K \cdot u_d(t) \Rightarrow K = 0 \Rightarrow e_1 = 0,$$

due to the hypothesis $u_d(t) \neq 0, \forall t \geq 0$. The result follows from La Salle's Theorem for periodic systems [21]. \square

Remark 7. It is immediate from Proposition 5.3 that the set U defined in (13) is now

$$U = \{u \in \mathcal{C}([0, T]); u(t) \neq 0, \forall t \geq 0\}. \quad (31)$$

Assumption 4. The output command profile $x_{2d}(t)$ is T -periodic, smooth and positive, $\forall t \in [0, T]$, and is such that

$$\inf \{g(x_{2d}(t))\} > \sup \{\dot{x}_{2d}(t) + \lambda x_{2d}(t)\}. \quad (32)$$

Theorem 5.1. *If Assumption 4 is verified, the feedforward control input $u = u_d(t)$ given by (30) solves the exact tracking problem $y_d = x_{2d}(t)$ for system (24)-(25)-(26).*

Proof. Assumption 4 ensures the fulfillment of Assumptions 1 (see Remark 6.i) and 2, as well as $0 < u_d(t) < 1, \forall t$, because of Proposition 5.2. Hence, the result follows from Proposition 2.1. \square

So far, the control problem regarding the exact tracking of periodic references by the output voltage of nonlinear DC-to-DC power converters is theoretically solved. Once at this point it is worth recalling that, according to Theorem 5.1, the internal dynamics periodic reference $x_{1d}(t) = \eta_d(t)$ that is to be obtained as a solution of (28) is unstable. Besides, in the general case $\eta_d(t)$ is not available in closed-form. It is therefore immediate that any control law of the form (30) should use a more or less accurate approximation of $\eta_d(t)$, instead of $\eta_d(t)$ itself. For such cases, next result ensures the existence of GAS periodic outputs whenever the actually used control $u(t)$ does not saturate and belongs to the set U defined in (31).

Theorem 5.2. *Let $x_{2d}(t)$ be such that Assumption 4 is fulfilled. Let also $\eta_d^a(t)$ be a continuous, T -periodic approximation of $\eta_d(t)$ satisfying $\eta_d^a(t) \neq 0, \forall t \geq 0$, and such that the control action*

$$u_d^a(t) = \frac{\dot{x}_{2d}(t) + \lambda x_{2d}(t)}{\eta_d^a(t)} \quad (33)$$

verifies $0 < u_d^a(t) < 1, \forall t \geq 0$. Then, the use of $u_d^a(t)$ in (24)-(25)-(26) produces one and only one T -periodic, GAS output response $y_d^a(t)$.

Proof. The control input $u_d^a(t)$ can be easily written as:

$$u_d^a(t) = u_d(t) + e_u(t) = u_d(t) + [\dot{x}_{2d}(t) + \lambda x_{2d}(t)] \frac{\eta_d(t) - \eta_d^a(t)}{\eta_d(t)\eta_d^a(t)}.$$

Assumption 4 and the hypotheses over $\eta_d^a(t)$, together with Proposition 5.3, guarantee that the hypotheses of Theorem 3.1. As it is also assumed that the control action $u_d^a(t)$ does not saturate, the result follows. \square

Remark 8. Notice that, as $0 < u_d(t) < 1$ follows from the assumptions in Theorem 5.2, continuity arguments ensure that for $\eta_d^a(t)$ sufficiently close to $\eta_d(t)$, it also happens that $0 < u_d^a(t) < 1$.

A rather simple and reliable option to compute $\eta_d^a(t)$ is numeric integration in backwards time; alternatively, after a careful determination of $\eta_d(0)$, one can integrate (28) in $[0, T]$ and then extend the function periodically in $[T, +\infty)$. However, the use of such noncausal references results in a lack of robustness due to high sensitivity in front of parametric perturbations or uncertainties.

An alternative way to proceed with the computation of $\eta_d^a(t)$ is given in Section 4: as (28) can be readily expressed as an Abel equation in normal form (15), the output voltage references $\xi_d(t)$ for which Assumption A is fulfilled allow a straightforward use of Theorem 4.3. Hence, for this cases it is possible to obtain a uniformly convergence, iterative sequence of periodic approximations of $\eta_d(t)$. Consequently, approximate tracking is achievable following Section 3:

Theorem 5.3. *Let $x_{2d}(t)$ be a T -periodic output reference such that Assumption A and*

$$\min \left\{ \frac{T}{2}, \inf \{g(x_{2d}(t))\} \right\} > \sup \{\dot{x}_{2d}(t) + \lambda x_{2d}(t)\} \quad (34)$$

are fulfilled. Then, the iterative mapping (19) allows to obtain a uniformly convergent sequence $\{\eta_n(t)\}$ of continuous, T -periodic approximations of $\eta_d(t)$. Furthermore, the sequence of feedforward control actions $\{u_n(t)\}$, with

$$u_n(t) = \frac{\dot{x}_{2d}(t) + \lambda x_{2d}(t)}{\eta_n(t)}, \quad (35)$$

is such that $0 < u_n < 1$, $\forall n$, and its use in (24)-(25)-(26) produces a sequence of T -periodic, GAS outputs $\{y_n(t)\}$ that converges uniformly to $y = x_{2d}(t)$.

Proof. According to Theorem 4.3, Assumption A ensures that the sequence of continuous, T -periodic functions $\{\eta_n(t)\} = \{g_0 + \bar{\eta}_n(t)\}$, where $\{\bar{\eta}_n(t)\}$ is iteratively obtained using (19), converges uniformly to $\eta_d(t)$. Moreover, using Theorem 4.3 and Lemma 4.2 it is possible to establish a uniform, positive lower bound for $\{\eta_n(t)\}$:

$$\eta_n(t) = g_0 + \bar{\eta}_n(t) \geq g_0 - \|\bar{\eta}_n(t)\| \geq g_0 - L \geq g_0 - L_a \geq g_0 - \left(g_0 - \frac{T}{2}\right) \geq \frac{T}{2}.$$

On the other hand, Assumption A and (34) are sufficient conditions for Assumption 4 to be satisfied. In turn, as $\dot{x}_{2d}(t) + \lambda x_{2d}(t) > 0$ follows from Assumption 4 [20], it is immediate that

$$0 < u_n(t) = \frac{\dot{x}_{2d}(t) + \lambda x_{2d}(t)}{\eta_n(t)} < \frac{\dot{x}_{2d}(t) + \lambda x_{2d}(t)}{\frac{T}{2}} < 1, \quad \forall n \geq 0.$$

Furthermore,

$$\|u_n(t) - u_d(t)\| = \left\| \frac{u_d(t)}{\eta_n(t)} (\eta_d(t) - \eta_n(t)) \right\| \leq \frac{2}{T} \|\eta_d(t) - \eta_n(t)\|,$$

which indicates that $\{u_n(t)\}$ converges uniformly to $u_d(t)$. Then, Theorem 3.2 guarantees that when (24)-(25)-(26) undergoes $\{u_n(t)\}$, a sequence of T -periodic, GAS output responses $\{y_n\}$ that converges uniformly to $y = x_{2d}(t)$ is produced. \square

Remark 9. Notice that the output error $y_n(t) - x_{2d}(t)$ may be lowered at will by using a sufficiently high order element of $\{\eta_n(t)\}$ as an internal variable reference in the feedforward control law (35).

Additionally, in sight of Corollary 1, in case that $\eta_d(t)$ possesses a finite Fourier series expansion and $\bar{\eta}_0$ is appropriately selected, the approximations can be computed in closed form, with explicitly parameter-dependent expressions. Thus, it

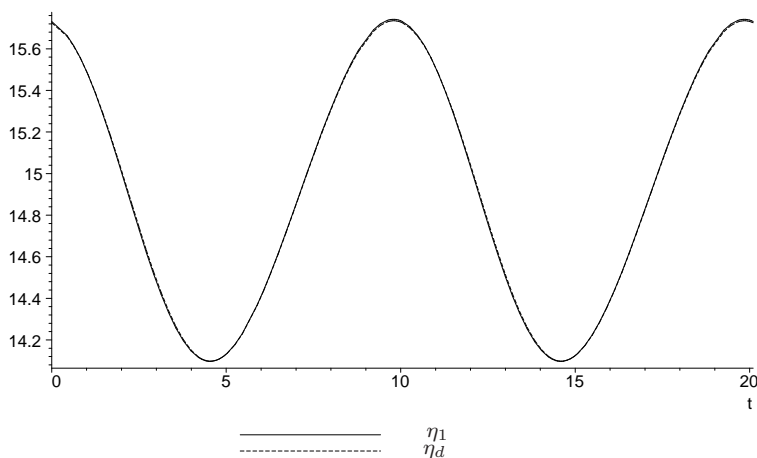


FIGURE 1. The input current approximate reference $\eta_1(t) = g_0 + \bar{A}[\bar{\eta}_{1G}(t)]$ matching the exact reference $\eta_d(t)$.

is possible to dynamically compensate the effect of piecewise constant load disturbances lying in a known compact set Λ through a real-time updating of the selected current reference $x_{1d}(t) = \eta_n(t) = \eta_n(t, \lambda)$ according to the instantaneous variation of λ , which is assumed to be estimated (using, for example, the algebraic estimator [22]) or measured. Hence, success is subject to the fulfillment of the Assumptions established in Theorem 5.3 for all $\lambda \in \Lambda$.

6. Simulation results. The above developed technique has been tested on a boost converter with $V_g = 15V$, $L = 0.018H$, $C = 0.00022F$ and $R = 10\Omega$. The selected output voltage reference profile has been $v_C(\tau) = 60 + 15 \sin(2\pi\nu\tau)$, with $\nu = 50Hz$, this yielding a normalized

$$x_{2d}(t) = \xi_d(t) = A + B \sin(\omega t) = 4 + \sin(\omega t), \quad (36)$$

where $\omega = 0.6252$ and $\lambda = 0.9045$.

With these settings, the hypotheses stated in Theorem 5.3 are fulfilled, in particular inequalities (17) and (34). The contractive constant a is to be selected in $(0.4268, 1)$; once a is fixed, the possible radius L of the set M_L defined in (18) lie in $(L_-, L_a]$, with $L_- = 1.3438$ and $L_a = 14.9248a - 5.0252$. Let $a = 0.9$ and $L = 8.4$. Furthermore, the iterative procedure of Theorem 4.3 provides better convergence rates with initial conditions closer to $\bar{\eta}_d$ [12]. Hence, let us pick $\bar{\eta}_0 = \bar{\eta}_{1G}$, with η_{1G} denoting the periodic component of the first Galerkin approximation of $\phi(t)$, namely [23]:

$$\bar{\eta}_{1G}(t) = \frac{4AB\omega(1 + \lambda^2Q)}{4 + \lambda^2\omega^2Q^2} \cos \omega t + \frac{2\lambda AB(4 - \omega^2Q)}{4 + \lambda^2\omega^2Q^2} \sin \omega t, \quad (37)$$

with $Q = 2A^2 + B^2$. The fact that $\bar{\eta}_{1G}(t)$ has a λ -dependent closed-form analytic expression maintains the possibility of achieving robustness by means of dynamic compensation. Finally, $\|\bar{\eta}_{1G}\| = 0.75 < L$.

The goodness of the input current reference approximations provided by the iterative procedure (19) introduced in Theorem 4.3 is revealed in Figure 1, where $\eta_1(t) = g_0 + \bar{A}[\bar{\eta}_{1G}(t)]$ appears matching the exact solution $\eta_d(t)$ of (28). The reason

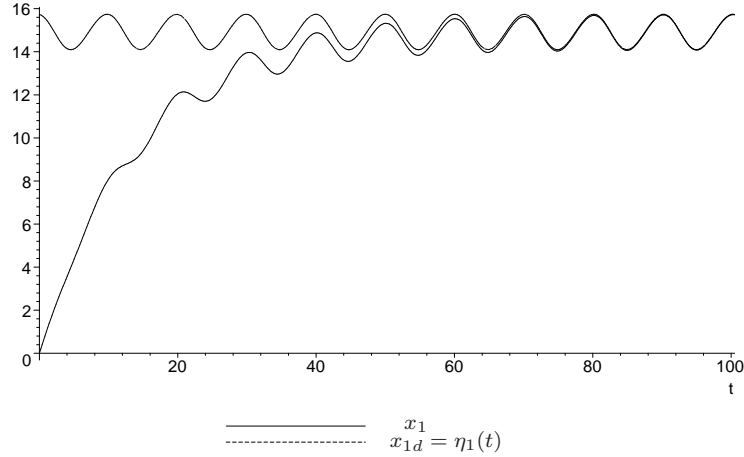


FIGURE 2. The input current $x_1(t)$ tracking the approximate reference $\eta_1(t)$.

why approximations η_n of order 2 and higher are not included Figure 1 is because they are visually indistinguishable among them and from η_d .

The dynamical behavior of system (24)-(25)-(26) subject to the continuous feed-forward control law (35), with current reference $\eta_n(t) = \eta_1(t) = g_0 + \bar{A}(\bar{\eta}_{1G})$, has been simulated with MAPLE. Initial conditions have been set to $x_1(0) = x_2(0) = 0$. Figures 2 and 3 show the input current x_1 and the output voltage x_2 approximately tracking, respectively, the internal reference $x_{1d} = \eta_1(t)$ and the output reference $y_d = x_{2d}(t)$, in an asymptotic fashion.

The robustness of the control approach in front of piecewise constant load disturbances is observed as follows: at $t = 15$ normalized time units (ntu), the output resistance R is assumed to undergo an additive perturbation of a 50% of the nominal value $R = R_N = 10\Omega$, thus growing up to $R = R_P = 15\Omega$. Assuming output

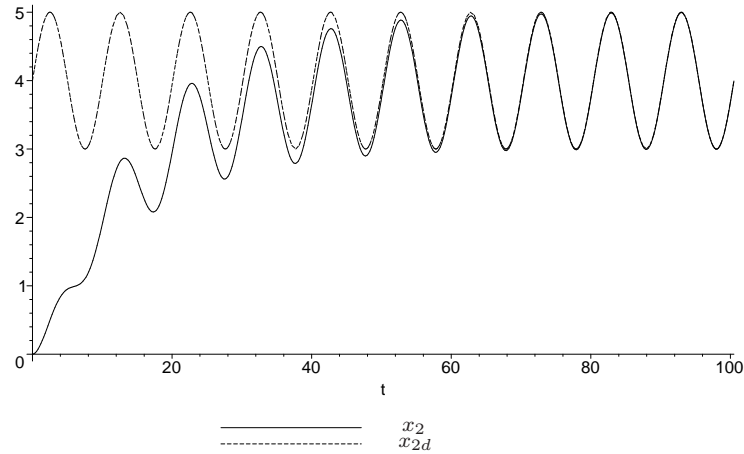


FIGURE 3. The output voltage $x_2(t)$ approximately tracking the reference $y_d = x_{2d}(t)$.

load measurement, a delay of 0.01 ntu between the appearance of the disturbance and the incorporation of the actual value of λ in the inductor current reference $x_{1d} = \eta_1(t)$ is considered. This value is in accordance with the sampling frequency of commercially available sensing devices.

According to Remark 9, the hypotheses stated in Theorem 5.3 have to be now verified $\forall \lambda \in \Lambda = [\lambda_-, \lambda_+] = [\lambda_P, \lambda_N] = [0.6030, 0.9045]$. Then, recalling (36), let us define the auxiliary functions:

$$\bar{B}(\lambda) = B\sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2}, \quad K(\lambda) = A\sqrt{\bar{B}^2(\lambda) + 3B^2} + \frac{B\bar{B}(\lambda)}{2},$$

where $\bar{B}_\pm = \bar{B}(\lambda_\pm)$, $K_\pm = K(\lambda_\pm)$, is adopted for simplicity.

Proposition 6.1. *Let $A > B > 0$.*

(i) *It is sufficient for the fulfillment of Assumption A in Λ that $A > \bar{B}_-$ and also that:*

$$A^2 + \frac{B^2}{2} > \frac{T}{2\lambda_-} + \sqrt{\frac{T}{4\pi\lambda_-} (4K_- - B\bar{B}_-)}. \quad (38)$$

(ii) *It is sufficient for the fulfillment of (34) in Λ that:*

$$\frac{T}{2} > \lambda_+(A + \bar{B}_+) \quad \wedge \quad A - B > \frac{A + \bar{B}_-}{A - \bar{B}_-}. \quad (39)$$

Proof. Item (i) is proved in [13]. The first inequality in (39) arises from the fact that

$$f_1(\lambda) = \sup \{ \dot{x}_{2d}(t) + \lambda x_{2d}(t) \} = \lambda A + B\sqrt{\omega^2 + \lambda^2}$$

is an increasing function for $\lambda > 0$. Finally, the second inequality in (39) stems from Chapter 4.6 in [24], after taking into account that $\bar{B}_- \geq \bar{B}_+$ and realizing that

$$f_2(\bar{B}) = \frac{A + \bar{B}}{A - \bar{B}}$$

is an increasing function for all \bar{B} . \square

The actual settings verify the fulfillment of $A > \bar{B}_-$ and also of (38) and (39). Finally, it has to be checked that $\bar{\eta}_{1G}$ is selected in the corresponding set M_L , for all $\lambda \in \Lambda$. Firstly, it follows from Lemma 4.2 and (37), respectively, that

$$L_a(\lambda) = ag_0(\lambda) - \frac{T}{2} = a\lambda \left(A^2 + \frac{B^2}{2} \right) - \frac{T}{2}$$

and

$$\|\bar{\eta}_{1G}(\lambda)\| = \frac{2AB\sqrt{4\omega^2Q^2\lambda^4 + (16 + \omega^2Q^2)\lambda^2 + 4\omega^2}}{4 + \lambda^2\omega^2Q^2}.$$

As $\|\bar{\eta}_{1G}(\lambda)\|$ is a decreasing function of λ for all

$$\lambda > \sqrt{\frac{64 + 4\omega^2Q^2 - 8\omega^4Q^2}{(\omega^2Q^2 - 16)}} = 0.05,$$

which encompasses Λ , it must happen that $L_{a=1}(\lambda_-) > \|\bar{\eta}_{1G}(\lambda_+)\|$, and this is indeed true in the present case.

Figure 4 depicts the input current x_1 tracking the command profile $x_{1d}(t)$, which has been updated at $t = 15.01$ ntu. Figure 5 portrays the output voltage reference $x_{2d}(t)$ and the output voltage state variable x_2 . Notice that dynamic compensation allows effectiveness of the tracking task to be recovered in no more than four periods.

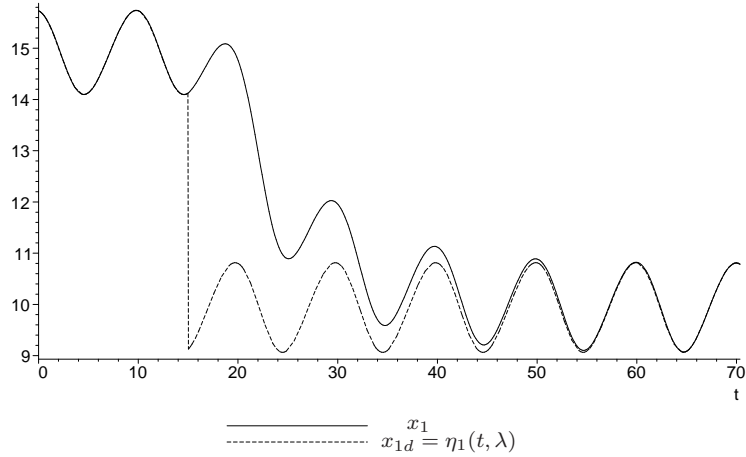


FIGURE 4. The input current x_1 tracking the approximate reference $\eta_1(t, \lambda)$, which is updated according to a load step change

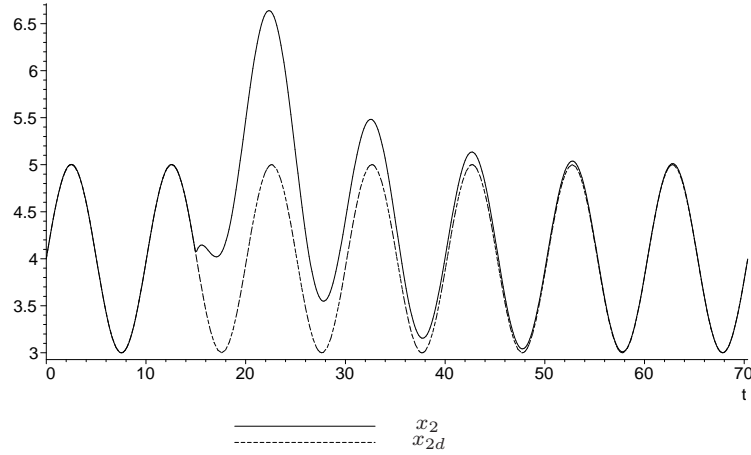


FIGURE 5. The output voltage x_2 accommodating a load step change by means of dynamic compensation

7. Appendix. Consider the set of linear, forced periodic systems of the form:

$$\dot{x} = A_n(t)x + b_n(t), \quad (40)$$

with $A_n(t)$, $b_n(t)$, continuous and T -periodic, for all $n \in \mathbb{N}$, and such that the sequences (A_n) , (b_n) , converge uniformly to a certain matrix $A(t)$ and to 0, respectively, i.e. $A_n(t) \rightarrow A(t)$, $b_n(t) \rightarrow 0$. For, let $\Phi_n(t)$, $\Phi(t)$ be, respectively, fundamental matrices of the homogeneous systems

$$\dot{x} = A_n(t)x, \quad (41)$$

$$\dot{x} = A(t)x, \quad (42)$$

satisfying $\Phi_n(0) = \Phi(0) = \mathbb{I}$.

Lemma 7.1. *The sequence (Φ_n) converges uniformly to Φ in $[0, T]$.*

Proof. Notice that

$$\dot{\Phi}_n(t) - \dot{\Phi}(t) = A_n(t)\Phi_n(t) - A(t)\Phi(t) = [A_n(t) - A(t)]\Phi(t) + A_n(t)[\Phi_n(t) - \Phi(t)].$$

Integrating between 0 and t , $t \in [0, T]$, yields

$$\Phi_n(t) - \Phi(t) = \int_0^t [A_n(\tau) - A(\tau)]\Phi(\tau)d\tau + \int_0^t A_n(\tau)[\Phi_n(\tau) - \Phi(\tau)]d\tau.$$

Taking any usual matrix norm $|\cdot|$,

$$|\Phi_n(t) - \Phi(t)| \leq \int_0^t |A_n(\tau) - A(\tau)| \cdot |\Phi(\tau)|d\tau + \int_0^t |A_n(\tau)| \cdot |\Phi_n(\tau) - \Phi(\tau)|d\tau.$$

Denote $\|\cdot\| := \sup_{t \in [0, T]} \{|\cdot|\}$. As (A_n) is uniformly convergent by hypothesis, it is also uniformly bounded. Hence, there exists $M > 0$ such that $\|A_n\| \leq M$, and

$$|\Phi_n(t) - \Phi(t)| \leq T\|A_n - A\| \cdot \|\Phi\| + M \int_0^t |\Phi_n(\tau) - \Phi(\tau)|d\tau.$$

Using now Gronwall's inequality,

$$|\Phi_n - \Phi| \leq T\|\Phi\| \cdot \|A_n - A\|e^{Mt}.$$

Therefore,

$$\|\Phi_n - \Phi\| \leq Te^{MT}\|\Phi\| \cdot \|A_n - A\| \rightarrow 0,$$

and $\Phi_n \rightarrow \Phi$ uniformly in $[0, T]$. \square

Theorem 7.2. *Assume that $x = 0$ is a hyperbolic equilibrium solution of (41), (42). Then:*

(i) *For every $n \in \mathbb{N}$, (40) has one and only one periodic solution x_n , which is T -periodic and hyperbolic.*

(ii) *The sequence of T -periodic solutions (x_n) of (40) converges uniformly to 0.*

Proof. On the one hand, (i) is a straightforward consequence of standard results on linear periodic systems [16]. On the other hand, it is known from [16] that,

$$x_n(t) = \int_0^T K_n(t, \tau)b_n(\tau)d\tau, \quad (43)$$

where $K_n(t, \tau)$ is the Green matrix function

$$K_n(t, \tau) = \begin{cases} \Phi_n(t) [\mathbb{I} - \Phi_n(T)]^{-1} \Phi_n^{-1}(\tau), & 0 \leq \tau \leq t \leq T, \\ \Phi_n(t+T) [\mathbb{I} - \Phi_n(T)]^{-1} \Phi_n^{-1}(\tau), & 0 \leq t \leq \tau \leq T, \end{cases}$$

The nonsingularity of $\Phi_n(t)$, $\Phi(t)$, $\forall t$, because of their fundamental matrices condition guarantees the existence and continuity of $\Phi_n^{-1}(t)$, $\Phi^{-1}(t)$, $\forall t$. Then, $\Phi_n^{-1} \rightarrow \Phi^{-1}$ uniformly in $[0, T]$ from Lemma 7.1. Moreover, the fact that $x = 0$ be the only T -periodic solution of (41), (42), entails that $\mu = 1$ is not a characteristic multiplier of both systems, i.e. an eigenvalue of $\Phi_n(T)$, $\Phi(T)$ [16]. Then, the existence of $[\mathbb{I} - \Phi_n(T)]^{-1}$, $[\mathbb{I} - \Phi(T)]^{-1}$ is ensured, while the uniform convergence $[\mathbb{I} - \Phi_n(T)]^{-1} \rightarrow [\mathbb{I} - \Phi(T)]^{-1}$ is newly achieved through Lemma 7.1. Finally, recalling the fundamental matrix elementary property

$$\Phi_n(t+T) = \Phi_n(t)\Phi_n(T), \quad \Phi(t+T) = \Phi(t)\Phi(T),$$

and using again Lemma 7.1, it is immediate that the sequence $(K_n(t, \tau))$ converges uniformly to the Green matrix function

$$K(t, \tau) = \begin{cases} \Phi(t) [\mathbb{I} - \Phi(T)]^{-1} \Phi^{-1}(\tau), & 0 \leq \tau \leq t \leq T, \\ \Phi(t+T) [\mathbb{I} - \Phi(T)]^{-1} \Phi^{-1}(\tau), & 0 \leq t \leq \tau \leq T. \end{cases}$$

Now, taking norms in (43) results in

$$\|x_n\| \leq T \|K_n\| \cdot \|b_n\|$$

and, for $n \rightarrow \infty$, it is

$$\lim_{n \rightarrow \infty} \|x_n\| \leq T \cdot \lim_{n \rightarrow \infty} \|K_n\| \cdot \lim_{n \rightarrow \infty} \|b_n\| = T \cdot \|K\| \cdot 0 = 0,$$

which proves statement (ii). \square

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