Stable inversion of Abel equations: application to tracking control in DC-DC nonminimum phase boost converters ⋆

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Abstract

Stable inversion plays a key role in the solution of the exact tracking control problem in nonminimum phase systems. However, the general methods developed so far for the computation of stable inverses require backwards time numeric integration of the internal dynamics equation, which yields high sensitivity to external disturbances and/or structured uncertainties. This article introduces an iterative technique that provides periodic, closed-form analytic expressions uniformly convergent to the exact periodic solution of a certain class of Abel ODE written in the normal form. The method is then applied to the output voltage tracking of periodic references in DC-DC boost power converters through a state feedback indirect control scheme. The procedure lies on a number of assumptions for which sufficient conditions involving system parameters and reference candidates are derived. It also allows to attenuate the effect of bounded, piecewise constant load disturbances using dynamic compensation. Simulation results validate the proposed algorithm.

Key words: Stable inversion; Abel equations; Tracking control; Switched power converters

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1 Introduction

Nonminimum phase output tracking is a challenging, theoretically sound control problem with a variety of applications that include controlling DC-DC boost and buck-boost converters [1–3]. Stable inversion of unstable internal dynamics in nonminimum phase plants plays a crucial role in developing the output tracking control algorithms [4,5]. Hence, the exact tracking of a known output reference $y = y_d(t)$ in nonminimum phase, time-invariant systems by means of stable inversion was addressed in [4]. Since then, further studies have completed the method by relaxing original hypotheses, extending it to discrete-time systems and introducing new approaches (see [5] and the references therein). However, these general methods require backwards time numeric integration for the computation of a stable inverse, which is the main reason of the well-known sensitivity of inversion-based exact-output tracking controllers to external perturbations and/or plant parameter uncertainties.

The solution of the exact tracking control problem of periodic references in nonminimum phase DC-DC nonlinear switched power converters requires stable inversion of the internal dynamics equation satisfied by the unstable variable, i.e. the inductor current, which takes the *normal form* of Abel ODE [1]. A variety of approaches for this purpose are available in the literature. In [2], the stable inverse is computed from the expression of the equilibrium current in the regulation case, just replacing the setpoint voltage reference by the actual time-varying one: this yields a severe trade-off between system parameters and command profile in order to keep the tracking error between acceptable bounds. A bounded reference for the inductor current is obtained in [3] as a solution of the unstable linearized internal dynamics, which reduces its effectiveness to a vicinity of the operating point. The method introduced in [6] exploits the differential flatness of the system to derive an iterative sequence of bounded approximations of the nonminimum phase variable reference; however, no convergence proof is provided. Finally, a uniformly convergent sequence of Galerkin approximations of the inductor current reference is proposed as a solution in [7]. However, this approach suffers of two major drawbacks. Firstly, only the first Galerkin approximation can be obtained in a closed-form and, therefore, used for dynamic compensation of the disturbances. Secondly, the performance of the control setup depends on a number of hypotheses which are difficult to verify. The last issues have been partially overcome in [8] with the introduction of a Banach's fixed-point theorem-based iterative technique that allows generating a uniformly convergent sequence of periodic functions. It is worth noting that these functions are analytically computable in the closed form. However, although the set of hypotheses is reduced with respect to [7], the verification problem still remains.

In this article, the procedure developed in [8] for obtaining a stable inverse of the Abel equation is improved. Now, sufficient, easily verifiable conditions are imposed on the output voltage reference profile and the system parameters in order to fulfill the required assumptions. Namely, the method solves the stable inversion problem in a class of Abel ODE providing a uniformly convergent, closed-form analytic iterative sequence of periodic approximations of its periodic solution that shows an explicit dependence on the system parameters. The proposed technique is then applied to nonminimum phase output voltage tracking in DC-DC boost power converters via indirect control. Furthermore, robustness to bounded, piecewise constant load disturbances may be achieved by dynamic compensation once the disturbed/unknown parameters are measured or estimated. This estimation can be accomplished, for instance, using Higher Order Sliding Mode (HOSM) observation and input reconstruction techniques developed for nonminimum phase systems in [9] or algebraic estimation [10]. In turn, the steady-state output voltage tracking error should be reduced at will using a sufficiently high order iteration for the current reference.

The structure of the paper is as follows. In Section 2 stable inversion of Abel differential equation is rigorously studied. The output voltage tracking in nonminimum phase DC-DC boost converters using the developed stable inversion technique is presented in Section 3. Section 4 contains the simulation study, which confirms the efficacy of the proposed control algorithm. Conclusions are presented in Section 5. In order to improve readability, proofs are concentrated in an Appendix.

2 Stable inversion of a class of Abel equations

Consider the affine single-input, single-output system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $f, g : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ are smooth vector fields and $h : D \subseteq \mathbb{R}^n \to R$ is a smooth scalar map.

Let the control problem be the exact tracking of the function $y_d(t)$ by the output y. Assume that (1) has relative degree $n-1$ on $D_0 \subseteq D$ and also that its internal dynamics equation can be written as (or transformed into) an Abel equation in the normal form [11]

$$\dot{\eta} = 1 - \frac{g(t)}{\eta}, \tag{2}$$

where $g(t) = g(y_d(t))$ and $\eta = \varphi(x)$ is to be selected in such a way that the mapping $T^\top(x) = (\varphi(x), h(x), L_f h(x), \dots, L_f^{n-2} h(x))$ is a diffeomorphism on D_0 [4]. Here $L_f^k h(x)$ denotes the k-th order Lie derivative of $h(x)$ along f.

Theorem 1 *[1] Let $g(t)$ be T-periodic, smooth and such that $g(t) > 0$, $\forall t \geq 0$. Then, (2) has one and only one T-periodic solution $\phi(t)$, which is positive and unstable.* $\square$

Let us obtain an iterative sequence of approximations of the periodic solution of (2). For, let $T \in \mathbb{R}^+$ and denote $\mathcal{C}^n_{per}([0,T]) = \{\eta \in \mathcal{C}^n([0,T]);\ \eta(0) = \eta(T)\}$ the subset of elements of $\mathcal{C}^n([0,T])$ that allow a continuous and T-periodic extension in $\mathbb{R}$, with $\mathcal{C}_{per}([0,T]) = \mathcal{C}^0_{per}([0,T])$. Recall that $(\mathcal{C}_{per}([0,T]), \|\cdot\|)$, where $\|\cdot\|$ is the uniform norm, is a Banach space with respect to the metric induced by $\|\cdot\|$.

Consider now the projection operator $P_0 : \mathcal{C}_{per}([0,T]) \to \mathbb{R}$, that extracts the mean value of periodic functions, namely, $P_0(\eta) = (1/T)\int_0^T \eta(t)dt$, and let $\bar{X}$ denote the subset of $\mathcal{C}_{per}([0,T])$ that contains the elements with zero mean value, i.e. $\bar{X} = \{\eta \in \mathcal{C}_{per}([0,T]);\ P_0(\eta) = 0\}$. Then, it is immediate that any $\eta \in \mathcal{C}_{per}([0,T])$ can be uniquely decomposed as $\eta = \eta_0 + \bar{\eta}$, with $\eta_0 = P_0(\eta)$ and $\bar{\eta} \in \bar{X}$. Finally, $\bar{X}$ being closed by integration, for all $\bar{\eta} \in \bar{X}$ there exists a unique element $\hat{\eta} \in \bar{X}$ such that $\dot{\hat{\eta}} = \bar{\eta}$.

Assumption A. Let $g(t)$ be positive, $C^\infty_{per}([0,T])$ and verify:

$$g_0 > \frac{T}{2} + \sqrt{2\|\hat{g}\|}.$$

Let us define

$$\alpha = 1 - \sqrt{\left(1 - \frac{T}{2g_0}\right)^2 - \frac{2\|\hat{g}\|}{g_0^2}}, \quad L(z) := zg_0 - \frac{T}{2}. \tag{3}$$

Theorem 2 *If Assumption A holds, then $\forall a \in (\alpha, 1)$ and $\forall L \in (L(\alpha), L(a)]$, there exists a closed, nonempty subset $M_L = \{\bar{\eta} \in \bar{X};\ \|\bar{\eta}\| \leq L\} \subset \mathcal{C}_{per}([0,T])$ such that the sequence $\{\phi_n\}$=$\{g_0 + \bar{\phi}_n\}$, obtained by means of the iterative procedure*

$$\bar{\phi}_{n+1} = \bar{A}(\bar{\phi}_n) = \frac{1}{g_0}\left[\hat{\phi}_n - \hat{g} - \frac{\bar{\phi}_n^2 - P_0\left(\bar{\phi}_n^2\right)}{2}\right], \tag{4}$$

with $\bar{\phi}_0 \in M_L$, converges uniformly to the T-periodic solution $\phi(t)$ of (2).

Next Corollary establishes that if both g and $\bar{\phi}_0$ have finite Fourier expansions, then the successive approximations ϕ_n also have finite Fourier expansions. Moreover, its coefficients depend explicitly on those of g and $\bar{\phi}_0$ and are analytically computable in the closed form. The proof requires tedious algebraic manipulation and has been omitted.

Corollary 3 *Let Assumption A hold, let*

$$g(t) = g_0 + \bar{g}(t) = g_0 + \sum_{k=1}^{r} A_k \cos k\omega t + B_k \sin k\omega t$$

and let also $\bar{\phi}_0 \in M_L$ *be*

$$\bar{\phi}_0(t) = \sum_{k=1}^{s} \alpha_{0k} \cos k\omega t + \beta_{0k} \sin k\omega t.$$

Then, the successive approximations $\phi_n = g_0 + \bar{\phi}_n$ *obtained from (4) are such that, for all* $n \geq 1$, ϕ_n *has* $m = 2^{n-1} \cdot \max\{r, 2s\}$ *harmonics and*

$$\phi_{n+1}(t) = g_0 + \sum_{k=1}^{2m} \alpha_{n+1,k} \cos k\omega t + \beta_{n+1,k} \sin k\omega t,$$

where the coefficients follow the recursive assignment

$$\begin{aligned}
\alpha_{n+1,k} =& \frac{B_k - \beta_{nk}}{k g_0 w} - \frac{1}{2g_0} \sum_{\substack{i-j=k \\ 1\leq i,j\leq m}} (\alpha_{ni}\alpha_{nj} + \beta_{ni}\beta_{nj}) + \\
&+ \frac{1}{4g_0} \sum_{\substack{i+j=k \\ 1\leq i,j\leq m}} (\beta_{ni}\beta_{nj} - \alpha_{ni}\alpha_{nj}), \\
\beta_{n+1,k} =& \frac{\alpha_{nk} - A_k}{k g_0 w} + \frac{1}{2g_0} \sum_{\substack{i-j=k \\ 1\leq i,j\leq m}} (\alpha_{ni}\beta_{nj} - \alpha_{nj}\beta_{ni}) + \\
&- \frac{1}{4g_0} \sum_{\substack{i+j=k \\ 1\leq i,j\leq m}} (\alpha_{ni}\beta_{nj} + \alpha_{nj}\beta_{ni}).
\end{aligned}$$

Remark 4 *The quality of the approximations provided by Theorem 2 depends on the contractive constant and the distance between the initial condition* $\phi_0 = g_0 + \bar{\phi}_0$ *and the periodic solution* $\phi = g_0 + \bar{\phi}$*, namely [12]:*

$$\|\phi_n - \phi\| = \|\bar{\phi}_n - \bar{\phi}\| \leq a^n \|\bar{\phi}_0 - \bar{\phi}\| = a^n \|\phi_0 - \phi\|, \quad \forall n \geq 0.$$

Remark 5 *Notice that the recurrence (4) coincides with the one derived in [8], the only apparent difference being in Assumption A. However, in the latter approach the problem is posed and solved in* L_2 *and uses the* L_2 *norm, which hides a-priori information about uniform bounds on* ϕ_n*. As an example, notice that with the procedure arising from Theorem 2 it is ensured that* $\phi_n > 0$*,* $\forall n \geq 0$ *(see relation B.1 in Appendix B), which is essential for its use in the control scheme developed in next section.*

3 Approximate output tracking control in nonminimum phase boost converters

The state-space averaged model of the boost DC-to-DC switched power converter [13] may be written in dimensionless variables as [1]:

$$\dot{x}_1 = 1 - u x_2, \tag{5}$$
$$\dot{x}_2 = -\lambda x_2 + u x_1, \tag{6}$$

where

$$x_1 = \frac{1}{V_{\mathrm{g}}}\sqrt{\frac{L}{C}}\, i_{\mathrm{L}}, \quad x_2 = \frac{v_{\mathrm{C}}}{V_{\mathrm{g}}}, \quad t = \frac{\tau}{\sqrt{LC}}, \quad \lambda = \frac{1}{R}\sqrt{\frac{L}{C}}.$$

Notice that the inductor current i_{L} and the capacitor voltage v_{C} are proportional to the dimensionless state variables x_1 and x_2, respectively, while $u : [0, +\infty) \longrightarrow (0, 1)$. Recall also that the control action in the physical converter is actually carried out by means of a switch; hence, $u(t)$ is implemented through a PWM signal. The constant voltage source V_g, the inductance L and the capacitance C are considered known parameters, while perturbations may affect the load resistance R.

Let the control objective be the tracking of a smooth, T-periodic reference $x_{2d}(t)$ by the state variable x_2 such that Assumption A is fulfilled. It is well known that $y = x_2$ is a nonminimum phase output with relative degree 1 in $D_0 = \{x \in \mathbb{R}^2;\ x_1 \neq 0\}$. Therefore, after selecting $\eta = x_1$, the standard stable inversion procedure previously sketched yields the internal dynamics equation

$$\dot{x}_1 = 1 - \frac{x_{2d}(t)\,[\dot{x}_{2d}(t) + \lambda x_{2d}(t)]}{x_1}. \tag{7}$$

It is worth noting that equation (7) falls into a format of Abel ODE (2) with the assignment $g(t) = x_{2d}(t)\,[\dot{x}_{2d}(t) + \lambda x_{2d}(t)]$. Hence, Theorem 1 ensures that, for $g(t) > 0$, (7) has a positive, T-periodic, unstable solution $\phi(t)$.

Consider that Assumption A is fulfilled and let system (5)-(6) undergo the indirect state feedback control action

$$u = \frac{1 - \dot{\phi}_n + \gamma(x_1 - \phi_n)}{x_2}, \quad \gamma \in \mathbb{R}^+, \tag{8}$$

where ϕ_n stands for an approximation of the periodic solution of (7) obtained through Theorem 2. It is then straightforward from (5) that (8) forces a steady-state in which x_1 tracks the reference signal $\phi_n(t)$. Hence, (6)-(8) yield the dynamics of x_2 for $x_1 = \phi_n(t)$, namely,

$$x_{2n}\,(\dot{x}_{2n} + \lambda x_{2n}) = \phi_n(1 - \dot{\phi}_n). \tag{9}$$

Assumption B. Assumption A holds and $\bar{\phi}_0$ is selected in $M_L \supset M_D = \{\bar{\eta} \in M_L;\ \|\dot{\bar{\eta}}\| < D\}$, with

$$L < \frac{g_0 - \|\bar{g}\|}{2} \quad \text{and} \quad \frac{\|\bar{g}\| + L}{g_0 - L} \leq D < 1. \tag{10}$$

Remark 6 *The existence of such a positive D is indeed ensured. Notice that (10) yields $(\|\bar{g}\| + L)(g_0 - L)^{-1} < 1$, while Lemma 11 and Theorem 2 entail*

$$g_0 - L \geq g_0 - L(a) = (1-a)g_0 + \frac{T}{2} > 0. \tag{11}$$

Next result characterizes the output responses $x_{2n}(t)$.

Theorem 7 *Let Assumption B hold. Then, for every $n \geq 1$, (9) has one and only one T-periodic solution $x_{2n}(t)$ in $\mathbb{R}^+$, which is asymptotically stable. Moreover, the sequence $\{x_{2n}\}$ converges uniformly to x_{2d}.*

The applicability of the control procedure is restricted to the fulfillment of an obviously necessary condition: the steady-state control law $u = u(t, \phi_n, x_{2n})$ must lie in $(0,1)$, $\forall t \geq 0$. As $u \in (0,1)$, its isolation in (5)-(6) yields that the steady-state trajectories that do not saturate the control action are those satisfying:

$$0 < \frac{1 - \dot{\phi}_n}{x_{2n}} < 1, \quad \text{or} \quad 0 < \frac{\dot{x}_{2n} + \lambda x_{2n}}{\phi_n} < 1. \tag{12}$$

It is immediate from the positive character of ϕ_n (established in relation (B.1) of Appendix B) and x_{2n} (derived from Theorem 7) that, under Assumption B, the unsaturated region (12) is equivalently defined by

$$0 < 1 - \dot{\phi}_n < x_{2n}, \quad \text{or} \quad 0 < \dot{x}_{2n} + \lambda x_{2n} < \phi_n. \tag{13}$$

Hence, the key demand of unsaturation of the control action in the steady-state is claimed straightforward:

Assumption C. Assumption B holds and the steady-state of (5)-(6) under the control law (8) remains in the unsaturated region defined by (13), for all $n \in \mathbb{N}$.

Restrictions involving x_{2d} and the system parameters for the fulfillment of Assumptions B and C are established below.

Proposition 8 *Let Assumption A hold. Then, it is necessary for the fulfillment of Assumption B that*

$$\frac{g_0 + \|\bar{g}\| - T}{2} < \sqrt{\left(g_0 - \frac{T}{2}\right)^2 - 2\|\hat{g}\|}. \tag{14}$$

Proposition 9 *Let Assumption B hold. Then, it is sufficient for the fulfillment of Assumption C that:*

$$g_0 - L > \lambda \frac{(1+D)^2}{1-D} \tag{15}$$

Remark 10 *(i) Unsaturation of the control action in the steady state is also assumed in [8] from a certain $n_1 \in \mathbb{N}$. However, no discussion about how to find n_1, or at least, how to prove that it actually exists, was provided. Notice that, with the present approach, the fulfillment of (15) ensures steady-state unsaturation for any input current reference ϕ_n obtained from Theorem 2.*

(ii) It is also worth pointing out that the fulfillment of Assumption C assures unsaturation of the control action (8) during transients if the initial conditions $x_1(0)$, $x_2(0)$ are set close enough to those of their respective reference profiles. Otherwise, the performance can not be guaranteed.

Now, all assumptions are easily verifiable and the stable inverse algorithm developed for the generic Abel ODE (2) can be applied to equation (7) for the tracking of time-varying, periodic output voltage references in the DC-DC boost converter (5)-(6).

The proposed procedure can also be used to tackle the robust tracking control problem under sudden load changes. This is because, from Corollary 3, the current references ϕ_n are available in the closed form and show explicit dependence on λ, which is proportional to the output load. Then, it is possible to dynamically compensate the effect of piecewise constant load disturbances belonging to a known compact set Λ through a real-time updating of the selected current reference $\phi_n(t) = \phi_n(t, \lambda)$ according to the instantaneous variation of λ, which is assumed to be estimated (using, for example, HOSM observation [9] or algebraic estimation [10]) or measured. Hence, success is subject to the fulfillment of Assumption C for all $\lambda \in \Lambda$ and, according to Remark 10.ii, to the unsaturation of the control action during the transients that occur because of load jumps. For specific situations, such as the case of DC periodic output voltage command profiles, it is possible to obtain sufficient conditions over the system parameters and reference candidates that guarantee Assumption A and (14) to be verified not only for a fixed λ, but also for all $\lambda \in \Lambda = [\lambda_-, \lambda_+] \subset \mathbb{R}^+$ (see [14]).

4 Simulation results

The technique has been tested on a boost converter with $V_g = 50V$, $L = 0.018H$, $C = 0.00022F$ and $R_N = 10\Omega$. The output voltage reference profile has been set to:

$$v_C(\tau) = 210 + 50\sin(2\pi\nu\tau),$$

with $\nu = 50Hz$. At a certain time instant, the load resistance is assumed to undergo an additive perturbation of a 50% of the nominal value R_N, thus growing up to $R_P = 15\Omega$. The corresponding values in normalized variables are $\Lambda = [\lambda_-, \lambda_+] = [0.6030, 0.9045]$ and $x_{2d}(t) = 4.2 + \sin\omega t$, where $\omega = 0.6252$.

With these settings, Assumptions A, B and C have been numerically verified $\forall\lambda \in \Lambda$. Indeed, Assumption A is fulfilled because

$$\min_{\lambda\in\Lambda}\left\{g_0 - \frac{T}{2} - \sqrt{2\|\hat{g}\|}\right\} = 1.62 > 0.$$

Regarding Assumption B, it is worth remarking that the contractive constant a is to be selected in $(0.5359, 1)$; once a is fixed, the possible radius L of M_L belong to $(L(\alpha), L(a)]$, with $L(\alpha) \leq 0.8371$ and $L(a) \geq 10.9288a - 5.0252$. Let $a = 0.9$ and $L = 1$. This entails $D \in (0.724, 1]$, so we choose $D = 0.8 < 1 = L$. Hence, as

$$\min_{\lambda\in\Lambda}\left\{\frac{g_0 - \|\bar{g}\|}{2} - L\right\} = 1.40 > 0, \quad \min_{\lambda\in\Lambda}\left\{D - \frac{\|\bar{g}\| + L}{g_0 - L}\right\} = 0.08 > 0,$$

(10) hold $\forall\lambda \in \Lambda$. Finally,

$$\min_{\lambda\in\Lambda}\left\{g_0 - L - \lambda\frac{(1+D)^2}{1-D}\right\} = 0.17 > 0,$$

which ensures the fulfillment of (15) and, therefore, of Assumption C $\forall\lambda \in \Lambda$.

Furthermore, according to Remark 4, the iterative procedure of Theorem 2 provides better convergence rates with initial conditions closer to $\bar{\phi}$. Thence, let us pick $\bar{\phi}_0 = \bar{\phi}_{1G}$, with $\bar{\phi}_{1G}$ denoting the periodic component of the first Galerkin approximation of $\phi(t)$, namely [15]:

$$\bar{\phi}_{1G}(t) = \frac{4AB\omega(1+\lambda^2 Q)}{4+\lambda^2\omega^2 Q^2}\cos\omega t + \frac{2\lambda AB(4-\omega^2 Q)}{4+\lambda^2\omega^2 Q^2}\sin\omega t,$$

where $Q = 2A^2 + B^2$ and A, B are the offset and amplitude of x_{2d}, i.e. $x_{2d}(t) = A + B\sin\omega t$. The fact that $\bar{\phi}_{1G}(t)$ has a λ-dependent closed-form analytic expression maintains the possibility of achieving robustness by means of dynamic compensation. Finally, $\|\bar{\phi}_{1G}\| \leq 0.8255 < L$, $\|\dot{\bar{\phi}}_{1G}\| \leq 0.5161 < D$, $\forall\lambda \in \Lambda$.

The dynamical behavior of system (5), (6) subject to the continuous state feedback control law (8), with current reference $x_{1d} = \phi_1 = g_0 + \bar{A}(\bar{\phi}_{1G})$ and $\gamma = 0.5$, has been simulated with MAPLE. The sensitivity to initial conditions is checked setting $x_1(0) = 15 \neq \phi_1(0)$, $x_2(0) = 1 \neq x_{2d}(0)$. In

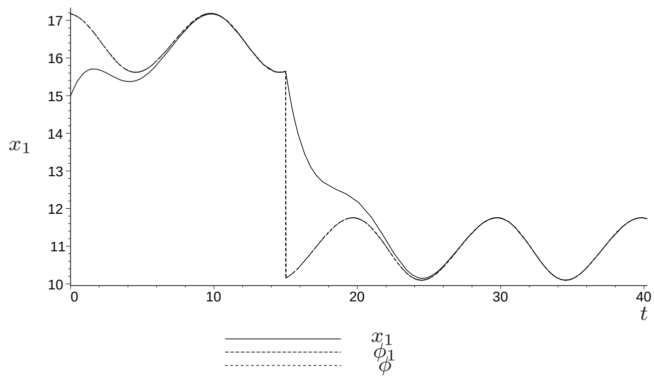

Fig. 1. The input current x_1 tracking ϕ_1 under dynamic compensation of a load disturbance occurring at $t = 15$ ntu.

turn, the robustness of the control approach in front of piecewise constant load disturbances is tested as follows: at $t = 15$ normalized time units (ntu), the output resistance undergoes the above described step change; assuming output load measurement, a delay of 0.01 ntu between the appearance of the disturbance and the incorporation of the actual value of λ in the inductor current reference ϕ_1 is considered.

Figure 1 depicts the input current x_1 tracking the command profile $x_{1d}(t)$. The plot includes the exact solution $\phi(t)$ of equation (2), which appears to be indistinguishable from its approximation $\phi_1(t)$. Figure 2 depicts the output voltage reference $x_{2d}(t)$ and the output voltage state variable x_2. Notice that both state variables exhibit asymptotic tendency to their respective references, while dynamic compensation allows effectiveness of the tracking task to be recovered immediately after the disturbance occurs. Figure 3 shows that the control action (8) does not saturate during the entire process.

The main drawbacks of the method are: (a) the peaking effect undergone by the output voltage and the control action, which is inherent to dynamic compensation in indirect control schemes, and (b) the performance decay associated to mismatches between the supposed or estimated value of λ and its

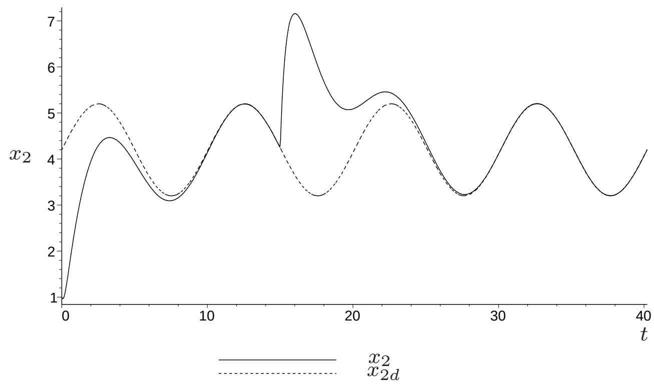

Fig. 2. The output voltage x_2 tracking x_{2d} under dynamic compensation of a load disturbance occurring at $t = 15$ ntu.

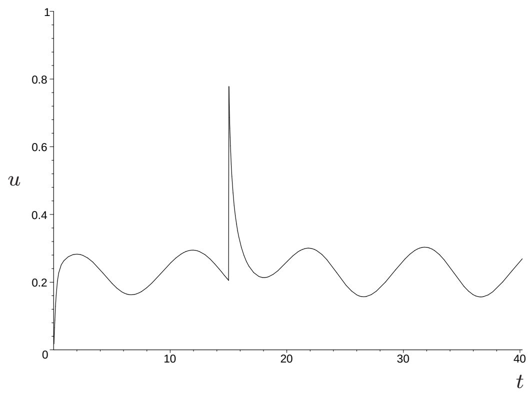

Fig. 3. The control action u accommodating a load disturbance occurring at $t = 15$ ntu.

actual value, because this implies that x_1 would be following an erroneous reference ϕ_n and, consequently, x_2 would not track the expected output reference but a different signal. Indeed, it is immediate from the proof of Theorem 7 that the internal dynamics equation (9) has a positive, T-periodic, asymptotically stable solution whenever $G_n = \phi_n(1 - \dot{\phi}_n)$ is positive and T-periodic.

5 Conclusions

This article provides a stable inversion method for a class of Abel equations in case of DC periodic tracking. The procedure allows to find periodic, closed-form analytic approximations uniformly convergent to the exact periodic solution of the inverse problem. The method is applied to the output voltage tracking control of nonminimum phase DC-DC boost converters: a state feedback indirect control scheme that uses the approximate references of the nonminimum phase variable yields asymptotic tracking of the output voltage reference target. Furthermore, bounded piecewise constant load disturbances belonging to an a priori known compact set can be dynamically compensated. Sufficient conditions for the fulfillment of the technical assumptions and physical restrictions arising in the procedure are provided. The efficacy of the proposed algorithm is verified via computer simulation.

Further research is devoted to the extension of the stable inversion method to the tracking of a broader class of functions that includes periodic functions with non-definite sign and non-periodic functions. The experimental implementation of the control technique for a boost converter is under study as well.

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A Proof of Theorem 2

Follow Section 4 in [16] and replace Lemmas 5 and 6 therein by Lemmas 11 and 12, respectively, which are stated below. The proof of Lemma 11 is straightforward.

Lemma 11 *If Assumption A holds, then* $0 < T(2g_0)^{-1} \leq \alpha$. *Moreover,* $\forall a \in (\alpha, 1)$, $0 \leq L(\alpha) < L(a)$. □

Lemma 12 *Let* $\bar{x}, \hat{x} \in \bar{X}$ *be such that* $\dot{\hat{x}} = \bar{x}$. *Then,* $\|\hat{x}\| \leq (T/2)\|\bar{x}\|$ *for all* $\bar{x} \in \bar{X}$.

Proof. $\hat{x}$ being continuous and with zero mean value in $[0, T]$, it is straightforward that there exists, at least, $t_0 \in [0, T]$ such that $\hat{x}(t_0) = 0$. Therefore, assuming that both $\bar{x}(t)$ and $\hat{x}(t)$ are naturally extended to $[t_0, t_0 + T]$, one has that

$$\int_{t_0}^{t_0+T} \hat{x}(t)dt = 0.$$

Moreover, the T-periodicity of $\hat{x}(t)$ ensures the existence of $c \in [t_0, t_0 + T]$ such that $\|\hat{x}\| = |\hat{x}(c)|$; hence,

$$\|\hat{x}\| = |\hat{x}(c)| = \left| \int_{t_0}^{c} \bar{x}(t)dt \right|.$$

Furthermore, it is also straightforward that

$$\int_0^T \bar{x}(t)dt = 0 \Longrightarrow \int_{t_0}^{t_0+T} \bar{x}(t)dt = 0;$$

thus,

$$\int_{t_0}^{c} \bar{x}(t)dt = \int_{t_0+T}^{t_0} \bar{x}(t)dt + \int_{t_0}^{c} \bar{x}(t)dt = \int_{t_0+T}^{c} \bar{x}(t)dt,$$

this yielding

$$\|\hat{x}\| = \left|\int_{t_0}^{c} \bar{x}(t)dt\right| \leq (c - t_0)\|\bar{x}\|, \quad \text{and}$$
$$\|\hat{x}\| = \left|\int_{t_0+T}^{c} \bar{x}(t)dt\right| \leq (t_0 + T - c)\|\bar{x}\|.$$

The result follows adding both inequalities . □

B Proof of Theorem 7

Lemma 13 *Let Assumption B hold. Then, the sequence* $\{G_n\} = \{\phi_n(1-\dot{\phi}_n)\}$ *verifies that* $G_n(t) > 0$, $\forall n \geq 0$.

Proof. On the one hand, let $\bar{\phi}_0 \in M_L$; then, by Theorem 2, $\bar{\phi}_n \in M_L$, $\forall n \geq 0$. Consequently, $\forall n \geq 0$, (11) entails

$$\phi_n = g_0 + \bar{\phi}_n \geq g_0 - \|\bar{\phi}_n\| \geq g_0 - L > 0. \tag{B.1}$$

On the other hand one has $\bar{\phi}_0 \in M_L$ and $\|\dot{\phi}_0\| = \|\dot{\bar{\phi}}_0\| \leq D < 1$ by hypothesis. In accordance to the induction principle, assume that $\|\dot{\phi}_n\| = \|\dot{\bar{\phi}}_n\| \leq D < 1$. Now, using (4) and (10),

$$\|\dot{\phi}_{n+1}\| \leq \frac{1}{g_0}\left(g_0 D - LD + LD\right) \leq D < 1.$$

Therefore, $1 - \dot{\phi}_n \geq 1 - \|\dot{\phi}_n\| \geq 1 - D > 0$ and, finally, $G_n = \phi_n(1 - \dot{\phi}_n) > 0$, $\forall n \geq 0$. □

Equation (9) can be written as

$$x_{2n}\left(\dot{x}_{2n} + \lambda x_{2n}\right) = G_n \tag{B.2}$$

The change of variables $y = 1/2x_{2n}^2$ linearizes (B.2), which allows to obtain its general solution:

$$x_{2n}(t) = \pm\left[x_{2n}^2(0)e^{-2\lambda t} + 2e^{-2\lambda t}\int_0^t e^{2\lambda s}G_n(s)ds\right]^{\frac{1}{2}}. \tag{B.3}$$

Lemma 13 guarantees that, under Assumption B, $G_n > 0$, for all $n \geq 1$. As it is also $\lambda > 0$, all the elements inside the square root in (B.3) are positive and

x_{2n} is well defined: it is $x_{2n} > 0$ for $x_{2n}(0) > 0$ and $x_{2n} < 0$ for $x_{2n}(0) < 0$. Notice that $x_{2n}(0) = 0$ does not define a solution because it requires $G_n(0) = 0$, which is impossible by hypothesis. Furthermore, the periodic solution (abusively denoted) x_{2n} may be found demanding $x_{2n}(0) = x_{2n}(T)$, this yielding two periodic solutions, one in $\mathbb{R}^+$ and another one in $\mathbb{R}^-$. The positive solution is:

$$x_{2n}(t) = \left[\frac{2e^{-2\lambda t}}{e^{2\lambda T} - 1} \int_0^T e^{2\lambda s} G_n(s) ds + 2e^{-2\lambda t} \int_0^t e^{2\lambda s} G_n(s) ds \right]^{\frac{1}{2}},$$

and its asymptotic stability follows immediately from the fact that $\lambda > 0$.

Let us now prove the uniform convergence. For, consider the change of variables $z_n = 1/2(x_{2n}^2 - x_{2d}^2)$. Therefore, (B.2) becomes

$$\dot{z}_n + 2\lambda z_n = F\phi_n(t), \tag{B.4}$$

where

$$F\phi_n(t) = G_n - x_{2d} \left(\dot{x}_{2d} + \lambda x_{2d} \right) = \phi_n(1 - \dot{\phi}_n) - \phi(1 - \dot{\phi}),$$

ϕ being the positive, T-periodic solution of (7). It is then immediate that (B.4) has a T-periodic, asymptotically stable solution (abusively denoted) z_n which may be found by a procedure equivalent to the one followed to find x_{2n}:

$$z_n(t) = \frac{e^{-2\lambda t}}{e^{2\lambda T} - 1} \int_0^T e^{2\lambda s} F\phi_n(s) ds + e^{-2\lambda t} \int_0^t e^{2\lambda s} F\phi_n(s) ds. \tag{B.5}$$

Notice also that

$$\begin{aligned}
&\int_0^t e^{2\lambda s} F\phi_n(s) ds = \\
&= \int_0^t e^{2\lambda s} \left(\phi_n(s) - \phi(s) \right) ds - \frac{1}{2} \int_0^t e^{2\lambda s} \frac{d}{ds} \left(\phi_n^2(s) - \phi^2(s) \right) ds = \\
&= \int_0^t e^{2\lambda s} \left[\phi_n(s) - \phi(s) \right] ds - \frac{e^{2\lambda t}}{2} \left[\phi_n^2(t) - \phi^2(t) \right] + \frac{1}{2} \left[\phi_n^2(0) - \phi^2(0) \right] + \\
&+ \lambda \int_0^t e^{2\lambda s} \left[\phi_n^2(s) - \phi^2(s) \right] ds.
\end{aligned}$$

Then, taking uniform norms in (B.5),

$$\begin{aligned}
\|z_n\| \leq & NTe^{2\lambda T} \|\phi_n - \phi\| + \frac{N}{2} \left(1 + e^{2\lambda T} \right) \left(\|\phi_n\| + \|\phi\| \right) \|\phi_n - \phi\| + \\
& + \lambda NTe^{2\lambda T} \left(\|\phi_n\| + \|\phi\| \right) \|\phi_n - \phi\|,
\end{aligned}$$

where

$$N = \frac{e^{2\lambda T}}{e^{2\lambda T} - 1}.$$

On the one hand, $\phi_n \to \phi$ uniformly and ϕ is continuous and periodic, thus bounded. On the other hand, these two facts entail the uniform boundedness of $\{\phi_n\}$. Consequently, taking limits for $n \to \infty$ yields $\{z_n\} \to 0$ uniformly and, being $x_{2d} > 0$ and $x_{2n} > 0$, for all $n \geq 1$, then $x_{2n} \to x_{2d}$ uniformly. □

C Proof of Proposition 8

Theorem 2 indicates that the radius L of M_L must satisfy $L(\alpha) < L$. Hence, $L_- < 2^{-1}(g_0 - \|\bar{g}\|)$ is a necessary condition for Assumption B. Relation (14) follows re-writing the later inequality using (3). □

D Proof of Proposition 9

Assumption B ensures that $1 - \dot{\phi}_n > 0$, $\forall n \in \mathbb{N}$; thus, $x_{2n} > 1 - \dot{\phi}_n$ if

$$\inf_{t\in[0,T]} \{x_{2n}\} > \|1 - \dot{\phi}_n\|. \tag{D.1}$$

Recall that x_{2n} is $\mathcal{C}^1_{per}([0,T])$ and satisfies (9). Hence, there exists $t_m \in [0,T]$, with $\dot{x}_{2n}(t_m) = 0$, where $x_{2n}(t)$ attains minimum value, this yielding

$$\phi_n(t_m)\left[1 - \dot{\phi}_n(t_m)\right] = \lambda x_{2n}^2(t_m).$$

Thus,

$$x_{2n}(t_m) = \sqrt{\frac{1}{\lambda}\phi_n(t_m)\left[1 - \dot{\phi}_n(t_m)\right]},$$

which means that (D.1) is guaranteed by

$$\inf_{t\in[0,T]} \left\{\phi_n(t)\left[1 - \dot{\phi}_n(t)\right]\right\} > \lambda \|\left(1 - \dot{\phi}_n\right)\|^2.$$

Finally, (B.1) and Assumption B entail

$$\inf_{t\in[0,T]} \left\{\phi_n(t)\left[1 - \dot{\phi}_n(t)\right]\right\} \geq (g_0 - L)(1 - D),$$
$$\lambda\|1 - \dot{\phi}_n\|^2 \leq \lambda(1 + D)^2.$$

Then, (15) is a sufficient condition for (D.1) and, therefore, of (13). □