An Osgood criterion for stochastic differential equations with an additive noise

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Content

1. Introduction and Osgood test
2. Comparison Theorem
3. Blow-up for a class of integral equations
4. Stochastic differential equations driven by an additive noise
5. Comparison with the Feller test for explosions
Consider

\[
\begin{aligned}
\frac{dv(t)}{dt} &= b(v(t)), \quad t > 0, \\
v(0) &= a.
\end{aligned}
\]
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\begin{align*}
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v(0) &= a.
\end{align*}
\]

We define the explosion time as

\[
T = \sup\{ t > 0 : |v(t)| < \infty \}.
\]
Osgood criterion

Consider

\[ \begin{align*}
\frac{dv(t)}{dt} &= b(v(t)), \quad t > 0, \\
v(0) &= a.
\end{align*} \]

(H1) \( a \in \mathbb{R} \) and \( b : \mathbb{R} \to \mathbb{R}_+ \) is a positive, non-decreasing and locally Lipschitz function.
Osgood criterion

Consider

\[
\begin{cases}
\frac{dv(t)}{dt} = b(v(t)), & t > 0, \\
v(0) = a.
\end{cases}
\]

(H1) \( a \in \mathbb{R} \) and \( b : \mathbb{R} \rightarrow \mathbb{R}_+ \) is a positive, non-decreasing and locally Lipschitz function.

Remark

Under (H1), there exists a maximal interval on which above equation has a unique solution.
Osgood criterion

Consider

\[
\begin{aligned}
\frac{dv(t)}{dt} &= b(v(t)), \quad t > 0, \\
v(0) &= a.
\end{aligned}
\]

\((H1)\) \(a \in \mathbb{R}\) and \(b : \mathbb{R} \to \mathbb{R}_+\) is a positive, non-decreasing and locally Lipschitz function.

Observe that

\[
\frac{v'(t)}{b(v(t))} = 1 \Rightarrow \int_0^t \frac{v'(s)}{b(v(s))} ds = \int_0^t 1 ds = t.
\]
Osgood criterion

Consider

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\begin{cases}
\frac{dv(t)}{dt} = b(v(t)), & t > 0, \\
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\]

\textbf{(H1)} \ a \in \mathbb{R} \text{ and } b : \mathbb{R} \to \mathbb{R}_+ \text{ is a positive, non-decreasing and locally Lipschitz function.}

Observe that

\[
\frac{v'(t)}{b(v(t))} = 1 \Rightarrow \int_0^t \frac{v'(s)}{b(v(s))} ds = \int_0^t 1 ds = t.
\]

Hence

\[
\int_a^{v(t)} \frac{dy}{b(y)} = \int_{v(0)}^{v(t)} \frac{dy}{b(y)} = t.
\]
Osgood criterion

\[ \int_a^{v(t)} \frac{dy}{b(y)} = t. \]

Define

\[ B(x) = \int_a^x \frac{dy}{b(y)}, \quad x \geq a. \]

Therefore \( B(v(t)) = t. \) Thus

\[ v(t) = B^{-1}(t), \quad 0 < t < B(\infty). \]
Osgood criterion

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Therefore \( B(v(t)) = t. \) Thus

\[ v(t) = B^{-1}(t), \quad 0 < t < B(\infty). \]

In this case the explosion time is

\[ B(\infty) = \int_a^\infty \frac{ds}{b(s)}. \]
A basic example

Consider the following non-linear ordinary differential equation

\[
\begin{align*}
\frac{dv(t)}{dt} &= (v(t))^2, & t > 0, \\
v(0) &= a.
\end{align*}
\]
Consider the following non-linear ordinary differential equation

\[
\begin{aligned}
\frac{dv(t)}{dt} &= (v(t))^2, \quad t > 0, \\
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\end{aligned}
\]

For \( a > 0 \), there is a unique solution \( v \), in the interval \( 0 < t < 1/a \):

\[
v(t) = \frac{1}{1/a - t}.
\]
A basic example

Consider the following non-linear ordinary differential equation

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For \( a > 0 \), there is a unique solution \( v \), in the interval \( 0 < t < 1/a \):

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The number \( T = 1/a \) is the explotion time.
A basic example

Consider the following non-linear ordinary differential equation

\[ \begin{aligned}
\frac{dv(t)}{dt} &= (v(t))^2, \quad t > 0, \\
v(0) &= a.
\end{aligned} \]

For \( a < 0 \), there is a unique solution \( v \), in the interval \( 0 < t < 1/a \):

\[ v(t) = -\frac{1}{t - 1/a}. \]

In this case the explosion time is \( T = \infty \).
A basic example

Consider the following non-linear ordinary differential equation

\[
\begin{align*}
\frac{dv(t)}{dt} &= (v(t))^2, \quad t > 0, \\
v(0) &= a.
\end{align*}
\]

Remark

Note that

\[
\int_a^\infty \frac{dx}{x^2}
\]

is finite if and only if \( a > 0 \).
Suppose that $g$ is a measurable function such that

$$\limsup_{t \to \infty} \left( \inf_{0 \leq h \leq 1} g(t + h) \right) = \infty,$$

and $b$ is a positive and nondecreasing function. The solution of the integral equation

$$X_t = a + \int_0^t b(X_s) ds + g(t), \quad t \geq 0,$$
Main result

Suppose that $g$ is a measurable function such that

$$\limsup_{t \to \infty} \left( \inf_{0 \leq h \leq 1} g(t + h) \right) = \infty,$$

and $b$ is a positive and nondecreasing function. The solution of the integral equation

$$X_t = a + \int_0^t b(X_s)ds + g(t), \quad t \geq 0,$$

explodes in finite time if and only if $\int_\infty^\infty \frac{ds}{b(s)} < \infty$.

As an example, we see that $g$ can represent the paths of some stochastic processes.
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Comparison Theorem

Lemma

Assume that $b$ satisfies Hypothesis (H1), $a_1 > a_2$ and $T > 0$. Also assume that $u$ and $v$ are two measurable functions on $[0, T]$ such that

$$v(t) \geq a_1 + \int_0^t b(v(s))ds, \quad t \in [0, T],$$

and

$$u(t) = a_2 + \int_0^t b(u(s))ds, \quad t \in [0, T].$$

Then, $v \geq u$ on $[0, T]$. 

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Comparison Theorem

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$$v(t) \leq a_1 + \int_0^t b(v(s)) \, ds, \quad t \in [0, T],$$

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Comparison Theorem

**Lemma**

Assume that $b$ satisfies Hypothesis $(H1)$, $a_1 > a_2$ and $T > 0$. Also assume that $u$ and $v$ are two measurable functions on $[0, T]$ such that

$$v(t) \geq a_1 + \int_0^t b(v(s)) \, ds, \quad t \in [0, T],$$

and

$$u(t) = a_2 + \int_0^t b(u(s)) \, ds, \quad t \in [0, T].$$

Then, $v \geq u$ on $[0, T]$.

**Proof**: Note that the facts that $a_1 > a_2$ and $u$ is continuous yield

$$\{ t \in [0, T] : v(s) \geq u(s), \ s \in [0, t] \} \neq \emptyset.$$
Proof

Thus, we only need to show

\[ \tilde{T} = \sup\{ t \in (0, T] : v(s) \geq u(s), \ s \in [0, t] \} = T. \]
Proof

\[ \tilde{T} = \sup\{ t \in (0, \, T] : v(s) \geq u(s), \, s \in [0, \, t]\} = T. \]

But, if it is not so, then Hypothesis (H1) and the continuity of the integral lead to write

\[
v(\tilde{T} + t) - u(\tilde{T} + t) \geq a_1 - a_2 + \int_0^{\tilde{T} + t} [b(v(s)) - b(u(s))] \, ds
\]

\[
\geq a_1 - a_2 + \int_{\tilde{T}}^{\tilde{T} + t} [b(v(s)) - b(u(s))] \, ds
\]

\[
\geq \frac{a_1 - a_2}{2} > 0,
\]

for \( t \) small enough. Therefore \( \tilde{T} \) cannot be the supremum. \( \square \)
Contents

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Main result

Now we study the explosion in finite time of the solution to

\[ X_t = a + \int_0^t b(X_s)ds + g(t), \quad t \geq 0. \]
Main result

Now we study the explosion in finite time of the solution to

$$X_t = a + \int_0^t b(X_s)ds + g(t), \quad t \geq 0.$$  

Here $a \in \mathbb{R}$, $b$ satisfies Hypothesis (H1) and (H2) $g : [0, \infty) \to \mathbb{R}$ is a function with left and right-limits such that

$$\limsup_{t \to \infty} \left( \inf_{0 \leq h \leq 1} g(t + h) \right) = \infty.$$
Main result

**Theorem**

Let Hypotheses \((H1)\) and \((H2)\) hold. Then, the solution of equation

\[
X_t = a + \int_0^t b(X_s)ds + g(t), \quad t \geq 0.
\]

exploits in finite time if and only if

\[
\int_{\cdot}^{\infty} \frac{ds}{b(s)} < \infty.
\]
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Theorem

Let Hypotheses (H1) and (H2) hold. Then, the solution of equation

\[ X_t = a + \int_0^t b(X_s)ds + g(t), \quad t \geq 0. \]

exploits in finite time if and only if

\[ \int_0^\infty \frac{ds}{b(s)} < \infty. \]

Proof: Necessity: Suppose that \( X \) exploits at the time \( T_e < \infty \).
Set \( M := \sup_{0 \leq t \leq T_e} |g(t)| \). Then,

\[ X_t \leq (a + M) + \int_0^t b(X_s)ds, \quad t \in [0, T_e]. \]
Proof

Suppose that \( X \) exploits at the time \( T_e < \infty \). Set 
\[
M := \sup_{0 \leq t \leq T_e} |g(t)|.
\]
Then,
\[
X_t \leq (a + M) + \int_0^t b(X_s)ds, \quad t \in [0, T_e].
\]

Hence, our comparison result implies that the solution of
\[
u(t) = (a + M + 1) + \int_0^t b(u(s))ds, \quad t \geq 0,
\]
exploits in the interval \([0, T_e]\).
Proof

Suppose that $X$ exploits at the time $T_e < \infty$. Set $M := \sup_{0 \leq t \leq T_e} |g(t)|$. Then,

$$X_t \leq (a + M) + \int_0^t b(X_s)ds, \quad t \in [0, T_e].$$

Hence, our comparison result implies that the solution of

$$u(t) = (a + M + 1) + \int_0^t b(u(s))ds, \quad t \geq 0,$$

exploits in the interval $[0, T_e]$, which allows to conclude that

$$\int_{\cdot}^\infty \frac{ds}{b(s)} < \infty$$

because of Osgood criterion.
Theorem

Let Hypotheses (H1) and (H2) hold. Then, the solution of equation

\[ X_t = a + \int_0^t b(X_s)ds + g(t), \quad t \geq 0. \]

exploits in finite time if and only if

\[ \int_{\cdot}^{\infty} \frac{ds}{b(s)} < \infty. \]

Proof: Sufficiency: Now assume that the solution \( X \) of does not exploit in finite time.
Proof

Now assume that the solution $X$ of does not exploit in finite time. So, using Hypothesis (H2), we can find an increasing sequence $\{t_n : n \in \mathbb{N}\}$ such that $t_n \uparrow \infty$ and

$$\inf_{0 \leq h \leq 1} g(t_n + h) \uparrow \infty, \quad \text{as } n \to \infty.$$
Proof

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$$\inf_{0 \leq h \leq 1} g(t_n + h) \uparrow \infty, \quad \text{as } n \rightarrow \infty.$$ 

On the other hand, Hypothesis (H1) yields that

$$X_{t+t_n} \geq a + \int_{t_n}^{t+t_n} b(X_s)ds + g(t + t_n)$$

$$\geq a + \int_{0}^{t} b(X_{s+t_n})ds + \inf_{0 \leq h \leq 1} g(t_n + h), \quad t \in [0, 1].$$
Proof

Now assume that the solution $X$ of does not exploit in finite time. So, using Hypothesis (H2), we can find an increasing sequence \( \{t_n : n \in \mathbb{N}\} \) such that $t_n \uparrow \infty$ and

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\]

\[
\geq a + \int_{0}^{t} b(X_{s+t_n})ds + \inf_{0 \leq h \leq 1} g(t_n + h), \quad t \in [0, 1].
\]

Then, the solution of the equation

\[
u(t) = \frac{1}{2} \left( a + \inf_{0 \leq h \leq 1} g(t_n + h) \right) + \int_{0}^{t} b(u(s))ds, \quad t \geq 0,
\]

cannot exploit in the interval $[0, 1]$ due to comparison Lemma.
Proof

Now assume that the solution $X$ of does not exploit in finite time. So, using Hypothesis (H2), we can find an increasing sequence \( \{t_n : n \in \mathbb{N}\} \) such that $t_n \uparrow \infty$ and

$$\inf_{0 \leq h \leq 1} g(t_n + h) \uparrow \infty, \quad \text{as } n \to \infty.$$ 

Then, the solution of the equation

$$u(t) = \frac{1}{2} \left( a + \inf_{0 \leq h \leq 1} g(t_n + h) \right) + \int_0^t b(u(s))ds, \quad t \geq 0,$$

cannot exploit in the interval $[0, 1]$ due to comparison Lemma. In other words, the time of explosion of equation

$$\int_0^\infty \frac{ds}{2^{-1}(a + \inf_{0 \leq h \leq 1} g(t_n + h)) b(s)}$$

is bigger than 1.
Proof

Now assume that the solution \( X \) of \( X \) does not exploit in finite time. So, using Hypothesis (H2), we can find an increasing sequence \( \{t_n : n \in \mathbb{N}\} \) such that \( t_n \uparrow \infty \) and

\[
\inf_{0 \leq h \leq 1} g(t_n + h) \uparrow \infty, \quad \text{as } n \to \infty. \tag{1}
\]

Then, the solution of the equation

\[
u(t) = \frac{1}{2} \left( a + \inf_{0 \leq h \leq 1} g(t_n + h) \right) + \int_0^t b(u(s))ds, \quad t \geq 0,
\]

cannot exploit in the interval \([0, 1]\) due to comparison Lemma. In other words, the time of explosion

\[
\int_0^\infty \int_{2^{-1}(a+\inf_{0 \leq h \leq 1} g(t_n+h))}^{\infty} \frac{ds}{b(s)}
\]

of this equation is bigger than 1.

Finally, (1) gives \( \int_0^\infty \frac{ds}{b(s)} = \infty \).
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Main result

Here we show two classes of processes $Z$, whose paths satisfy Hypothesis ($H2$), with probability 1. Consequently, we can analyze the explosion in finite time of the solution to

$$X_t = a + \int_0^t b(X_s)ds + Z_t, \quad t \geq 0,$$

where $a \in \mathbb{R}$. 
Bifractional Brownian motion

The bifractional Brownian motion (bBm) with parameters $H \in (0, 1)$ and $K \in (0, 1]$ was introduced by Houdré and Villa (2003). It is a centered Gaussian process $B_{H,K} = \{B_{H,K}(t) : t \geq 0\}$ with covariance function

$$R_{H,K}(t, s) = \frac{1}{2^K} \left\{ (t^{2H} + s^{2H})^K - |t - s|^{2HK} \right\}$$

such that, for $s, t \geq 0$, the inequalities

$$2^{-K}|t - s|^{2HK} \leq E \left[ (B_{H,K}(t) - B_{H,K}(s))^2 \right] \leq 2^{1-K}|t - s|^{2HK}$$

hold.

$B_H := B_{H,1}$ is the fractional Brownian motion with Hurst parameter $H$ and $W := B_{1/2,1}$ is the Brownian motion.
Bifractional Brownian motion: The law of the iterated logarithm

**Lemma**

*With probability 1,*

\[
\limsup_{t \to \infty} \frac{B_{H,K}(t)}{\psi_{H,K}(t)} = 1,
\]

*with*

\[
\psi_{H,K}(t) := \sqrt{2t^{2HK} \log \log t}, \quad t > e.
\]
The law of the iterated logarithm

**Proof**: In order to see that the result is true, we proceed as in the Brownian case. That is, we only need to observe that the definition of $B_{H,K}$ yields that the process

$$\tilde{B}_{H,K}(t) = \begin{cases} 
0, & t = 0, \\
t^{2HK} B_{H,K}(\frac{1}{t}), & t > 0
\end{cases}$$

is also a bBm with parameters $H$ and $K$, and that

$$\limsup_{t \to \infty} \frac{B_{H,K}(t)}{\psi_{H,K}(t)} = \limsup_{h \downarrow 0} \frac{\tilde{B}_{H,K}(h)}{\sqrt{2h^{2HK} \log \log h^{-1}}} = 1.$$ 

The last equality is an immediate consequence of Arcones (1995) (Corollary 3.1).
Lemma

With probability 1,

$$\sup_{s,t \in [n, n+2]} \frac{|B_{H,K}(t) - B_{H,K}(s)|}{\psi_{H,K}(n)} \to 0, \quad \text{as } n \to \infty,$$

where

$$\psi_{H,K}(t) := \sqrt{2t^{2HK} \log \log t}, \quad t > e.$$
Proof

For each \( n \in \mathbb{N} \) consider the centered Gaussian process 
\( \{ B_{H,K}(t + n) - B_{H,K}(n) : t \in [0, 2] \} \). Then, from inequality (2) and Carmona et al. (2003) (Lemma 5.2), we have that for \( p \geq 1/HK \), there exists a constant \( C > 0 \), that only depends on \( H, K \) and \( p \), such that

\[
E \left[ \left( \sup_{s,t \in [n,n+2]} |B_{H,K}(t) - B_{H,K}(s)| \right)^p \right] \leq C2^{pHK}.
\]

Hence,

\[
E \left[ \sum_{n=1}^{\infty} \left( \sup_{s,t \in [n,n+2]} \frac{|B_{H,K}(t) - B_{H,K}(s)|}{\psi_{H,K}(n)} \right)^p \right] \leq \sum_{n=1}^{\infty} \frac{C2^{pHK}}{\psi_{H,K}(n)^p} < \infty.
\]
Bifractional Brownian motion

\[ X_t = a + \int_0^t b(X_s)\,ds + Z_t, \quad t \geq 0, \]
Bifractional Brownian motion

\[ X_t = a + \int_0^t b(X_s)ds + Z_t, \quad t \geq 0, \]

**Theorem**

Assume that Hypothesis (H1) is satisfied and that the process Z is the bifractional Brownian motion \( B_{H,K} \). Then, the solution of above equation explodes in finite time with probability 1 if and only if

\[ \int_{-\infty}^{\infty} \frac{ds}{b(s)} < \infty. \]
Bifractional Brownian motion

\[ X_t = a + \int_0^t b(X_s)ds + Z_t, \quad t \geq 0, \]

**Theorem**

Assume that Hypothesis (H1) is satisfied and that the process \( Z \) is the bifractional Brownian motion \( B_{H,K} \). Then, the solution of above equation explodes in finite time with probability 1 if and only if

\[ \int_{-\infty}^{\infty} \frac{ds}{b(s)} < \infty. \]

**Proof**: we only need to see that the paths of \( B_{H,K} \) are as in Hypothesis (H2) with probability 1.
Proof

Toward this end, let $\omega_0 \in \Omega$. Then,

$$\inf_{0 \leq h \leq 1} B_{H,K}(\omega_0, t + h) = B_{H,K}(\omega_0, t) + \inf_{0 \leq h \leq 1} (B_{H,K}(\omega_0, t + h) - B_{H,K}(\omega_0, t))$$
Proof

Toward this end, let $\omega_0 \in \Omega$. Then,

$$\inf_{0 \leq h \leq 1} B_{H,K}(\omega_0, t + h)$$

$$= B_{H,K}(\omega_0, t) + \inf_{0 \leq h \leq 1} (B_{H,K}(\omega_0, t + h) - B_{H,K}(\omega_0, t))$$

$$\geq B_{H,K}(\omega_0, t) + \inf_{0 \leq h \leq 1} (- |B_{H,K}(\omega_0, t + h) - B_{H,K}(\omega_0, t)|)$$
Proof

Toward this end, let $\omega_0 \in \Omega$. Then,

$$\inf_{0 \leq h \leq 1} B_{H,K}(\omega_0, t + h)$$

$$= B_{H,K}(\omega_0, t) + \inf_{0 \leq h \leq 1} (B_{H,K}(\omega_0, t + h) - B_{H,K}(\omega_0, t))$$

$$\geq B_{H,K}(\omega_0, t) + \inf_{0 \leq h \leq 1} ( - |B_{H,K}(\omega_0, t + h) - B_{H,K}(\omega_0, t)|)$$

$$= B_{H,K}(\omega_0, t) - \left( \sup_{0 \leq h \leq 1} \frac{|B_{H,K}(\omega_0, t + h) - B_{H,K}(\omega_0, t)|}{\psi_{H,K}([t])} \right) \psi_{H,K}([t]),$$

where $[t]$ is the integer part of $t$. 
Proof

Toward this end, let $\omega_0 \in \Omega$. Then,

$$\inf_{0 \leq h \leq 1} B_{H,K}(\omega_0, t + h)$$

$$\geq B_{H,K}(\omega_0, t) - \left( \sup_{0 \leq h \leq 1} \frac{|B_{H,K}(\omega_0, t + h) - B_{H,K}(\omega_0, t)|}{\psi_{H,K}([t])} \right) \psi_{H,K}([t]),$$

Lemma

With probability 1,

$$\sup_{s, t \in [n, n+2]} \frac{|B_{H,K}(t) - B_{H,K}(s)|}{\psi_{H,K}(n)} \to 0, \quad \text{as } n \to \infty,$$

where

$$\psi_{H,K}(t) := \sqrt{2t^{2HK} \log \log t}, \quad t > e.$$
Proof

Therefore, for $t$ large enough,

$$\inf_{0 \leq h \leq 1} B_{H,K}(\omega_0, t + h) \geq B_{H,K}(\omega_0, t) - \frac{1}{4} \psi_{H,K}([t]).$$
Proof

Therefore, for $t$ large enough,

$$\inf_{0 \leq h \leq 1} B_{H,K}(\omega_0, t + h) \geq B_{H,K}(\omega_0, t) - \frac{1}{4} \psi_{H,K}([t])$$

$$= \frac{B_{H,K}(\omega_0, t)}{\psi_{H,K}(t)} \psi_{H,K}(t) - \frac{1}{4} \psi_{H,K}([t]).$$
Proof

Therefore, for $t$ large enough,

$$\inf_{0 \leq h \leq 1} B_{H,K}(\omega_0, t + h) \geq B_{H,K}(\omega_0, t) - \frac{1}{4} \psi_{H,K}([t])$$

$$= \frac{B_{H,K}(\omega_0, t)}{\psi_{H,K}(t)} \psi_{H,K}(t) - \frac{1}{4} \psi_{H,K}([t]).$$

Lemma

With probability 1,

$$\limsup_{t \to \infty} \frac{B_{H,K}(t)}{\psi_{H,K}(t)} = 1,$$

with

$$\psi_{H,K}(t) := \sqrt{2t^{2HK} \log \log t}, \quad t > e.$$
Proof

Hence, there exists a sequence $0 < t_n \uparrow \infty$ such that

$$
\inf_{0 \leq h \leq 1} B_{H,K}(\omega_0, t_n + h) \geq \frac{1}{2}\psi_{H,K}(t_n) - \frac{1}{4}\psi_{H,K}([t_n])
$$

$$
\geq \frac{1}{4}\psi_{H,K}([t_n]) \to \infty.
$$
Proof

Hence, there exists a sequence $0 < t_n \uparrow \infty$ such that

$$\inf_{0 \leq h \leq 1} B_{H,K}(\omega_0, t_n + h) \geq \frac{1}{2} \psi_{H,K}(t_n) - \frac{1}{4} \psi_{H,K}([t_n])$$

$$\geq \frac{1}{4} \psi_{H,K}([t_n]) \rightarrow \infty.$$ 

Thus,

$$\limsup_{t \rightarrow \infty} \left( \inf_{0 \leq h \leq 1} B_{H,K}(\omega_0, t + h) \right) = \infty.$$
Let $\beta \in (0, 1)$. Now we consider an increasing $1/\beta$-self similar process $Z$, whose trajectories satisfies Hypothesis (H2). The reader can consult Rivero (2003) (and references therein) for details.
Self-similar processes

Let $\xi = \{\xi_t : t \geq 0\}$ be a Lévy process. Set

$$A_t = \int_0^t \exp \left( \frac{\xi_s}{\alpha} \right) ds, \quad t \geq 0,$$

where $\alpha > 0$, and

$$\tau(t) = \inf \{ s : A_s > t \},$$

the time change related to $A$. 
Self-similar processes

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where $\alpha > 0$, and

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the time change related to $A$. Then, the process

$$Z_t = x \exp\left(\xi_{\tau(tx^{-1/\alpha})}\right), \quad t \geq 0,$$

with $x > 0$, is an $\alpha$-self similar Markov process.
Self-similar processes

It is well-known that the law of $\xi$ is characterized by its Laplace transform

$$E \left( \exp(-\lambda \xi_t) \right) = \exp(-t\phi(\lambda)), \quad t \geq 0 \text{ and } \lambda \geq 0.$$
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Here, by the Lévy-Khintchine’s formula, $\phi$ has the form

$$\phi(\lambda) = d\lambda + \int_{(0,\infty)} (1 - \exp(-\lambda x))\Pi(dx),$$

where $d$ is called the drift coefficient and $\Pi$ the Lévy measure associated with $\xi$. 
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Now consider $\beta \in (0, 1)$ and the measure

$$ \Pi(dx) = \frac{\beta \exp(x)}{\Gamma(1 - \beta)(\exp(x) - 1)^{1+\beta}} dx. \quad (3) $$
Theorem

Let $Z$ be the $1/\beta$-self similar Markov process related to a subordinator with zero drift and Lévy measure (3). Then, under Hypothesis (H1), the solution of equation

$$X_t = a + \int_0^t b(X_s) ds + Z_t, \quad t \geq 0,$$

explodes in finite time with probability 1 if and only if

$$\int_0^\infty \frac{ds}{b(s)} < \infty.$$
Proof

The result follows from Theorem M and from the fact that

\[
\liminf_{t \to \infty} \frac{Z_t}{t^{1/\beta}(\log \log t)^{(\beta-1)/\beta}} = \beta(1 - \beta)^{(1-\beta)/\beta},
\]

which is proven in Rivero (2003) (see p. 469). \(\square\)
Contents

1 Introduction and Osgood test
2 Comparison Theorem
3 Blow-up for a class of integral equations
4 Stochastic differential equations driven by an additive noise
5 Comparison with the Feller test for explosions
Feller test

Let $\mathcal{W} = \{\mathcal{W}_t : t \geq 0\}$ be a Brownian motion. We can use the Feller test to see if the solution of a stochastic differential equation of the form

$$dX_t = b(X_t)dt + \sigma(X_t)d\mathcal{W}_t, \quad t > 0,$$

$$X_0 = a,$$

explodes in finite time, with probability 1, knowing only the coefficients $b$ and $\sigma$. 
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In our case (i.e., $\sigma \equiv 1$), this test can be expressed as follows:
Feller test

Let $\rho(x) = \int_0^x \exp \left( -2 \int_0^s b(r)dr \right) ds$ and $v(x) = 2 \int_0^x \frac{\rho(x) - \rho(y)}{\rho'(y)} dy$.

Proposition (Feller test)

The explosion time $T_e$ of the solution $X$ of the equation

$$dX_t = b(X_t)dt + dW_t, \quad t > 0,$$

$$X_0 = a,$$

is finite with probability 1 if and only if, one of the following conditions holds: (i) $v(\infty) < \infty$ and $v(-\infty) < \infty$,

(ii) $v(\infty) < \infty$ and $\rho(-\infty) = -\infty$,

(iii) $v(-\infty) < \infty$ and $\rho(\infty) = \infty$. 
Example

Let us consider

\[ dX_t = X_t^2 \, dt + dW_t, \quad X_0 = -1. \]
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\[
\rho(x) = \int_0^x \exp \left( -2 \int_0^s r^2 dr \right) ds
= \int_0^x e^{-\frac{2}{3}s^3} ds.
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\]

\[
= \int_0^x e^{-\frac{2}{3} s^3} \, ds.
\]

Hence

\[
v(x) = 2 \int_0^x \int_0^y e^{\frac{2}{3} (z^3 - y^3)} \, dz \, dy \Rightarrow v(\infty) = 2 \int_0^\infty \int_0^y e^{\frac{2}{3} (z^3 - y^3)} \, dz \, dy.
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Example

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Hence

\[
v(x) = 2 \int_0^x \int_0^y e^{\frac{2}{3}(z^3 - y^3)} \, dz \, dy \Rightarrow v(\infty) = 2 \int_0^\infty \int_0^y e^{\frac{2}{3}(z^3 - y^3)} \, dz \, dy.
\]

So we study

\[
\int_0^\infty \int_0^y e^{\frac{2}{3}(z^3 - y^3)} \, dz \, dy = \left( \int_0^1 + \int_1^\infty \right) \left( \int_0^y e^{\frac{2}{3}(z^3 - y^3)} \, dz \right) \, dy.
\]
\[ \int_1^\infty \int_0^y e^{\frac{2}{3}(z^3-y^3)} \, dz \, dy = \int_0^1 e^{\frac{2}{3}z^3} \, dz \int_1^\infty e^{-\frac{2}{3}y^3} \, dy \]
\[ + \int_1^\infty e^{\frac{2}{3}z^3} \left( \int_z^\infty e^{-\frac{2}{3}y^3} \, dy \right) \, dz. \]
Example

\[
\int_1^\infty \int_0^y e^{\frac{2}{3}(z^3-y^3)} \, dz \, dy = \int_0^1 e^{\frac{2}{3}z^3} \, dz \int_1^\infty e^{-\frac{2}{3}y^3} \, dy \\
+ \int_1^\infty e^{\frac{2}{3}z^3} \left( \int_z^\infty e^{-\frac{2}{3}y^3} \, dy \right) \, dz.
\]

On the other hand

\[
\int_z^\infty e^{-\frac{2}{3}y^3} \, dy \leq \int_z^\infty \frac{y^2}{z^2} e^{-\frac{2}{3}y^3} \, dy = \frac{1}{z^2} \int_z^\infty y^2 e^{-\frac{2}{3}y^3} \, dy \\
= \left. \frac{1}{z^2} \left( -\frac{e^{-\frac{2}{3}y^3}}{2} \right) \right|_{z}^{\infty} = \frac{e^{-\frac{2}{3}z^3}}{2z^3}.
\]

Using this,

\[
\int_1^\infty e^{\frac{2}{3}z^3} \left( \int_z^\infty e^{-\frac{2}{3}y^3} \, dy \right) \, dz \leq \int_1^\infty e^{\frac{2}{3}z^3} \left( \frac{e^{-\frac{2}{3}z^3}}{2z^2} \right) \, dz = \frac{1}{2} < \infty.
\]
Example

This implies $v(\infty) < \infty$. Moreover

$$\rho(-\infty) = \int_{0}^{-\infty} e^{-\frac{2}{3}s^3} ds = -\infty.$$ 

By Feller test, $P(T_e < \infty) = 1$. 
Theorem

Let \( b \) satisfy Hypothesis (\( H1 \)). Then we have

(i) \( \rho(-\infty) = -\infty \).

(ii) \( \int_{0}^{\infty} \frac{ds}{b(s)} < \infty \) if and only if \( v(\infty) < \infty \).
Feller test

Theorem

Let $b$ satisfy Hypothesis (H1). Then we have

(i) $\rho(-\infty) = -\infty$.

(ii) $\int_0^\infty \frac{ds}{b(s)} < \infty$ if and only if $v(\infty) < \infty$.

Remark: The Feller test is proven using the Itô’s calculus. However, when $Z B_{H,K}$, we cannot use this important tool because, in general, $B_{H,K}$ is not a semimartingale.
Proof

(i) Observe that

$$
\rho(-\infty) = -\int_{-\infty}^{0} \exp \left( 2 \int_{s}^{0} b(r) dr \right) ds \\
\leq -\int_{-\infty}^{0} \exp \left( 2b(s)(-s) \right) ds \\
\leq -\int_{-\infty}^{0} \exp (0) ds = -\infty.
$$
Proof

(ii) Suppose $\int_0^\infty \frac{ds}{b(s)} < \infty$. Then

$$v(\infty) \leq 2 \int_0^\infty \int_y^\infty \frac{b(s)}{b(y)} \exp\left(-2 \int_0^s b(r) dr\right) \exp\left(2 \int_0^y b(t) dt\right) ds dy$$

$$= 2 \int_0^\infty \frac{1}{b(y)} \exp\left(2 \int_0^y b(t) dt\right) \left(\int_y^\infty b(s) \exp\left(-2 \int_0^s b(r) dr\right) ds\right) dy$$

$$= \int_0^\infty \frac{ds}{b(s)},$$

where, in order to evaluate the integral, we used that

$$\int_0^\infty b(r) dr \geq b(0) \int_0^\infty dr = \infty.$$
Conversely, now let us assume that $v(\infty) < \infty$. We first note that

$$\int_0^\infty \frac{1}{b(s)} \exp \left( -2 \int_0^s b(r)dr \right) ds \leq \int_0^\infty \frac{1}{b(s)} \exp (-2b(0)s) ds \leq \frac{1}{b(0)} \int_0^\infty \exp (-2b(0)s) ds < \infty.$$
Proof

Conversely, now let us assume that \( v(\infty) < \infty \). We first note that

\[
\int_0^\infty \frac{1}{b(s)} \exp \left( -2 \int_0^s b(r) \, dr \right) \, ds \\
\leq \frac{1}{b(0)} \int_0^\infty \exp (-2b(0)s) \, ds < \infty. \tag{4}
\]

On the other hand, Fubini theorem yields

\[
v(\infty) = 2 \int_0^\infty \int_0^s \exp \left( -2 \int_0^s b(r) \, dr \right) \exp \left( 2 \int_0^y b(t) \, dt \right) \, dy \, ds \\
\geq \int_0^\infty \frac{1}{b(s)} \exp \left( -2 \int_0^s b(r) \, dr \right) \left[ \exp \left( 2 \int_0^s b(r) \, dr \right) - 1 \right] \, ds \\
= \int_0^\infty \frac{1}{b(s)} \left[ 1 - \exp \left( -2 \int_0^s b(r) \, dr \right) \right] \, ds.
\]

Hence, (4) implies that \( \int_0^\infty \frac{ds}{b(s)} < \infty \). Thus the proof is complete. \( \square \)
References:


Gracias por la Atención