Functional Limit theorems for the quadratic variation of a continuous time random walk and for certain stochastic integrals

Noèlia Viles Cuadros
Universitat de Barcelona

joint work with Prof. Enrico Scalas

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Outline

1. Introduction

2. FCLT for the quadratic variation of Compound Renewal Processes

3. FCLT for the stochastic integrals driven by a time-changed symmetric $\alpha$-stable Lévy process
Scaling Limits

Consider a sequence of i.i.d. centered random variables $\xi_i$. Define the centered random walk:

$$S_n := \sum_{i=1}^{n} \xi_i.$$ 

(a) How does $S_n$ behave when $n$ is large?

(b) What is the limit after rescaling?

Lévy-Lindeberg Central Limit Theorem (CLT)

Given a sequence of random variables $(\xi_i)_{i \in \mathbb{N}}$ i.i.d. with mean $\mu$ and finite, positive variance $\sigma^2$, defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Then

$$\frac{S_n - n\mu}{\sqrt{n}} \xrightarrow{d} Y, \quad \text{with} \quad Y \sim N(0, \sigma^2).$$
Donsker’s Theorem

The classical CLT was generalized to a FCLT by Donsker (1951).

Donsker’s Theorem (1951)

Given a sequence of random variables \((\xi_i)_{i \in \mathbb{N}}\) i.i.d. with mean 0 and finite, positive variance \(\sigma^2\), defined on a probability space \((\Omega, \mathcal{F}, P)\). The random functions defined by

\[
X_n(t, \omega) := \frac{1}{\sigma \sqrt{n}} S_{\lfloor nt \rfloor} (\omega) + (nt - \lfloor nt \rfloor) \frac{1}{\sigma \sqrt{n}} \xi_{\lfloor nt \rfloor + 1}(\omega)
\]

satisfy that

\[
(X_n(t), t \in [0, T]) \overset{L}{\Rightarrow} (B(t), t \in [0, T])
\]

where \(B\) is a standard Brownian motion.
The Skorokhod space

The **Skorokhod space**, denoted by $\mathbb{D} = D([0, T], \mathbb{R})$ (with $T > 0$), is the space of real functions $x : [0, T] \to \mathbb{R}$ that are right-continuous with left limits:

1. For $t \in [0, T)$, $x(t+) = \lim_{s \downarrow t} x(s)$ exists and $x(t+) = x(t)$.
2. For $t \in (0, T]$, $x(t-) = \lim_{s \uparrow t} x(s)$ exists.

Functions satisfying these properties are called **cadlàg functions**.
Skorokhod topologies

The Skorokhod space provides a natural and convenient formalism for describing the trajectories of stochastic processes with jumps: Poisson process, Lévy processes, martingales and semimartingales, empirical distribution functions, discretizations of stochastic processes, etc.

It can be assigned a topology that, intuitively allows us to wiggle space and time a bit (whereas the traditional topology of uniform convergence only allows us to wiggle space a bit).

Skorokhod (1965) proposed four metric separable topologies on $\mathbb{D}$, denoted by $J_1$, $J_2$, $M_1$ and $M_2$.

A. Skorokhod.

Limit Theorems for Stochastic Processes.

The Skorokhod $J_1$-topology

For $T > 0$, let

$$
\Lambda := \{ \lambda : [0, T] \to [0, T], \text{ strictly increasing and continuous} \}.
$$

If $\lambda \in \Lambda$, then $\lambda(0) = 0$ and $\lambda(T) = T$.

For $x, y \in \mathbb{D}$, the Skorokhod $J_1$-metric is

$$
d_{J_1}(x, y) := \inf_{\lambda \in \Lambda} \left\{ \sup_{t \in [0, T]} |\lambda(t) - t|, \sup_{t \in [0, T]} |x(t) - y(\lambda(t))| \right\} \tag{1}
$$

Convergence in $J_1$-topology

The sequence $x_n(t) \in \mathbb{D}$ converges to $x_0(t) \in \mathbb{D}$ in the $J_1$-topology if there exists a sequence of increasing homeomorphisms $\lambda_n : [0, T] \to [0, T]$ such that

$$
\sup_{t \in [0, T]} |\lambda_n(t) - t| \to 0, \quad \sup_{t \in [0, T]} |x_n(\lambda_n(t)) - x_0(t)| \to 0, \tag{2}
$$

as $n \to \infty$. 
The Skorokhod $M_1$-topology

We use the $M_1$-topology in order to be able to establish stochastic process limits with unmatched jumps in the limit process.

We define the $M_1$-metric using the completed graph of the functions. For $x \in \mathbb{D}$, the completed graph of $x$ is

$$\Gamma_x^{(a)} = \{(t, z) \in [0, T] \times \mathbb{R} : z = ax(t-) + (1-a)x(t) \text{ for some } a \in [0, 1]\},$$

where $x(t-)$ is the left limit of $x$ at $t$ and $x(0-) := x(0)$.

A function in $D([0, 1], \mathbb{R})$ and its completed graph
The Skorokhod $M_1$-topology

We define the $M_1$ metric using the uniform metric defined on parametric representations of the completed graphs of the functions. A parametric representation of $\Gamma^{(a)}_x$ is a continuous nondecreasing function 

$$(r, u) : [0, 1] \rightarrow \Gamma^{(a)}_x,$$

with $r$ being the time component and $u$ being the spatial component.

Denote $\Pi(x)$ the set of parametric representations of $\Gamma^{(a)}_x$ in $\mathbb{D}$.

For $x_1, x_2 \in \mathbb{D}$, the Skorokhod $M_1$-metric on $\mathbb{D}$ is

$$d_{M_1}(x_1, x_2) := \inf_{(r_i, u_i) \in \Pi(x_i)} \left\{ \| r_1 - r_2 \|_{[0, T]} \lor \| u_1 - u_2 \|_{[0, T]} \right\}. \quad (3)$$
Convergence in $M_1$-topology

The sequence $x_n(t) \in \mathbb{D}$ converges to $x_0(t) \in \mathbb{D}$ in the $M_1$-topology if

$$\lim_{n \to +\infty} d_{M_1}(x_n(t), x_0(t)) = 0. \quad (4)$$

In other words, we have the convergence in $M_1$-topology if there exist parametric representations $(y(s), t(s))$ of the graph $\Gamma_{x_0(t)}$ and $(y_n(s), t_n(s))$ of the graph $\Gamma_{x_n(t)}$ such that

$$\lim_{n \to \infty} \| (y_n, t_n) - (y, t) \|_{[0, \tau]} = 0. \quad (5)$$
Characterization for the $M_1$-convergence (Silvestrov(2004))

If the following two conditions are satisfied:

(i) Let $A$ be a dense subset in $[0, +\infty)$ which contains 0.

\[
\{X_n(t)\}_{t \in A} \overset{\mathcal{L}}{\Rightarrow} \{X(t)\}_{t \in A} \text{ as } n \to +\infty.
\]

(ii) Condition on $M_1$-compactness:

\[
\lim_{\delta \to 0} \limsup_{n \to +\infty} w(X_n, \delta) = 0,
\]

where $w(X_n, \delta) := \sup_{t \in A} w(X_n, t, \delta)$, and

\[
w(X_n, t, \delta) := \sup_{0 \vee (t-\delta) \leq t_1 < t_2 < t_3 \leq (t+\delta) \wedge T} \{\|X_n(t_2) - [X_n(t_1), X_n(t_3)]\|\}.
\]

Then,

\[
\{X_n(t)\}_{t \geq 0} \overset{M_1-top}{\Rightarrow} \{X(t)\}_{t \geq 0} \text{ as } n \to +\infty.
\]
Some remarks

For \( x, y \in \mathbb{R} \) denote the standard segment as

\[
[x, y] := \{ax + (1 - a)y, \ a \in [0, 1]\}.
\]

The modulus of \( M_1 \)-compactness plays the same role for càdlàg functions as the modulus of continuity for continuous functions.

D.S. Silvestrov.
A continuous time random walk (CTRW) is a pure jump process given by a sum of i.i.d. random jumps \((Y_i)_{i \in \mathbb{N}}\) separated by i.i.d. random waiting times (positive random variables) \((J_i)_{i \in \mathbb{N}}\).
It is called the compound Poisson process. It is a Markov and Lévy process. The functional limit is an $\alpha$-stable Lévy process. Then the position of a particle at time $t > 0$ can be expressed as the sum of the jumps up to time $t$ of the corresponding counting process $N(t)$ defined by

$$N(t) \overset{\text{def}}{=} \max\{n : T_n \leq t\}.$$ 

Let $X_n = \sum_{i=1}^n Y_i$ denote the position of a diffusing particle after $n$ jumps and $T_n = \sum_{i=1}^n J_i$ be the epoch of the $n$-th jump.
\(\alpha\)-stable Lévy processes

A continuous-time process \( L = \{ L_t \}_{t \geq 0} \) with values in \( \mathbb{R} \) is called a Lévy process if its sample paths are càdlàg at every time point \( t \), and it has stationary, independent increments, that is:

(a) For all \( 0 = t_0 < t_1 < \cdots < t_k \), the increments \( L_{t_i} - L_{t_{i-1}} \) are independent.

(b) For all \( 0 \leq s \leq t \) the random variables \( L_t - L_s \) and \( L_{t-s} - L_0 \) have the same distribution.

An \( \alpha\)-stable process is a real-valued Lévy process \( L_{\alpha} = \{ L_{\alpha}(t) \}_{t \geq 0} \) with initial value \( L_{\alpha}(0) \) that satisfies the self-similarity property

\[
\frac{1}{t^{1/\alpha}} L_{\alpha}(t) \overset{\mathcal{L}}{=} L_{\alpha}(1), \quad \forall t > 0.
\]

If \( \alpha = 2 \) then the \( \alpha\)-stable Lévy process is the Wiener process.
Consider a CTRW whose i.i.d. jumps \((Y_i)_{i \in \mathbb{N}}\) have symmetric \(\alpha\)-stable distribution with \(\alpha \in (1, 2]\), and whose i.i.d waiting times \((J_i)_{i \in \mathbb{N}}\) satisfy
\[
\mathbb{P}(J_i > t) = E_\beta(-t^\beta),
\] (9)
for \(\beta \in (0, 1]\), where
\[
E_\beta(z) = \sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(1 + \beta j)},
\]
denotes the Mittag-Leffler function.

If \(\beta = 1\), the waiting times are exponentially distributed with parameter \(\lambda = 1\) and the counting process is the Poisson process.
Compound Fractional Poisson Process

The counting process associated is called the fractional Poisson process

\[ N_\beta(t) = \max\{n : T_n \leq t\}. \]

If we subordinate a CTRW to the fractional Poisson process, we obtain the compound fractional Poisson process, which is not Markov

\[ X_{N_\beta(t)} = \sum_{i=1}^{N_\beta(t)} Y_i. \]  \hspace{1cm} (10)

The functional limit of the compound fractional Poisson process is an \( \alpha \)-stable Lévy process subordinated to the fractional Poisson process.

These processes are possible models for tick-by-tick financial data.
\(\beta\)-stable subordinator and its functional inverse

A \(\beta\)-stable subordinator \(\{D_\beta\}_{t \geq 0}\) is a real-valued \(\beta\)-stable Lévy process with nondecreasing sample paths.

The functional inverse of \(\{D_\beta\}_{t \geq 0}\) can be defined as

\[
D_\beta^{-1}(t) := \inf\{x \geq 0 : D_\beta(x) > t\}.
\]

It has almost surely continuous non-decreasing sample paths and without stationary and independent increments.

Magdziard & Weron, 2006
About scaling limits

- **Scaling limit of a CTRW**: the limit process resulting from appropriate scaling in time and space according to a functional central limit theorem (FCLT).

The limit behavior of the CTRW depends on the distribution of the jumps and the waiting times.

- If the **waiting times have finite mean**, the CTRW behaves like a random walk in the limit. So, by Donskers Theorem, if the waiting times have finite mean and the jumps have finite variance then the scaled CTRW converges in distribution to a Brownian motion.

- If the **waiting times have finite mean** and the **jumps** are in the **DOA of an $\alpha$-stable random variable, with $\alpha \in (0, 2)$**, then the appropriately scaled CTRW converges in distribution to an $\alpha$-stable Lévy motion.
About scaling limits

- If the **waiting times have an infinite mean**, the CTRW limit behavior is more complex. **Meerschaert and Scheffler** proved a FCLT which identifies the limit process as a composition of an $\alpha$-stable Lévy motion $L_\alpha(t)$ and the inverse of a $\beta$-stable subordinator, $D_{\beta}^{-1}(t)$, where $\alpha \in (0, 2]$ and $\beta \in (0, 1)$

M. Meerschaert, H. P. Scheffler.
Limit Theorems for continuous time random walks.

M. Meerschaert, H. P. Scheffler.
Convergence to the inverse $\beta$-stable subordinator

For $t \geq 0$, we define

$$T_t := \sum_{i=1}^{\lfloor t \rfloor} J_i.$$

We have

$$\left\{ c^{-1/\beta} T_{ct} \right\}_{t \geq 0} \xrightarrow{c} \left\{ D_\beta(t) \right\}_{t \geq 0}, \quad \text{as} \quad c \to +\infty.$$

For any integer $n \geq 0$ and any $t \geq 0$: $\{ T_n \leq t \} = \{ N_\beta(t) \leq n \}$.

**Theorem (Meerschaert & Scheffler (2001))**

$$\left\{ c^{-1/\beta} N_\beta(ct) \right\}_{t \geq 0} \xrightarrow{c} \left\{ D_{-1}^{-1}(t) \right\}_{t \geq 0}, \quad \text{as} \quad c \to +\infty.$$

**Theorem (Meerschaert & Scheffler (2001))**

$$\left\{ c^{-1/\beta} N_\beta(ct) \right\}_{t \geq 0} \xrightarrow{J_{1-top}} \left\{ D_{-1}^{-1}(t) \right\}_{t \geq 0}, \quad \text{as} \quad c \to +\infty.$$
Convergence to the symmetric $\alpha$-stable Lévy process

Assume the jumps $Y_i$ belong to the strict generalized domain of attraction of some stable law with $\alpha \in (0, 2)$, then $\exists a_n > 0$ such that

$$a_n \sum_{i=1}^{n} Y_i \xrightarrow{\mathcal{L}} \tilde{L}_\alpha, \text{ as } c \rightarrow +\infty.$$

Theorem (Meerschaert & Scheffler (2001))

$$\left\{ c^{-1/\alpha} \sum_{i=1}^{\lfloor ct \rfloor} Y_i \right\}_{t \geq 0} \xrightarrow{\mathcal{L}} \{ L_\alpha(t) \}_{t \geq 0}, \text{ when } c \rightarrow +\infty.$$

Corollary (Meerschaert & Scheffler (2004))

$$\left\{ c^{-1/\alpha} \sum_{i=1}^{\lfloor ct \rfloor} Y_i \right\}_{t \geq 0} \xrightarrow{J_{1-top}} \{ L_\alpha(t) \}_{t \geq 0}, \text{ when } c \rightarrow +\infty.$$
Theorem (Meerschaert & Scheffler (2004))

Under the distributional assumptions considered above for the waiting times $J_i$ and the jumps $Y_i$, we have

$$
\left\{ c^{-\beta/\alpha} \sum_{i=1}^{N_\beta(t)} Y_i \right\}^t \overset{M_1-top}{\to} \{ L_\alpha(D_\beta^{-1}(t)) \}^t_{t \geq 0}, \quad \text{when} \quad c \to +\infty, \quad (11)
$$

in the Skorokhod space $D([0, +\infty), \mathbb{R})$ endowed with the $M_1$-topology.

M. Meerschaert, H. P. Scheffler.

Limit theorems for continuous-time random walks with infinite mean waiting times.

Idea of the proof

Apply

\[ \{ c^{-1/\beta} N_\beta(ct) \}_{t \geq 0} \overset{J_{1-top}}{\Rightarrow} \{ D_\beta^{-1}(t) \}_{t \geq 0}, \text{ as } c \to +\infty. \]

and

\[ \left\{ c^{-1/\alpha} \sum_{i=1}^{[ct]} Y_i \right\}_{t \geq 0} \overset{J_{1-top}}{\Rightarrow} \{ L_\alpha(t) \}_{t \geq 0}, \text{ when } c \to +\infty. \]

\[ \left\{ \left( c^{-1/\alpha} \sum_{i=1}^{[ct]} Y_i, c^{-1/\beta} N_\beta(ct) \right) \right\}_{t \geq 0} \overset{J_{1-top}}{\Rightarrow} \{(L_\alpha(t), D_\beta^{-1}(t))\}_{t \geq 0}. \]
Idea of the proof

The proof uses a continuous mapping approach.

**Continuous Mapping Theorem (Whitt 2002)**

Suppose that \((x_n, y_n) \to (x, y)\) in \(D([0, a], \mathbb{R}^k) \times D_1^\uparrow\) (where \(D_1^\uparrow\) is the subset of functions nondecreasing and with \(x^i(0) \geq 0\)). If \(y\) is continuous and strictly increasing at \(t\) whenever \(y(t) \in Disc(x)\) and \(x\) is monotone on \([y(t^-), y(t)]\) and \(y(t^-), y(t) \notin Disc(x)\) whenever \(t \in Disc(y)\), then \(x_n \circ y_n \to x \circ y\) in \(D([0, a], \mathbb{R}^k)\), where the topology throughout is \(M_1\) or \(M_2\).

The convergence result only holds in weaker \(M_1\)-topology since the composition map is continuous in \(M_1\)-topology but not in \(J_1\) at \((L_\alpha, D_\beta^{-1})\).

- **W. Whitt**,  
  *Stochastic-Process Limits: An Introduction to Stochastic-Process Limits and Their Application to Queues.*  
FCLT for the quadratic variation of Compound Renewal Processes
Quadratic Variation

Let \( \{ Y_i \}_{i=1}^{\infty} \) be a sequence of i.i.d. random variables (also independent of the \( J_i \)'s) then the compound process \( X(t) \) defined by

\[
X(t) = \sum_{i=1}^{N_\beta(t)} Y_i
\]  

(12)

The quadratic variation of \( X \) is

\[
[X](t) = [X, X](t) = \sum_{i=1}^{N_\beta(t)} [X(T_i) - X(T_{i-1})]^2 = \sum_{i=1}^{N_\beta(t)} Y_i^2.
\]  

(13)
FCLT for the Quadratic Variation

**Theorem (Scalas & V. (2012))**

Under the distributional assumptions considered above for the waiting times $J_i$ and the jumps $Y_i$, we have

$$\left\{ \left( \frac{1}{n^{2/\alpha}} \sum_{i=1}^{[nt]} Y_i^2, \frac{1}{n^{1/\beta}} T_{nt} \right) \right\}_{t \geq 0} \xrightarrow{n \to +\infty} J_{1-top} \{ (L^{+}_{\alpha/2}(t), D_{\beta}(t)) \}_{t \geq 0}, \quad (14)$$

in the Skorokhod space $D([0, +\infty), \mathbb{R}_+ \times \mathbb{R}_+)$ endowed with the $J_1$-topology. Moreover, we have also

$$\sum_{i=1}^{N_\beta(nt)} \frac{Y_i^2}{n^{2\beta/\alpha}} \xrightarrow{M_{1-top}} L^{+}_{\alpha/2}(D_{\beta}^{-1}(t)), \quad \text{as} \quad n \to +\infty,$$

in the Skorokhod space $D([0, +\infty), \mathbb{R}_+)$ with the $M_1$-topology, where $L^{+}_{\alpha/2}(t)$ denotes an $\alpha/2$-stable positive Lévy process.
E. Scalas, N. Viles,
On the Convergence of Quadratic variation for Compound Fractional Poisson Processes.
FCLT for the stochastic integrals driven by a time-changed symmetric $\alpha$-stable Lévy process
Damped harmonic oscillator subject to a random force

The equation of motion is informally given by

\[ \ddot{x}(t) + \gamma \dot{x}(t) + kx(t) = \xi(t), \] (15)

where \( x(t) \) is the position of the oscillating particle with unit mass at time \( t \), \( \gamma > 0 \) is the damping coefficient, \( k > 0 \) is the spring constant and \( \xi(t) \) represents white Lévy noise (formal derivative symmetric \( L_\alpha(t) \)).

I. M. Sokolov,
Harmonic oscillator under Lévy noise: Unexpected properties in the phase space.
The formal solution is

\[ x(t) = F(t) + \int_{-\infty}^{t} G(t - t')\xi(t')dt', \quad (16) \]

where \( G(t) \) is the Green function for the homogeneous equation. The solution for the velocity component can be written as

\[ v(t) = F_v(t) + \int_{-\infty}^{t} G_v(t - t')\xi(t')dt', \quad (17) \]

where \( F_v(t) = \frac{d}{dt} F(t) \) and \( G_v(t) = \frac{d}{dt} G(t) \).
Consider the compound renewal process given by

\[
X(t) = \sum_{i=1}^{N_\beta(t)} Y_i = \sum_{i \geq 1} Y_i \mathbf{1}_{\{T_i \leq t\}}
\]

The corresponding white noise can be formally written as

\[
\Xi(t) = \frac{dX(t)}{dt} = \sum_{i=1}^{N_\beta(t)} Y_i \delta(t - T_i) = \sum_{i \geq 1} Y_i \delta(t - T_i) \mathbf{1}_{\{T_i \leq t\}}.
\]
Our goal

To study the convergence of the integral of a deterministic continuous and bounded function with respect to a properly rescaled CTRW.

We aim to prove that under a proper scaling and distributional assumptions:

\[ \left\{ \sum_{i=1}^{N_\beta(nt)} G \left( t - \frac{T_i}{n} \right) \frac{Y_i}{n^{\beta/\alpha}} \right\}_{t \geq 0} \overset{M_1\text{-top}}{\Rightarrow} \left\{ \int_0^t G(t - s) dL_\alpha(D^{-1}_\beta(s)) \right\}_{t \geq 0}, \]

and

\[ \left\{ \sum_{i=1}^{N_\beta(nt)} G_v \left( t - \frac{T_i}{n} \right) \frac{Y_i}{n^{\beta/\alpha}} \right\}_{t \geq 0} \overset{M_1\text{-top}}{\Rightarrow} \left\{ \int_0^t G_v(t - s) dL_\alpha(D^{-1}_\beta(s)) \right\}_{t \geq 0}, \]

when \( n \to +\infty \), in the Skorokhod space \( D([0, +\infty), \mathbb{R}) \) endowed with the \( M_1 \)-topology.
Rescaled CTRW

Let $h$ and $r$ be two positive scaling factors such that

$$\lim_{h,r \to 0} h^\alpha r^{-\beta} = 1,$$

with $\alpha \in (1, 2]$ and $\beta \in (0, 1]$.

We rescale the duration $J$ and the jump by positive scaling factors $r$ and $h$:

$$J_r := rJ, \quad Y_h := hY.$$

The rescaled CTRW denoted:

$$X_{r,h}(t) = \sum_{i=1}^{N_\beta(t/r)} hY_i,$$

where $N_\beta = \{N_\beta(t)\}_{t \geq 0}$ is the fractional Poisson process.
Distributional assumptions

- **Jumps** \( \{Y_i\}_{i \in \mathbb{N}} \): i.i.d. symmetric \( \alpha \)-stable random variables such that \( Y_1 \) belongs to DOA of an \( \alpha \)-stable random variable with \( \alpha \in (1, 2] \).

- **Waiting times** \( \{J_i\}_{i \in \mathbb{N}} \): i.i.d. random variables such that \( J_1 \) belongs to DOA of some \( \beta \)-stable random variables with \( \beta \in (0, 1) \).
Theorem (Scalas & V.)

Let $f \in C_b(\mathbb{R})$. Under the distributional assumptions and the scaling,

$$\left\{ \sum_{i=1}^{N_\beta(nt)} f \left( \frac{T_i}{n} \right) \frac{Y_i}{n^{\beta/\alpha}} \right\}_{t \geq 0} \xrightarrow{M_1-top} \left\{ \int_0^t f(s) dL_\alpha(D_{\beta}^{-1}(s)) \right\}_{t \geq 0},$$

in $D([0, +\infty), \mathbb{R})$ with $M_1$-topology.
Sketch of the proof

✓ Check $M_1$-compactness condition for the integral process

\[
\left\{ l_n(t) := \sum_{k=1}^{N_{\beta}(nt)} f\left(\frac{T_k}{n}\right) \frac{Y_k}{n^{\beta/\alpha}} \right\}_{t \geq 0}
\]

✓ Prove the convergence in law of the family of processes $\{l_n(t)\}_{t \geq 0}$ when $n \to +\infty$.

   ✓ $\{X^{(n)}(t)\}_{t \geq 0}$ is uniformly tight or a good sequence.
   ✓ Apply the Continuous Mapping Theorem (CMT) taking as a continuous mapping the composition function.

✓ Apply Characterization of the $M_1$-convergence.
**M$_1$-compactness condition**

**Lemma (Scalas & V.)**

Let $f \in C_b(\mathbb{R})$. Let $\{Y_i\}_{i \in \mathbb{N}}$ be i.i.d. symmetric $\alpha$-stable random variables. Assume that $Y_1$ belongs DOA of $S_\alpha$, with $\alpha \in (1, 2]$. Let $\{J_i\}_{i \in \mathbb{N}}$ be i.i.d. such that $J_1$ belongs to the strict DOA of $S_\beta$ with $\beta \in (0, 1)$. Consider

$$I_n(t) := \sum_{k=1}^{N_\beta(nt)} f \left( \frac{T_k}{n} \right) \frac{Y_k}{n^{\beta/\alpha}}.$$  \hspace{1cm} (19)

If

$$\lim_{\delta \to 0} \limsup_{n \to +\infty} w_s(X_n, \delta) = 0,$$

where $X_n(t) := \sum_{k=1}^{N_\beta(nt)} \frac{Y_k}{n^{\beta/\alpha}}$. Then,

$$\lim_{\delta \to 0} \limsup_{n \to +\infty} w_s(I_n, \delta) = 0.$$  \hspace{1cm} (20)
Now, to see the convergence in the $M_1$-topology it only remains to prove

\[
\sum_{k=1}^{N_{\beta}(nt)} f \left( \frac{T_k}{n} \right) \frac{Y_k}{n^{\beta/\alpha}} \xrightarrow{\mathcal{L}} \int_0^t f(s) dL_\alpha(D_{\beta}^{-1}(s)), \quad n \to +\infty.
\]

A fundamental question is to know under what conditions the convergence in law of $(H^n, X^n)$ to $(H, X)$ implies that $X$ is a semimartingale and that $\int_0^t H^n(s-) dX^n_s$ converges in law to $\int_0^t H(s-) dX_s$. 
Good sequence

Let \((X^n)_{n \in \mathbb{N}}\) be an \(\mathbb{R}^k\)-valued process defined on \((\Omega^n, \mathcal{F}_n, \mathbb{P}^n)\) s.t. it is a \(\mathcal{F}_t^n\)-semimartingale. Assume that \((X^n)_{n \in \mathbb{N}} \xrightarrow{\mathcal{L}} X\) in the Skorokhod topology.

The sequence \((X^n)_{n \in \mathbb{N}}\) is said to be good if for any sequence \((H^n)_{n \in \mathbb{N}}\) of \(\mathbb{M}^{km}\)-valued, càdlàg processes, \(H^n \mathcal{F}_t^n\)-adapted, such that

\[
(H^n, X^n) \xrightarrow{\mathcal{L}} (H, X)
\]

in the Skorokhod topology on \(D_{\mathbb{M}^{km} \times \mathbb{R}^m}(0, \infty))\), \(\exists\) a filtration \(\mathcal{F}_t\) such that \(H\) is \(\mathcal{F}_t\)-adapted, \(X\) is a \(\mathcal{F}_t\)-semimartingale, and

\[
\int_0^t H^n(s-)dX^n_s \xrightarrow{\mathcal{L}} \int_0^t H(s-)dX_s,
\]

when \(n \to \infty\).
Lemma

If \((X^n)_{n \in \mathbb{N}}\) is a sequence of local martingales and the following condition holds for each \(t < +\infty\), where

\[
\sup_n \mathbb{E}^n \left[ \sup_{s \leq t} |\Delta X^{(n)}(s)| \right] < +\infty,
\]

\(\Delta X^{(n)}(s) := X^{(n)}(s) - X^{(n)}(s^-)\) (21)

denotes the increment of \(X^{(n)}\) in \(s\), then the sequence is uniformly tight.
\((X^{(n)})_{n \in \mathbb{N}}\) uniformly tight

**Lemma (Scalas & V.)**

Assume that \((Y_i)_{i \in \mathbb{N}}\) be i.i.d. symmetric \(\alpha\)-stable random variables, with \(\alpha \in (1, 2]\). Let

\[
X^{(n)}(t) := \sum_{i=1}^{\lfloor n^{\beta}t \rfloor} \frac{Y_i}{n^{\beta/\alpha}}
\]  

(22)

be defined on the probability space \((\Omega^n, \mathcal{F}^n, \mathbb{P}^n)\). Then \(X^n(t)\) is a \(\mathcal{F}^n_t\)-martingale (with respect the natural filtration of \(X^{(n)}\)) and

\[
\sup_n \mathbb{E}^n \left[ \sup_{s \leq t} |\Delta X^{(n)}(s)| \right] < +\infty,
\]

for each \(t < +\infty\).
Convergence in law

**Proposition (Scalas & V.)**

Let $f \in C_b(\mathbb{R})$. Under the distributional assumptions and the scaling considered above we have that

$$\left\lfloor n^\beta t \right\rfloor \sum_{i=1}^{N_{\beta}(nt)} f \left( \frac{T_i}{n} \right) \frac{Y_i}{n^\beta/\alpha} \xrightarrow{\mathcal{L}} \int_0^t f(D_\beta(s))dL_\alpha(s), \quad n \to +\infty.$$ 

**Proposition (Scalas & V.)**

Let $f \in C_b(\mathbb{R})$. Under the distributional assumptions and scaling,

$$\left\{ \sum_{i=1}^{N_{\beta}(nt)} f \left( \frac{T_i}{n} \right) Y_i \right\} \xrightarrow{\mathcal{L}} \left\{ \int_0^{D_\beta^{-1}(t)} f(D_\beta(s))dL_\alpha(s) \right\}_{t \geq 0}$$

as $n \to +\infty$, where $\int_0^{D_\beta^{-1}(t)} f(D_\beta(s))dL_\alpha(s) \overset{a.s.}{=} \int_0^t f(s)dL_\alpha(D_\beta^{-1}(s))$. 
Applications

Corollary (Scalas & V.)

\[
\left\{ \sum_{i=1}^{N_\beta(nt)} G \left( t - \frac{T_i}{n} \right) \frac{Y_i}{n^{\beta/\alpha}} \right\} \quad \overset{M_1-\text{top}}{\Rightarrow} \quad \left\{ \int_0^t G(t-s) dL_{\alpha}(D^{-1}_\beta(s)) \right\} \quad t \geq 0,
\]

and

\[
\left\{ \sum_{i=1}^{N_\beta(nt)} G_v \left( t - \frac{T_i}{n} \right) \frac{Y_i}{n^{\beta/\alpha}} \right\} \quad \overset{M_1-\text{top}}{\Rightarrow} \quad \left\{ \int_0^t G_v(t-s) dL_{\alpha}(D^{-1}_\beta(s)) \right\} \quad t \geq 0,
\]

in \( D([0, +\infty), \mathbb{R}) \) with \( M_1 \)-topology.
Summary:

- We have studied the convergence of a class of stochastic integrals with respect to the Compound Fractional Poisson Process.
- Under proper scaling hypotheses, these integrals converge to the integrals w.r.t a symmetric $\alpha$-stable process subordinated to the inverse $\beta$-stable subordinator.

Future work:

- It is possible to approximate some of the integrals discussed in Kobayashi (2010) by means of simple Monte Carlo simulations. This will be the subject of a forthcoming applied paper.
- To extend this result to the integration of stochastic processes instead of deterministic functions.

K. Kobayashi.
Thank you for your attention
Future work:

The functional convergence of quadratic variation leads to the following conjecture on the integrals defined as:

\[
I_a(t) = \sum_{i=1}^{N_t} [(1 - a)G(X(T_{i-1})) + aG(X(T_i))](X(T_i) - X(T_{i-1}))
\]

\[
= I_{1/2}(t) + \left(a - \frac{1}{2}\right) [X, G(X)](t),
\]

where \( G(x) \) is a sufficiently smooth ordinary function and \( a \in [0, 1] \) and

\[
[X, G(X)](t) = \sum_{i=1}^{N_t} [X(T_i) - X(T_{i-1})][G(X(T_i)) - G(X(T_{i-1}))].
\]

It might be possible to prove that, under proper scaling, the integral converges in some sense to a stochastic integral driven by the semimartingale measure \( L_\alpha(D_\beta^{-1}(t)) \).