Density Analysis of BSDEs

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Joint work in progress with Dylan Possamaï and Anthony Réveillac

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Financial market model:

- \( W := (W_t)_{t \in [0, \tau]} \) a Brownian motion defined on the probability space \((\Omega, (\mathcal{F}_t)_{t \in [0, \tau]}, \mathbb{P})\)

- Risk-free asset \( S^0 := (S^0_t)_{t \in [0, \tau]} \),
  \[
  dS^0_t = S^0_t r \, dt
  \]

- Asset \( S := (S_t)_{t \in [0, \tau]} \),
  \[
  dS_t = S_t \left( \theta_t \, dt + dW_t \right),
  \]
  where \( \theta \) is predictable and bounded.
Motivation: pricing and hedging problems in finance

Investing strategy \((r = 0)\): \((x, (\Pi_t)_t)\) such that the associated wealth process denoted \((X_t^{x,\Pi})_t\) and defined for all \(t \in [0, T]\) by:

\[
X_t^{x,\Pi} := x + \int_0^t \Pi_u \frac{dS_u}{S_u} = x + \int_0^t \Pi_u (dW_u + \theta_u du).
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- Utility function $U(x) := -e^{-\alpha x}$
- Utility maximisation problem:

$$V(x) := \sup_{\Pi \in A} \mathbb{E}[U(X_T^{x,\Pi} - F)],$$

where $F$ is a $\mathcal{F}_T$ measurable variable (the liability of the investor).
Hu, Imkeller and Müller have showed that it can be reduced to solve a BSDE (Backward Stochastic Differential Equation) of the form:

\[ Y_t = F + \int_t^T h(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad Y_T = F \]

with an explicit formula for the generator \( h \), where \((Y, Z)\) is a pair of adapted processes "regular enough".
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with an explicit formula for the generator \( h \), where \((Y, Z)\) is a pair of adapted processes "regular enough".

The value is given by \( V(x) = -e^{-\alpha(x-Y_0)} \).

Optimal strategies are characterized by \( Z_t \).
If we are in the Markovian case, we consider the Forward BSDE:

\[
\begin{aligned}
X_t &= X_0 + \int_0^t b(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dW_s \\
Y_t &= g(X_T) + \int_t^T h(s, X_s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s, \quad t \in [0, T]
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\end{align*}
\]

**Problem:** Solve numerically this kind of equation.

**Idea:** Get the existence of densities for the $Y$ process and for the $Z$ process with estimates of these densities.
Let $\mathcal{H} = L^2([0, T], dt)$.
Malliavin calculus and densities estimates

- Let $\mathcal{H} = L^2([0, T], dt)$.
- Let $\mathcal{C}$ the space of random variables of the form:

$$F = f(W_{t_1}, ..., W_{t_n}), \ (t_1, ..., t_n) \in [0, T]^n, \ f \in C_b(\mathbb{R}^n).$$
Let $\mathcal{H} = L^2([0, T], dt)$.
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The Malliavin derivative $DF$ of $F$ is the $\mathcal{H}$-valued random variable defined as:

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W_{t_1}, \ldots, W_{t_n})1_{[0,t_i]}.$$
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Let $\mathcal{C}$ the space of random variables of the form:

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We denote by $\mathbb{D}^{1,2}$ the closure of $\mathcal{C}$ with respect to the Sobolev norm $\| \cdot \|_{1,2}$ defined as:

$$\|F\|_{1,2} := \mathbb{E}[|F|^2] + \mathbb{E} \left[ \int_0^T |D_tF|^2 dt \right].$$
Theorem (Bouleau-Hirsch)

Assume that $\|DF\|_{L^2([0,T])} > 0$ a.s., then $F$ has a probability distribution which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, denoted by $\rho_F$. 
Assume that $DF = \Phi_F(W)$ where $\Phi_F : \mathbb{R}^H \to \mathcal{H}$. We set:

$$g_F(x) = \int_0^{+\infty} e^{-u} \mathbb{E} \left[ \mathbb{E}^* \left[ \langle \Phi_F(W), \tilde{\Phi}_F^u(W) \rangle_{L^2([0,T])} \right| F = x \right] du$$
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\]

- Where \( \tilde{\Phi}_u^F(W) := \Phi_F(e^{-u} W + \sqrt{1 - e^{-2u}} W^*) \)
- With \( W^* \) an independent copy of \( W \) defined on a probability space \((\Omega^*, \mathcal{F}^*, \mathbb{P}^*)\)
- Where \( \mathbb{E}^* \) is the expectation under \( \mathbb{P}^* \).
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$$g_F(x) = \int_0^{+\infty} e^{-u} \mathbb{E} \left[ \mathbb{E}^* \left[ \langle \Phi_F(W), \tilde{\Phi}_u^F(W) \rangle_{L^2([0,T])} \right] | F = x \right] \, du$$

- Where $\tilde{\Phi}_u^F(W) := \Phi_F(e^{-u}W + \sqrt{1 - e^{-2u}}W^*)$
- With $W^*$ an independent copy of $W$ defined on a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$
- Where $\mathbb{E}^*$ is the expectation under $\mathbb{P}^*$.

**Theorem (Nourdin-Viens)**

$F$ has a density $\rho_F$ with respect to the Lebesgue measure if and only if the random variable $g_F(F)$ is positive a.s.. In this case, the support of $\rho_F$ is a closed interval of $\mathbb{R}$ and for all $x \in \text{supp}(\rho_F)$:

$$\rho_F(x) = \frac{\mathbb{E}(|F|)}{2g_F(x)} \exp \left( - \int_0^x \frac{udu}{g_F(u)} \right)$$
We make the classical assumption:

\[ h : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R} \text{ is Lipschitz in } (x, y, z) \text{ with Lipschitz constants respectively } k_x, k_y, k_z, \text{ i.e. for all } (x, x', y, y', z, z') \in \mathbb{R}^6: \]

\[ |h(x, y, z) - h(x', y', z')| \leq k_x |x - x'| + k_y |y - y'| + k_z |z - z'|. \]
Theorem (Antonelli-Kohatsu Higa (2005))

Assume that $L$ holds (plus some conditions on the coefficients $b, \sigma, g$ and $h$). We set $K := k_b + k_y + k_\sigma k_z$. Let $t \in (0, T]$. If for some $A \subset \mathbb{R}$ such that $\mathbb{P}(X_T \in A | X_t) > 0$:

\[
\begin{cases}
g e^{-\text{sgn}(g)KT} + h(t) \int_t^T e^{-\text{sgn}(h(s))Ks} ds & \geq 0 \\
g^A e^{-\text{sgn}(g^A)KT} + h(t) \int_t^T e^{-\text{sgn}(h(s))Ks} ds & > 0
\end{cases}
\]  

or

\[
\begin{cases}
\bar{g} e^{-\text{sgn}(\bar{g})KT} + \bar{h}(t) \int_t^T e^{-\text{sgn}(\bar{h}(s))Ks} ds & \leq 0 \\
\bar{g}^A e^{-\text{sgn}(\bar{g}^A)KT} + \bar{h}(t) \int_t^T e^{-\text{sgn}(\bar{h}(s))Ks} ds & < 0,
\end{cases}
\]

is met, then $Y_t$ has a probability distribution that is absolutely continuous with respect to the Lebesgue measure.
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Example (M., Possamaï, Réveillac).

Let $T = 1$, $g(x) = x$, $X = W$, $h(s, x, y, z) = (s - 2)x$.

(2) is not satisfied for any $t \in [0, T]$.

(1) is not satisfied for any $t \in [0, \frac{3 - \sqrt{5}}{2})$.

However for all $t \in [0, 1]$:

$$Y_t = \mathbb{E} \left[ W_1 + \int_t^1 (s - 2) W_s ds \bigg| \mathcal{F}_t \right]$$

$$= W_t (1 + \int_t^1 (s - 2) ds) = W_t \left( -\frac{1}{2} + 2t - \frac{t^2}{2} \right),$$

admits a density with respect to the Lebesgue measure except when $t = 2 - \sqrt{3}$. 
Antonelli and Kohatsu-Higa have proved an other theorem with upper order conditions on $h$ when it does not depend on $z$. Let:

$$\tilde{g}(x) := g'(x) + (T - t)h_x(T, x, g(x)),$$

$$\tilde{h}(s, x, y, z) := -\left(h_{xt} - hh_{xy} + \frac{1}{2}(h_{xxx} + 2zh_{xxy} + z^2h_{xxy})
+ h_y h_x + \sigma_x h_{xx} + z\sigma_x h_{xy}\right)(s, x, y).$$
Theorem (Antonelli-Kohatsu Higa)

Assume that $h$ does not depend on the variable $z$ and suppose that $L)$ holds (plus some conditions on the coefficients). Let $t \in (0, T]$. If for some $A \in \mathbb{R}$ such that $\mathbb{P}(X_T \in A | X_t) > 0$:

\[
\begin{align*}
\tilde{g} e^{-\text{sgn}(\tilde{g}) KT} + \tilde{h}(t) \int_t^T e^{-\text{sgn}(\tilde{h}(s)) KS} (T - s) ds & \geq 0 \\
\tilde{g}^A e^{-\text{sgn}(\tilde{g}^A) KT} + \tilde{h}(t) \int_t^T e^{-\text{sgn}(\tilde{h}(s)) KS} (T - s) ds & > 0
\end{align*}
\]

or

\[
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\tilde{g}^A e^{-\text{sgn}(\tilde{g}^A) KT} + \tilde{h}(t) \int_t^T e^{-\text{sgn}(\tilde{h}(s)) KS} (T - s) ds & < 0
\end{align*}
\]

is met, then $Y_t$ has a probability distribution which is absolutely continuous with respect to the Lebesgue measure.
We study now the existence of a density for $Y$ and Gaussian estimates of this density in the general case.

\[
\begin{align*}
H1 & : \text{For all } \theta \leq T, g \in C_b^1(\mathbb{R}), \ 0 < c \leq g'(X_T)D_\theta X_T \leq C, \text{ a.s.} \\
H2 & : 0 \leq h_x \leq C \\
H3 & : 0 \leq \sigma \leq C \text{ and } ||[b, \sigma]| \leq M\sigma
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**Theorem (Aboura-Bourguin (2012))**

Under the above assumptions $H1),H2) \text{ and } H3)$, $Y_t$ has a density for $t \in (0, T)$ denoted by $\rho_{Y_t}$ satisfying:

$$
\frac{\mathbb{E}[|Y_t - \mathbb{E}[Y_t]|]}{2ct} \exp \left( - \frac{(y - \mathbb{E}[Y_t])^2}{2Ct} \right) \leq \rho_{Y_t}(y)
$$

$$
\rho_{Y_t}(y) \leq \mathbb{E}[|Y_t - \mathbb{E}[Y_t]|] \exp \left( - \frac{(y - \mathbb{E}[Y_t])^2}{2ct} \right).
$$
We study the quadratic case under the following assumption:

Q) $h : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ such that for all $(t, x, y, z) \in ([0, T] \times \mathbb{R}) :$

$$|h(t, x, y, z)| \leq K(1 + |y| + |z|^2)$$

for some $K > 0$.

**Theorem (M., Possamaï, Réveillac)**

Assume that Q) holds with some conditions on the coefficients (but not on the sign of $DX_T$). Fix $t \in (0, T]$. If for some $A \subset \mathbb{R}$ such that $\mathbb{P}(X_T \in A | X_t) > 0$, $g' \geq 0$, $g'_A > 0$ and $h(t) \geq 0$ (resp. $g' \leq 0$, $g'_A < 0$ and $h(t) \leq 0$), then $Y_t$ has a probability distribution which is absolutely continuous with Lebesgue measure.
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Notice that in this theorem we do not need a sign for \( DX_T \).
We study the quadratic case under the following assumption:

\( h : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R} \) such that for all \((t, x, y, z) \in ([0, T] \times \mathbb{R}) : |h(t, x, y, z)| \leq K(1 + |y| + |z|^2) \) for some \( K > 0 \).

Theorem (M., Possamaï, Réveillac)

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Notice that in this theorem we do not need a sign for \( DX_T \).

In the proof we just need to control the norm of \( DX_T \).
Example  Consider the BSDE:

\[ Y_t = W_T + \int_t^T \frac{1}{2} |Z_s|^2 ds - \int_t^T Z_s dW_s. \]

Then:

according to the previous theorem, \( g' \equiv 1 > 0 \) so, \( Y_t \) admits a density for all \( t \in (0, T) \).
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Indeed, by the uniqueness of the solution to this BSDE: \( Y_t = W_t + \frac{1}{2} (T - t), \ Z_t = 1 \) and \( Y_t \) admits a density for all \( t \in (0, T] \).
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Density existence for the \( Z \) process?
Aboura and Bourguin proved that $Z_t$ admits a density under convexity and growth conditions for the terminal condition $g$ and for the generator $h$ when $h(x, y, z) = \tilde{f}(x, y) + \alpha z$ where $\alpha$ is constant.
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They have Gaussian estimates of this density when $h \in C^2_b(\mathbb{R})$ and $g \in C^2_b(\mathbb{R})$. 
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They have Gaussian estimates of this density when $h \in C^2_b(\mathbb{R})$ and $g \in C^2_b(\mathbb{R})$.

Using the fact that $Z_t$ can be represented by the Clark-Ocone formula and after, taking the Malliavin derivative of $Z_t$. 

Thibaut Mastrolia
Density Analysis of BSDEs
We consider the following FBSDE:

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\begin{align*}
X_t &= X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \\
Y_t &= g(X_T) + \int_t^T (\tilde{f}(s, X_s, Y_s) + h(Z_s))ds - \int_t^T Z_s dW_s.
\end{align*}
\]

**Theorem (M., Possamaï, Réveillac)**

Assume that $Q)$ holds with some conditions which ensure that $DX_T > 0$ and $D^2X_T \geq 0$ and assume that $\tilde{f}_x, \tilde{f}_{xx}, \tilde{f}_{xy}, \tilde{f}_{yy} \geq 0$. Then, if there exists $A \subset \mathbb{R}$ such that $\mathbb{P}(X_T \in A|X_t) > 0$, $g' \geq 0$, $g'' \geq 0$, $g'_A > 0$ and $h'' \geq 0$ then, for all $t \in (0, T]$, $Z_t$ has a density with respect to the Lebesgue measure.
Assume now that there exists a function $f \in C^2(\mathbb{R})$ such that for all $t \in [0, T]: X_t = f(t, W_t)$.  

Under this assumption, for all $0 \leq r, s \leq t \leq T$:  

$$D_r Y_t = D_s Y_t$$ 

and  

$$D_r Z_t = D_s Z_t, \quad \mathbb{P}\text{-a.s.}.$$ 

To simplify assume that $\tilde{f} \equiv 0$ (the generator of the BSDE depends only on $z$ through $h$). 

Theorem (M., Possamaï, Réveillac)  

Assume that $Q$) and conditions on coefficients hold. Assume that $h'' \geq 0$ and $(g \circ f)'' \geq 0$. Then, if there exists $A \subset \mathbb{R}$ such that $\mathbb{P}(X_T \in A | X_t) > 0$ and $(g \circ f)''|_A > 0$, then for all $t \in (0, T]$ $Z_t$ has a density with respect to the Lebesgue measure.
Assume now that there exists a function $f \in C^2(\mathbb{R})$ such that for all $t \in [0, T]$: $X_t = f(t, W_t)$.

Under this assumption, for all $0 \leq r, s \leq t \leq T$: $D_r Y_t = D_s Y_t$ and $D_r Z_t = D_s Z_t$, $\mathbb{P}$-a.s.
Densities existence: our contribution for $Z$

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**Theorem (M., Possamaï, Réveillac)**

Assume that Q) and conditions on coefficients hold. Assume that $h'' \geq 0$ and $(g \circ f)'' \geq 0$. Then, if there exists $A \subset \mathbb{R}$ such that $\mathbb{P}(X_T \in A|X_t) > 0$ and $(g \circ f)''_A > 0$, then for all $t \in (0, T]$ $Z_t$ has a density with respect to the Lebesgue measure.
Densities estimates: linear Feynman-Kac’s formula

\[
\begin{aligned}
\frac{\partial_t v(t, x)}{} + b(t, x) \cdot Dv(t, x) + \frac{1}{2} \text{Tr} \left[ \sigma \sigma^T (t, x) D^2 v(t, x) \right] &= 0 \\
v(T, \cdot) &= g(\cdot).
\end{aligned}
\]

"⇔"

\[
\begin{aligned}
dX_{s}^{t, x} &= b(s, X_s^{t, x}) ds + \sigma(s, X_s^{t, x}) dW_s \\
X_t^{t, x} &= x.
\end{aligned}
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\begin{aligned}
\begin{cases}
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    dX_{s,t,x} = b(s, X_{s,t,x}) ds + \sigma(s, X_{s,t,x}) dW_s \\
    X_{t,t,x} = x, \\
    v(t, x) = \mathbb{E}[g(X_{T,t,x}^t)] = P_{t,T} g(x), \quad (v \in C^{1,2})
\end{cases}
\end{aligned}
\]
Densities estimates: semi-linear Feynman-Kac's formula

\[
\begin{aligned}
\begin{cases}
\partial_t v(t, x) + b(t, x) \cdot Dv(t, x) + \frac{1}{2} \text{Tr.}[\sigma \sigma^T(t, x)D^2v(t, x)] = h(t, \cdot, v, \sigma^T \cdot Dv) \\
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\]

" ⇔ "

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\begin{aligned}
\begin{cases}
dX_{s}^{t, x} = b(s, X_{s}^{t, x})ds + \sigma(s, X_{s}^{t, x})dW_s; \quad X_{t}^{t, x} = x. \\
dY_{s}^{t, x} = h(t, X_{s}^{t, x}, Y_{s}^{t, x}, Z_{s}^{t, x})ds - Z_{s}^{t, x}dW_s; \quad Y_{T}^{t, x} = g(X_{T}^{t, x}).
\end{cases}
\end{aligned}
\]
Densities estimates: semi-linear Feynman-Kac’s formula

\[
\begin{cases}
\partial_t v(t, x) + b(t, x) \cdot Dv(t, x) + \frac{1}{2} \text{Tr} [\sigma \sigma^T (t, x) D^2 v(t, x)] = h(t, \cdot, v, \sigma^T \cdot Dv) \\
\n\vspace{0.3cm}
\n\nu(T, \cdot) = g(\cdot).
\end{cases}
\]

"⇔"

\[
\begin{cases}
\begin{aligned}
\quad dX_{s}^{t,x} &= b(s, X_{s}^{t,x}) ds + \sigma(s, X_{s}^{t,x}) dW_s; \quad X_{t}^{t,x} = x.
\quad \\
\quad dY_{s}^{t,x} &= h(t, X_{s}^{t,x}, Y_{s}^{t,x}, Z_{s}^{t,x}) ds - Z_{s}^{t,x} dW_s; \quad Y_{T}^{t,x} = g(X_{T}^{t,x}).
\end{aligned}
\end{cases}
\]

\[
\nu(t, x) = Y_{t}^{t,x}, \quad (\nu \in C^{1,2})
\]
\[
\begin{align*}
\begin{cases}
\partial_t v(t, x) + b(t, x) \cdot Dv(t, x) + \frac{1}{2} \text{Tr} [\sigma \sigma^T (t, x) D^2 v(t, x)] = h(t, \cdot, v, \sigma^T \cdot Dv) \\
v(T, \cdot) = g(\cdot).
\end{cases}
\end{align*}
\]

"⇔"

\[
\begin{align*}
\begin{cases}
\quad dX^{t,x}_s = b(s, X^{t,x}_s) ds + \sigma(s, X^{t,x}_s) dW_s; \quad X^{t,x}_t = x. \\
\quad dY^{t,x}_s = h(t, X^{t,x}_s, Y^{t,x}_s, Z^{t,x}_s) ds - Z^{t,x}_s dW_s; \quad Y^{t,x}_T = g(X^{t,x}_T).
\end{cases}
\end{align*}
\]

\[
v(t, x) = Y^{t,x}_t, \quad (v \in C^{1,2})
\]

Rk.: \( h \equiv 0 \implies Y^{t,x}_s = \mathbb{E}[g(X^{t,x}_T)|\mathcal{F}_s]. \)
Let \( u(t, W_t) := v(t, X_t) \) where \( X_t =: f(t, W_t) \). Then, 
\( u(t, W_t) := Y_t \) and \( u'(t, W_t) = v'(t, X_t)f'(t, W_t) := Z_t \).
Densities estimates

Let $u(t, W_t) := v(t, X_t)$ where $X_t := f(t, W_t)$. Then, $u(t, W_t) := Y_t$ and $u'(t, W_t) = v'(t, X_t) f'(t, W_t) := Z_t$.

Let $\alpha^*_u := \inf\{\alpha > 0, \ u(t, x) = \mathcal{O}(x^\alpha)\}$. 

Assume that $\alpha^*_u' \in (0, +\infty)$ and $\alpha^*_u'' \in (0, +\infty)$. Then, there exist $C > 0$, $\delta > 0$ and $\gamma \in (0, 1)$ such that for all $t \in (0, T]$ the probability distribution of $Z_t$ has a law which admits a density $\rho_{Z_t}$ such that for all $z \in \mathbb{R}$:

$$
E\left[|Z_t - E[Z_t]|^2\right] \leq \rho_{Z_t}(z) \leq E\left[|Z_t - E[Z_t]|^2\right] e^{-\frac{z^2}{2t\delta^2}} \leq \rho_{Z_t}(z) \leq E\left[|Z_t - E[Z_t]|^2\right] e^{-\frac{1}{2t\delta^2} \int_{E[Z_t]} z \ - \ E[Z_t] \ dx} \ 1 + |x|^2 \gamma
$$

A same result holds for $Y_t$ (we just use the first derivative of $u$).
Densities estimates

Let $u(t, W_t) := \nu(t, X_t)$ where $X_t =: f(t, W_t)$. Then, $u(t, W_t) := Y_t$ and $u'(t, W_t) = \nu'(t, X_t)f'(t, W_t) := Z_t$.

Let $\alpha_u^* := \inf\{\alpha > 0, \ u(t, x) = O(x^\alpha)\}$.

P) Assume that $\alpha_{u'}^* \in (0, +\infty)$ and $\alpha_{u''}^* \in (0, +\infty)$.
Densities estimates

Let \( u(t, W_t) := v(t, X_t) \) where \( X_t =: f(t, W_t) \). Then, 
\( u(t, W_t) := Y_t \) and \( u'(t, W_t) = v'(t, X_t)f'(t, W_t) := Z_t \).

Let \( \alpha_u^* := \inf\{\alpha > 0, \ u(t, x) = O(x^\alpha)\} \).

P) Assume that \( \alpha_u^* \in (0, +\infty) \) and \( \alpha_{u'}^* \in (0, +\infty) \).

---

**Theorem (M., Possamaï, Réveillac)**

Assume Q) and P) hold. Suppose that there exists \( \delta > 0 \) such that 
\( (g \circ f)'' \geq \delta > 0 \) and \( h'' \geq 0 \). Then, there exist \( C > 0, \delta > 0 \) and 
\( \gamma \in (0, 1) \) such that for all \( t \in (0, T] \) the probability distribution of \( Z_t \) 
has a law which admits a density \( \rho_{Z_t} \) such that for all \( z \in \mathbb{R} \):

\[
\frac{\mathbb{E}[|Z_t - \mathbb{E}[Z_t]|]}{2tC(1 + |z|^{2\gamma})} e^{-\frac{z^2}{2t\delta^2}} \leq \rho_{Z_t}(z) \leq \frac{\mathbb{E}[|Z_t - \mathbb{E}[Z_t]|]}{2t\delta^2} e^{-\frac{1}{2t\delta^2} \int_0^z \left[ 1 - e^{-\mathbb{E}[Z_t]} \right] \frac{xdx}{1+|x|^{2\gamma}}}.
\]

A same result holds for \( Y_t \) (we just use the first derivative of \( u \)).
Densities estimates

Let \( u(t, W_t) := v(t, X_t) \) where \( X_t =: f(t, W_t) \). Then, \( u(t, W_t) := Y_t \) and \( u'(t, W_t) = v'(t, X_t)f'(t, W_t) := Z_t \).

Let \( \alpha_u^* := \inf \{ \alpha > 0, \ u(t, x) = \mathcal{O}(x^\alpha) \} \).

P) Assume that \( \alpha_u^* \in (0, +\infty) \) and \( \alpha_u^{*'} \in (0, +\infty) \).

Theorem (M., Possamaï, Réveillac)

Assume Q) and P) hold. Suppose that there exists \( \delta > 0 \) such that \( (g \circ f)'' \geq \delta > 0 \) and \( h'' \geq 0 \). Then, there exist \( C > 0, \delta > 0 \) and \( \gamma \in (0, 1) \) such that for all \( t \in (0, T] \) the probability distribution of \( Z_t \) has a law which admits a density \( \rho_{Z_t} \) such that for all \( z \in \mathbb{R} \):

\[
\frac{\mathbb{E}[|Z_t - \mathbb{E}[Z_t]|]}{2tC(1 + |z|^{2\gamma})} e^{-\frac{z^2}{2t\delta^2}} \leq \rho_{Z_t}(z) \leq \frac{\mathbb{E}[|Z_t - \mathbb{E}[Z_t]|]}{2t\delta^2} e^{-\frac{1}{2t\delta^2} \int_0^z -\mathbb{E}[Z_t] \frac{xdx}{1 + |x|^{2\gamma}}}
\]

A same result holds for \( Y_t \) (we just use the first derivative of \( u \)).