ON THE ASYMMETRIC OF THE DENSITY IN
PERTURBED SPDE'S WITH SPATIALLY CORRELATED
NOISE

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Abstract

We consider a general type of perturbed stochastic partial differential equations:

$$Lu^\varepsilon(t, x) = \varepsilon a(u^\varepsilon(t, x)) \dot{F}(t, x) + b(u^\varepsilon(t, x)), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \varepsilon > 0,$$

with null initial conditions, $L$ a second-order partial differential operator and $\dot{F}$ a Gaussian noise, white in time and correlated in space. It has been proved that there exists a smooth density $p^\varepsilon_t(x, y)$, $t > 0, x \in \mathbb{R}^d$, for the law of the solution of above-mentioned equation. Here, we find the Taylor expansion of this density $p^\varepsilon_t(x, y)$ on the diagonal.

Keywords: Malliavin Calculus; spde's; asymptotics of densities; Gaussian noise.

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1 Presentation

In this paper we deal with the following general type of stochastic partial differential equations

$$\begin{cases}
L u^\varepsilon_{t, x} = \varepsilon a(u^\varepsilon_{t, x}) \dot{F}(t, x) + b(u^\varepsilon_{t, x}), \quad t \geq 0, \ x \in \mathbb{R}^d, \ \varepsilon > 0, \\
u^\varepsilon_{0, x} = 0, \quad \forall x \in \mathbb{R}^d, \\
\partial_t u^\varepsilon_{t, x}|_{t=0} = 0, \quad \forall x \in \mathbb{R}^d,
\end{cases}
$$

where $a, b : \mathbb{R} \rightarrow \mathbb{R}^d$ are smooth functions and $L$ is a second-order partial differential operator. This kind of spde's is driven by a noise $F = \{F(\varphi), \ \varphi \in \mathcal{D} (\mathbb{R}^{d+1})\}$ which is assumed to be an $L^2(\Omega, \mathcal{F}, P)$-valued centered Gaussian process with covariance functional

$$J(\varphi, \psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \Gamma(dx) \left[ \varphi(s, \cdot) * \tilde{\psi}(s, \cdot) \right](x),$$

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where \( \tilde{\psi}(s, x) = \psi(s, -x) \) and \( \Gamma \) is a non-negative and non-negative definite tempered measure, therefore symmetric. If we denote by \( \mu \) the spectral measure of \( \Gamma \), that is also a non-negative tempered measure, we have

\[
J(\varphi, \psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \mu(d\xi) \mathcal{F}\varphi(s, \bullet)(\xi) \overline{\mathcal{F}\psi(s, \bullet)}(\xi),
\]

with \( \mathcal{F}\varphi \) the Fourier transform of \( \varphi \) and \( \bar{z} \) the complex conjugate of \( z \). The Gaussian process \( F \) can be extended to a worthy martingale measure as in Dalang (1999) following the ideas given by Walsh (1986) (see also Dalang and Frangos (1998)).

A solution of (1.1) means a jointly measurable adapted process \( \{u^\varepsilon_{t,x}, (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\} \) such that

\[
u^\varepsilon_{t,x} = \varepsilon \int_0^t \int_{\mathbb{R}^d} S_{t-s}(x-y) a(u^\varepsilon_{s,y}) F(ds, dy) + \int_0^t ds \int_{\mathbb{R}^d} dy S_{t-s}(x-y) b(u^\varepsilon_{s,y}),
\]

with \( \varepsilon > 0 \), \( S_s(\bullet) \) the fundamental solution \( L_u = 0 \) satisfying some hypothesis which will be mentioned below and where the stochastic integral in (1.2) is defined with respect the previous worthy martingale measure.

We give the hypothesis assumed along this paper. First, on the coefficients:

(hc1) \( a(\cdot) \) and \( b(\cdot) \) are \( C^\infty \) functions with bounded derivatives of any order.

(hc2) There exists \( a_0 > 0 \) such that \( |a(z)| > a_0 \), for any \( z \in \mathbb{R} \).

Second, on the fundamental solution:

(hs1) The application \( t \to S_t \) is a deterministic function with values in the space of non-negative functions with rapid decrease such that for all \( T > 0 \)

\[
\int_{\mathbb{R}^d} S_t(y) \, dy \leq c_T < \infty, \quad 0 \leq t \leq T,
\]

and

\[
\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F} S_s(\bullet)(\xi)|^2 < +\infty.
\]

(hs2) There exist \( \alpha_1 > \alpha_2 > 0 \) and \( \alpha_3 > 0 \) such that \( \alpha_1 < (2 \alpha_2) \land (\alpha_2 + 2 \alpha_3) \), positive constants \( C_1, C_2 \) and \( C_3 \) and \( t_0 \in [0, t] \) such that, for all \( \rho \in [0, t] \),

\[
C_1 \rho^{\alpha_1} \leq \int_0^\rho ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F} S_s(\bullet)(\xi)|^2 \leq C_2 \rho^{\alpha_2},
\]

and

\[
\int_0^\rho ds \int_{\mathbb{R}^d} dy S_s(y) \leq C_3 \rho^{\alpha_3}.
\]
Dalang (1999) has proved the existence of a unique solution $u_{t,x}^\varepsilon$ of (1.2). As $\varepsilon$ converges to 0, the solution $u_{t,x}^\varepsilon$ tends obviously to the deterministic equation

$$
\Lambda_{t,x} = \int_0^t ds \int_{\mathbb{R}^d} S_{t-s} (x-y) b (\Lambda_{s,y}) , \quad t \geq 0, \ x \in \mathbb{R}^d .
$$

(1.3)

In Márquez-Carreras and Sarrà (2003), assuming (hc$_1$),(hc$_2$),(hs$_1$) and (hs$_2$), we prove that the law of $u_{t,x}^\varepsilon$ is absolutely continuous with respect to Lebesgue’s measure on $\mathbb{R}$ and that its density, denoted by $p_{t,x}^\varepsilon (y)$, is $C^\infty$. As $\varepsilon \to 0$, due to the convergence of the solution of (1.2) to a deterministic function, we expect that the density $p_{t,x}^\varepsilon (y)$ tends to a degenerate density with all the mass in a point $y = \Lambda_{t,x}$. In this paper we study the Taylor expansion of $p_{t,x}^\varepsilon (y)$ at $\varepsilon = 0$ for $y = \Lambda_{t,x}$. More precisely, assuming the same hypothesis as for the proof of the existence of a smooth density, we will find the expansion

$$
p_{t,x}^\varepsilon (y) = \frac{1}{\varepsilon} \left[ \frac{1}{\sqrt{2\pi \varepsilon Eg^2}} + \sum_{\ell=1}^n \frac{m(\ell)}{\ell!} \varepsilon^\ell + \varepsilon^{n+1} R_{n+1} (\varepsilon) \right],
$$

where $g$ will be a centered Gaussian variable, the coefficients $m(\ell)$ will be given and the remainder will be uniformly bounded with respect to $\varepsilon$.

Finally, we will observe that this general result can be applied to some particular examples, for instance, the $d$-dimensional spatial stochastic heat equation, $d \geq 1$, or the $d$-dimensional spatial stochastic wave equation, $d \in \{1, 2\}$. This paper generalizes the study of 1-dimensional spatial stochastic heat equation realized in Márquez-Carreras and Sanz-Solé (1998). In the setting of diffusions this particular case $y = \Lambda_{t,x}$ corresponds to the initial condition and the authors refer to this type of study as the behavior of the density on the diagonal, that is the reason of my tittle (see, for instance, Ben Arous (1989), Léandre (1988)). We also refer the reader to the papers of Léandre (1986), Léandre and Russo (1995) and Watanabe (1987) for some interesting results related to this type of study. More specifically, Léandre (1986) obtains estimates of the density for some processes by means of two renormalisations and using Malliavin’s calculus, Léandre and Russo (1995) study the behaviour in small time of the density of the robust Zakai equation under the weak Hörmander’s hypothesis and, finally, Watanabe (1987) carries out a presentation in a more general framework and then it is applied to Itô functionals in order to obtain asymptotic results for heat kernels.

2 The framework

Let $\mathcal{E}$ be the space of measurable functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that

$$
\int_{\mathbb{R}^d} \Gamma (dx) [\varphi (\bullet) * \tilde{\psi} (\bullet)] (x) < \infty ,
$$

where
where we write $H$ for the operations of the derivative operator $\partial_t$. As in Nualart (1995), the Sobolev spaces $H^k$ are defined by means of it-erations of the derivative operator $\partial_t$. Let $\mathcal{H}$ be the completion of $\mathcal{E}$ and $\mathcal{H}_T = L^2([0,T]; \mathcal{H})$. This last space is a real separable Hilbert one isomorphic to the reproducing kernel Hilbert space of the centered Gaussian noise (that is, if $\varphi, \psi \in \mathcal{D}([0,T] \times \mathbb{R}^d)$, $E[F(\varphi) F(\psi)] = \langle \varphi, \psi \rangle_{\mathcal{H}_T}$). This Gaussian noise $F$ can be identified with a Gaussian process $\{W(h), h \in \mathcal{H}_T\}$ as follows: let $\{e_j, j \geq 0\} \subset \mathcal{E}$ be a CONS of the Hilbert space $\mathcal{H}$, then

$$W_j(t) = \int_0^t e_j(x) F(ds, dx), \ j \in \mathbb{N}, \ t \in [0,T],$$

is a sequence of independent standard Brownian motions such that

$$F(\varphi) = \sum_{j=0}^{\infty} \int_0^T \langle \varphi(s, \bullet), e_j(\bullet) \rangle_{\mathcal{H}} dW_j(s), \ \varphi \in \mathcal{D}([0,T] \times \mathbb{R}^d),$$

and, for $h \in \mathcal{H}_T$,

$$W(h) = \sum_{j=0}^{\infty} \int_0^T \langle h(s, \bullet), e_j(\bullet) \rangle_{\mathcal{H}} dW_j(s).$$

As in Nualart (1995), the Sobolev spaces $\mathbb{D}^{k,p}$ are defined by means of it-erations of the derivative operator $\partial_t$. For a random variable $X$, $D^k X$ de-fines a $\mathcal{H}^k_T$-valued random variable whenever it exists. For $h \in \mathcal{H}_T$, set $D_h X = \langle D X, h \rangle_{\mathcal{H}_T}$ and for $r \in [0,T]$, $D_r X$ defines an element of $\mathcal{H}$, which is denoted by $D_r X$. Then, for any $h \in \mathcal{H}_T$,

$$D_h X = \int_0^T \langle D_r \bullet X, h(r) \rangle_{\mathcal{H}} dr,$$

and we write $D_r X = \langle D_r \bullet X, \varphi \rangle_{\mathcal{H}}$ for $\varphi \in \mathcal{H}$.

Using this framework and assuming $(hs_1)$ and $(hc_1)$, we have showed in Márquez-Carreras and Sarrà (2003) that, for any $t \geq 0$, $x \in \mathbb{R}^d$, $\varepsilon > 0$, $u_{t,x}^\varepsilon$ belongs to $\mathbb{D}^\infty = \bigcap_{p \geq 1} \bigcap_{k \geq 1} \mathbb{D}^{k,p}(\mathbb{R})$ and that, for any $k \geq 1$, $p \in [1, \infty)$,

$$\sup_{\varepsilon \in (0,1)} ||u_{t,x}^\varepsilon||_{k,p} < \infty,$$  \hspace{1cm} (2.4)

where $|| \cdot ||_{k,p}$ denotes the norm of the Sobolev space $\mathbb{D}^{k,p}$.

Moreover, for any $\varphi \in \mathcal{H}$, the derivatives satisfy, if $r \in [0,t]$

$$D_{r,\varphi} u_{t,x}^\varepsilon = \varepsilon \langle S_{t-r}(x - \bullet) a(u_{r,\bullet}^\varepsilon), \varphi \rangle_{\mathcal{H}}$$

$$+ \varepsilon \int_r^t \int_{\mathbb{R}^d} S_{t-s}(x - y) \ D_{r,\varphi} u_{s,y}^\varepsilon a'(u_{s,y}^\varepsilon) F(ds, dy)$$

$$+ \int_r^t \int_{\mathbb{R}^d} dy \ S_{t-s}(x - y) \ D_{r,\varphi} u_{s,y}^\varepsilon b'(u_{s,y}^\varepsilon),$$
and $D_{\tau,\nu}^\varepsilon u_{t,x}^{\varepsilon} = 0$, if $r > t$. In order to study the Malliavin derivatives we borrow the notations of Millet and Sanz-Solé (1999). Let $k \in \mathbb{N}$, $A_k = \{ \sigma_i = (r_i, \varphi_i) \in \mathbb{R}^+ \times \mathcal{H}, \ i = 1, \ldots, k \}$, $\bigvee_i r_i = r_1 \lor \cdots \lor r_k$, $\sigma = (\sigma_1, \ldots, \sigma_k)$ and $\hat{\sigma}_i = (\sigma_{i-1}, \sigma_i, \sigma_{i+1}, \ldots, \sigma_k)$. Let $P_m$ be the set of $m$ disjoints subsets $p_1, \ldots, p_m$, $m = 1, \ldots, k$, which are partitions of $A_k$, denote by $|p_i|$ the cardinal of $p_i$.

For $X$ belonging to $D^{k,p}$, $k \geq 1$, $p \geq 2$ and a smooth function $f$, rules of Malliavin’s derivatives yield

$$D_{\sigma}^k f(X) = \sum_{m=1}^{k} \sum_{P_m} c_m f^{(m)}(X) \prod_{i=1}^{m} D_{|p_i|} X,$$

with positive coefficients $c_m$, $m \geq 2$, and $c_1 = 1$. There

$$D_{\sigma}^k u_{t,x}^{\varepsilon} = \sum_{i=1}^{k} \langle S_{t-r_i} (x - \bullet) D_{\hat{\sigma}_i}^{k-1} a (u_{r_i,x}^{\varepsilon}, \varphi_i) \rangle_{\mathcal{H}}$$

$$+ \int_{\bigvee_i r_i}^{t} \int_{\mathbb{R}^d} S_{t-s} (x - y) [\Delta_{\sigma} (a, u_{s,y}^{\varepsilon}) F(ds, dy) + \Delta_{\sigma} (b, u_{s,y}^{\varepsilon}) ds dy]$$

$$+ \int_{\bigvee_i r_i}^{t} \int_{\mathbb{R}^d} S_{t-s} (x, y) D_{\sigma}^k u_{s,y}^{\varepsilon} [a' (u_{s,y}^{\varepsilon}) F(ds, dy) + b' (u_{s,y}^{\varepsilon}) ds dy],$$

where $\Delta_{\sigma} (f, X) = D_{\sigma}^k f(X) - f'(X) D_{\sigma}^k X$.

3 Preliminary lemmas

In this section we obtain some important previous results which will be needed in the proof of the main theorem of this paper.

**Lemma 3.1** Suppose $(\text{hs}_1)$ and $(\text{hc}_1)$. There exists a version of $\{u_{t,x}^{\varepsilon}, \ \varepsilon \in (0,1)\}$ which is a $C^\infty$ function with respect to $\varepsilon$. If we denote the derivatives $\partial_{x_j}^\varepsilon u_{t,x}^{\varepsilon}$ by $u_{t,x}^{\varepsilon,j}$, $j \geq 1$, then their are solutions of the following stochastic partial differential equations:

$$u_{t,x}^{\varepsilon,1} = \int_0^t \int_{\mathbb{R}^d} S_{t-s} (x - y) \left[ a (u_{s,y}^{\varepsilon}) + \varepsilon a' (u_{s,y}^{\varepsilon}) u_{s,y}^{\varepsilon,1} \right] F(ds, dy)$$

$$+ \int_0^t ds \int_{\mathbb{R}^d} dy \ S_{t-s} (x - y) \ b' (u_{s,y}^{\varepsilon}) u_{s,y}^{\varepsilon,1},$$

and, for $j \geq 2$,

$$u_{t,x}^{\varepsilon,j} = \mathcal{O}_{t,x}^{\varepsilon,j-1} + \int_0^t \int_{\mathbb{R}^d} S_{t-s} (x - y) \ c_j (j) \left[ \varepsilon a' (u_{s,y}^{\varepsilon}) u_{s,y}^{\varepsilon,j} \right] F(ds, dy)$$

$$+ b' (u_{s,y}^{\varepsilon}) u_{s,y}^{\varepsilon,j} ds dy, \quad (3.2)$$
where, if we use the shorthand \( \sum_{k=i}^{\ell} \sum_{\beta_1 + \cdots + \beta_k = \ell} \sum_{\beta_1, \ldots, \beta_k \geq 1} \),

\[
\mathcal{O}_{t,x}^{\varepsilon,j-1} = \int_0^t \int_{\mathbb{R}^d} S_{t-s}(x-y) \left[ \sum_{k=1}^j k_{j-1}(\beta_1, \ldots, \beta_k) a^{(k)}(u_{s,y}^\varepsilon) \right. \\
\times \prod_{n=1}^k u_{s,y}^{\varepsilon,\beta_n} F(ds, dy) \\
\left. + \sum_{(2,j)} c_j(\beta_1, \ldots, \beta_k) \left[ \varepsilon a^{(k)}(u_{s,y}^\varepsilon) \prod_{n=1}^k u_{s,y}^{\varepsilon,\beta_n} F(ds, dy) \\
+ b^{(k)}(u_{s,y}^\varepsilon) \prod_{n=1}^k u_{s,y}^{\varepsilon,\beta_n} ds dy \right] \right],
\]

with \( c_j(\beta_1, \ldots, \beta_k) \) and \( k_j(\beta_1, \ldots, \beta_k) \) are computed by induction.

Moreover, for \( j \geq 1 \),

\[
\lim_{\varepsilon \downarrow 0} u_{t,x}^{\varepsilon,j} = u_{t,x}^{0,j}, \quad a.s.,
\]

where \( u_{t,x}^{0,j} \) are the solutions of the stochastic partial differential equations (3.1) and (3.2) replacing \( \varepsilon \) by 0.

**Proof.** For \( \varepsilon \in [0,1] \) and \( \delta \) such that \( \varepsilon + \delta \in [0,1] \), we have

\[
\begin{align*}
\left( u_{t,x}^{\varepsilon+\delta} - u_{t,x}^{\varepsilon} \right) &= \delta A_{1,t,x}^{\varepsilon,\delta} + \varepsilon A_{2,t,x}^{\varepsilon,\delta} + A_{3,t,x}^{\varepsilon,\delta},
\end{align*}
\]

with

\[
\begin{align*}
A_{1,t,x}^{\varepsilon,\delta} &= \int_0^t \int_{\mathbb{R}^d} S_{t-s}(x-y) a(u_{s,y}^{\varepsilon+\delta}) F(ds, dy), \\
A_{2,t,x}^{\varepsilon,\delta} &= \int_0^t \int_{\mathbb{R}^d} S_{t-s}(x-y) \left[ a(u_{s,y}^{\varepsilon+\delta}) - a(u_{s,y}^{\varepsilon}) \right] F(ds, dy), \\
A_{3,t,x}^{\varepsilon,\delta} &= \int_0^t ds \int_{\mathbb{R}^d} dy S_{t-s}(x-y) \left[ b(u_{s,y}^{\varepsilon+\delta}) - b(u_{s,y}^{\varepsilon}) \right].
\end{align*}
\]

Burkholder’s and Hölder’s inequalities, (2.4), (hc1),(hs1) and Gronwall’s lemma yield

\[
\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} E \left| u_{t,x}^{\varepsilon+\delta} - u_{t,x}^{\varepsilon} \right|^p \leq c |\delta|^p.
\]

Kolmogorov’s theorem implies the existence of a continuous version of \( \{u_{t,x}^\varepsilon, \varepsilon \in [0,1]\} \) and

\[
\lim_{\varepsilon \downarrow 0} u_{t,x}^\varepsilon = \Lambda_{t,x},
\]
where $\Lambda_{t,x}$ is defined in (1.3).

Following the ideas given, for instance, by Kunita (1998) we now show some steps of the differentiability of order $j = 1$. For any $\varepsilon \in (0,1)$ and $\delta \in \mathbb{R} - \{0\}$ such that $0 \leq \varepsilon + \delta \leq 1$, set

$$\chi^{\varepsilon,\delta}_{t,x} = \frac{u^{\varepsilon,\delta}_{t,x} - u^{\varepsilon}_{t,x}}{\delta}.$$  

The mean value theorem gives

$$\chi^{\varepsilon,\delta}_{t,x} = A^{\varepsilon,\delta}_{t,x} + \int_0^t \int_{\mathbb{R}^d} S_{t-s}(x-y) \left[ \varepsilon \int_0^1 a'(z_{s,y}^{\varepsilon,\delta}(\lambda)) \, d\lambda \chi^{\varepsilon,\delta}_{s,y} F(ds,dy) ight. 
+ \left. \int_0^1 b'(Z_{s,y}^{\varepsilon,\delta}(\lambda)) \, d\lambda \chi^{\varepsilon,\delta}_{s,y} ds dy \right],$$  

with $Z_{s,y}^{\varepsilon,\delta}(\lambda) = u^{\varepsilon}_{s,y} + \lambda(u^{\varepsilon+\delta}_{s,y} - u^{\varepsilon}_{s,y}).$

Using the same arguments as in (3.5) we obtain, for any $p \in [1,\infty)$,

$$\sup_{\varepsilon,\delta,x,t} E |\chi^{\varepsilon,\delta}_{t,x}|^p \leq C,$$  

(3.6)

for some positive constant $C$ and where the supreme is over $\varepsilon \in [0,1]$, $\delta \neq 0$, $0 \leq \varepsilon + \delta \leq 1$, $x \in \mathbb{R}^d$ and $t \in [0,T]$.

Burkholder’s and Hölder’s inequalities, estimates (3.5) and (3.6), the hypothesis $(h_{s1})$ and $(h_{c1})$ and Gronwall’s lemma yield, for any $p > 1$,

$$\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} E |\chi^{\varepsilon,\delta}_{t,x} - \chi^{\varepsilon',\delta'}_{t,x}|^p \leq C \left[ |\varepsilon - \varepsilon'|^p + |\delta - \delta'|^p \right],$$  

for some positive constant $C$. Then, Kolmogorov’s theorem again implies that $u^{\varepsilon}_{t,x}$ has a differentiable version. We can easily check

$$\lim_{\delta \downarrow 0} E |\chi^{\varepsilon,\delta}_{t,x} - u^{\varepsilon,1}_{t,x}|^2 = 0,$$

and $\partial_\varepsilon u^{\varepsilon}_{t,x} = u^{\varepsilon,1}_{t,x}$, a.s. We can argument as before to obtain that $\{u^{\varepsilon,1}_{t,x}, \varepsilon \in [0,1]\}$ has a continuous version and

$$\lim_{\varepsilon \downarrow 0} u^{\varepsilon,1}_{t,x} = u^{0,1}_{t,x}, \text{ a.s.}$$

We can generalize all these results by induction on $j$. □

Using Kolmogorov’s criterium we can get, for any $j \geq 1$,

$$E \left[ \sup_{0 \leq t \leq 1} |u^{\varepsilon,j}_{t,x}| \right] < \infty.$$

(3.7)
The process \( u_{t,x}^\varepsilon \) is deterministic at \( \varepsilon = 0 \), for this reason we consider
\[
\hat{u}_{t,x}^\varepsilon = \frac{u_{t,x}^\varepsilon - \Lambda_{t,x}}{\varepsilon}, \quad 0 < \varepsilon \leq 1,
\]
and we denote \( \partial_j \hat{u}_{t,x}^\varepsilon \) by \( \hat{u}_{t,x}^{\varepsilon,j} \).

The relationship between the derivatives of \( u_{t,x}^\varepsilon \) and \( \hat{u}_{t,x}^\varepsilon \) is given by
\[
\hat{u}_{t,x}^{\varepsilon,j} = \frac{1}{j+1} \left[ u_{t,x}^{0,j+1} + \varepsilon \int_0^1 (1 - \tau)^{j+1} u_{t,x}^{\tau,j+2} d\tau \right].
\]

See Márquez-Carreras and Sanz-Solé (1998) for this proof.

Then, assuming \((hc_1)\) and \((hs_1)\), we have, for any \( j \geq 1 \),
\[
\hat{u}_{t,x}^{0,j} := \lim_{\varepsilon \downarrow 0} \hat{u}_{t,x}^{\varepsilon,j} = \frac{1}{j+1} u_{t,x}^{0,j+1}, \quad \text{a.s.} \tag{3.10}
\]

In the main proof we will also need

**Lemma 3.2** Assume \((hc_1)\) and \((hs_1)\). For any \( j \geq 1 \), \( k \in \mathbb{N} \), \( p \in [1, \infty) \), we have
\[
\sup_{0 < \varepsilon \leq 1} \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} ||\hat{u}_{t,x}^{\varepsilon,j}||_{k,p} \leq C. \tag{3.11}
\]

**Proof.** We only need to prove \((3.11)\) replacing \( \hat{u} \) by \( u \).

As in Márquez-Carreras and Sarrà (2003) we can easily check
\[
\sup_{0 \leq \varepsilon \leq 1} \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} E \left( ||u_{t,x}^{\varepsilon,j}||^p \right) \leq C'. \tag{3.12}
\]

The proof of \((3.11)\) for any \( j \geq 1 \), \( k \in \mathbb{N} \), \( p \in [1, \infty) \) is very tedious, we give the details for \( j = 1 \), \( k = 1 \) and any \( p \in [1, \infty) \), the other cases are checked by induction on \( j \) and \( k \). For any \( \varphi \in \mathcal{H} \) and \( r \in [0,T] \)
\[
D_{r,\varphi} u_{t,x}^{\varepsilon,1} = \mathbb{1}_{\{r < t\}} \left[ (S_{t-r}(t - \bullet) [a(u_{r,\bullet}^\varepsilon) + \varepsilon a'(u_{r,\bullet}^\varepsilon) u_{t,x}^{\varepsilon,1}])_j , \varphi \right]_{\mathcal{H}} + 
\]
\[
+ \int_0^t \int_{\mathbb{R}^d} S_{t-s} (x-y) [a'(u_{s,y}^\varepsilon) + \varepsilon a''(u_{s,y}^\varepsilon) u_{s,y}^{\varepsilon,1}] D_{r,\varphi} u_{s,y}^{\varepsilon,1} F(ds,dy) + 
\]
\[
+ \int_0^t \int_{\mathbb{R}^d} S_{t-s} (x-y) \varepsilon a'(u_{s,y}^\varepsilon) D_{r,\varphi} u_{s,y}^{\varepsilon,1} F(ds,dy) + 
\]
\[
+ \int_0^t \int_{\mathbb{R}^d} S_{t-s} (x-y) b''(u_{s,y}^\varepsilon) u_{s,y}^{\varepsilon,1} D_{r,\varphi} u_{s,y}^{\varepsilon,1} ds dy + 
\]
\[
+ \int_0^t \int_{\mathbb{R}^d} S_{t-s} (x-y) b'(u_{s,y}^\varepsilon) D_{r,\varphi} u_{s,y}^{\varepsilon,1} ds dy \right].
\]
Then, we have
\[
\| D u_t^{\varepsilon,1} \|_{L^p(\Omega; \mathcal{H}_T)} \leq C \sum_{i=1}^5 \gamma_{t,x}^{\varepsilon,i}, \tag{3.13}
\]
with
\[
\gamma_{t,x}^{\varepsilon,1} = \left\| S_{t-s} (x - \bullet) \left[ a (u_{s,\bullet}^{\varepsilon}) + \varepsilon a' (u_{s,\bullet}^{\varepsilon}) u_{s,\bullet}^{\varepsilon,1} \right] \right\|_{L^p(\Omega; \mathcal{H}_T)}^p,
\]
\[
\gamma_{t,x}^{\varepsilon,2} = \left\| \int_0^t \int_{\mathbb{R}^d} S_{t-s} (x - y) \left[ a' (u_{s,y}^{\varepsilon}) + \varepsilon a'' (u_{s,y}^{\varepsilon}) u_{s,y}^{\varepsilon,1} D u_{s,y}^{\varepsilon} F (ds, dy) \right] \right\|_{L^p(\Omega; \mathcal{H}_T)}^p,
\]
\[
\gamma_{t,x}^{\varepsilon,3} = \left\| \int_0^t \int_{\mathbb{R}^d} S_{t-s} (x - y) \varepsilon a' (u_{s,y}^{\varepsilon}) D u_{s,y}^{\varepsilon,1} F (ds, dy) \right\|_{L^p(\Omega; \mathcal{H}_T)}^p,
\]
\[
\gamma_{t,x}^{\varepsilon,4} = \left\| \int_0^t \int_{\mathbb{R}^d} S_{t-s} (x - y) b'' (u_{s,y}^{\varepsilon}) u_{s,y}^{\varepsilon,1} D, u_{s,y}^{\varepsilon} ds dy \right\|_{L^p(\Omega; \mathcal{H}_T)}^p,
\]
\[
\gamma_{t,x}^{\varepsilon,5} = \left\| \int_0^t \int_{\mathbb{R}^d} S_{t-s} (x - y) b' (u_{s,y}^{\varepsilon}) D u_{s,y}^{\varepsilon,1} ds dy \right\|_{L^p(\Omega; \mathcal{H}_T)}^p.
\]

Secondly we study the term \( \gamma_{t,x}^{\varepsilon,3} \). Set, for any \( \bar{t} \in [0, t] \),
\[
M_{\bar{t}} = \int_0^{\bar{t}} \int_{\mathbb{R}^d} S_{t-s} (x - y) \varepsilon a' (u_{s,y}^{\varepsilon}) D u_{s,y}^{\varepsilon,1} F (ds, dy).
\]
This process \( M_{\bar{t}} \) is a continuous \( \mathcal{H}_T \)-valued \( \mathcal{F}_{\bar{t}} \)-martingale and
\[
\langle M \rangle_{\bar{t}} = \sum_{i \geq 0} \| \mathbb{I}_{[0,\bar{t})} (\ast) S_{t-s} (x - \bullet) \varepsilon a' (u_{s,\bullet}^{\varepsilon}) D_{e_i} u_{s,\bullet}^{\varepsilon,1} \|_{\mathcal{H}_T}^2.
\]
is the unique predictable increasing process such that \( \langle M \rangle_0 = 0 \) and \( \| M_{\bar{t}} \|_{\mathcal{H}_T}^2 = \langle M \rangle_{\bar{t}} \) is a real \( \mathcal{F}_{\bar{t}} \)-martingale (\( \mathcal{F}_{\bar{t}} \) is the initial martingale measure associated to Dalang-Wash theory). Burkholder’s inequality for a \( \mathcal{H}_T \)-valued martingale, Parseval’s identity, Schwarz’s inequality applied to the scalar product
in $H_T$, Hölder’s inequality as before and assumption (hc$_1$) imply
\[
\gamma^{e,3}_{t,x} \leq C \left| \sum_{i \geq 0} \int_0^t ds \int_{\mathbb{R}^d} \Gamma (dy) \int_{\mathbb{R}^d} dy' \, S_{t-s}(x - y + y') \, D_{e_i} u^{e,1}_{s,y-y'} \right|^{p/2} \times \varepsilon \alpha'(u^{e}_{s,x-y'}) \, S_{t-s}(x + y') \, \alpha'(u^{e}_{s,x}) \right|^{p/2} \leq C \left| \int_0^t ds \int_{\mathbb{R}^d} \Gamma (dy) \int_{\mathbb{R}^d} dy' \, S_{t-s}(x - y + y') \, (D u^{e,1}_{s,y-y'} , D u^{e,1}_{s,y'})_{H_T} \right|^{p/2-1} \times \sup_{x \in \mathbb{R}^d} \sup_{0 \leq \tau \leq s} E \| D u^{e,1}_{\tau,x} \|^p_{H_T} ds .
\]

We can apply the same arguments together to (2.4) in order to deal with $\gamma^{e,2}_{t,x}$. Finally, we analyse the term $\gamma^{e,5}_{t,x}$ (the fourth term is similar to this one). Hölder’s inequality and (hc$_1$) give
\[
\gamma^{e,5}_{t,x} \leq C \left| \int_0^t ds \int_{\mathbb{R}^d} \mu (d\xi) |F S_s (\bullet) (\xi)|^2 \right|^{p/2-1} \times \sup_{x \in \mathbb{R}^d} \sup_{0 \leq \tau \leq s} E \| D u^{e,1}_{\tau,x} \|^p_{H_T} ds .
\]

Then, (3.14), (3.15) and (3.16), the hypothesis (hs$_1$) and Gronwall’s lemma imply (3.11) for $j = 1$ and $k = 1$. 

We need to enunciated a last ingredient proved in Márquez-Carreras and Sarrà (2003). Assume (hc$_1$), (hc$_2$), (hs$_1$) and (hs$_2$). For any $t > 0$ and $x \in \mathbb{R}^d$, there exists a finite and positive constant $C$ such that
\[
\left\| (D u^{e}_{t,x} , D u^{e}_{t,x})^{-1} \right\|_p \leq C \varepsilon^{-2} , \quad \text{for all } p > 1 , \ \varepsilon \in [0,1] .
\]

This bound yields, for some positive constant $C$,
\[
\sup_{0 < \varepsilon \leq 1} E \left( \left\| (D \hat{u}^{e}_{t,x} , D \hat{u}^{e}_{t,x})^{-1} \right\|_p \right) \leq C , \quad \text{for all } p > 1 .
\]

Using the same arguments as in (3.17) we can obtain that the centered Gaussian variable $u^{0,1}_{t,x}$ satisfies $E |u^{0,1}_{t,x}|^2 > 0$. 

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4 The main theorem

Before proving the main theorem we start this section by giving some im-
portant notions and results on Malliavin Calculus. We refer the reader to
Nualart (1998, 1995) for more information about these aspects.

Let $X : \Omega \rightarrow \mathbb{R}$ be a Wiener functional and we denote by $\Upsilon_X$ its
Malliavin matrix. The random variable $X$ is said to be non-degenerate
if $X \in D_\infty$ and $\Upsilon_X^{-1}X \in \bigcap_{p \geq 1} L^p(\Omega)$. Consider non-degenerate variables
$X, Y \in D_\infty$ and a smooth function $f$. For any $j \geq 1$, there exists a random
variable $H_j(X, Y) \in D_\infty$ such that

$$E[f^{(j)}(X)Y] = E[f(X)H_j(X, Y)], \quad (4.1)$$

where $H_j(X, Y)$ is found recursively as follows

$$H_1(X, Y) = D^*(X \Upsilon_X^{-1}DY),$$
$$H_j(X, Y) = H_{j-1}(X, H_{j-1}(X, Y)), \quad (4.2)$$

and $D^*$ denotes the Skorohod integral (see, for instance, Nualart (1998)).

It is not difficult to generalize Proposition 3.2.2 of Nualart (1998) in the
following sense: for any $p > 1$ and $k, j \in \mathbb{N}$, there exist a positive constant
$C(p, k, j)$ and positive real numbers $k_2, q_1, q_2, q_3, k_3, q_4, l_1, l_2, l_3$, depending on
$p, k$ and $j$, such that

$$\|H_j(X, Y)\|_{k,p} \leq C(p, k, j)\|\Upsilon_X^{-1}\|_{q_1}^{l_1} \|X\|_{k_2, q_2}^{l_2} \|Y\|_{k_3, q_3}^{l_3}. \quad (4.2)$$

Finally, using the same arguments as in Corollary 3.2.1 of Nualart (1998)
we obtain that the Radon measure defined by $f \mapsto E(f(X)Y)$ has a
bounded $C^\infty$ density

$$p(y) = E\left(\mathbb{1}_{\{X > y\}} H_1(X, Y)\right). \quad (4.3)$$

**Theorem 4.1** Assume (hc$_1$), (hc$_2$), (hs$_1$) and (hs$_2$). Let $p_{t,x}^\varepsilon(y)$ be the
density of the law of the process $u_{t,x}^\varepsilon$ (solution to (1.2)). Then, for any
$(t, x) \in (0, T] \times \mathbb{R}^d, \varepsilon > 0$,

$$p_{t,x}^\varepsilon(y) = \frac{1}{\varepsilon} \left[ \frac{1}{\sqrt{2\pi E[(u_{t,x}^{0,1})^2]}} + \sum_{\ell=1}^{n} \frac{m(\ell)}{\ell!} \varepsilon^{\ell} + \varepsilon^{n+1} R_{n+1}(\varepsilon) \right], \quad (4.4)$$

where $y = \Lambda_{t,x}$, $u_{t,x}^{0,1}$ is defined by means of (3.1) and (3.4); the coefficients
$m(\ell)$ for odd $\ell$ are null and for even $\ell$

$$m(\ell) = E\left[\mathbb{1}_{\{u_{t,x}^{0,1} > 0\}} M(\ell)\right],$$

$$p_{t,x}^\varepsilon(y) = \frac{1}{\varepsilon} \left[ \frac{1}{\sqrt{2\pi E[(u_{t,x}^{0,1})^2]}} + \sum_{\ell=1}^{n} \frac{m(\ell)}{\ell!} \varepsilon^{\ell} + \varepsilon^{n+1} R_{n+1}(\varepsilon) \right], \quad (4.4)$$

$$\Lambda_{t,x}, u_{t,x}^{0,1} \text{ is defined by means of (3.1) and (3.4); the coefficients}$$

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where $y = \Lambda_{t,x}$, $u_{t,x}^{0,1}$ is defined by means of (3.1) and (3.4); the coefficients
$m(\ell)$ for odd $\ell$ are null and for even $\ell$

$$m(\ell) = E\left[\mathbb{1}_{\{u_{t,x}^{0,1} > 0\}} M(\ell)\right],$$
with

\[ M(\ell) = \sum c_\ell(\beta_1, \ldots, \beta_k) H_{k+1} \left( u_{t,x}^{0,1}, \prod_{j=1}^{k} \frac{1}{\beta_j + 1} u_{t,x}^{0,\beta_j+1} \right). \]  

Moreover

\[ \sup_{\varepsilon \in (0,1]} |R_{n+1}(\varepsilon)| < \infty. \]

**Proof.** Let \( f \) be a \( C^\infty \) function with bounded support included in \( \mathbb{R} \). Lemma 3.1 implies that the mapping

\[ \varepsilon \in (0,1) \rightarrow f(\hat{u}_{t,x}^\varepsilon) \in \mathbb{R} \]

is \( C^\infty \), a.s. (recall that \( \hat{u}_{t,x}^\varepsilon \) is defined in (3.8)). Then, Taylor’s expansion and Leibniz’s formula together with (3.8)-(3.10) yield

\[
\begin{align*}
\hat{f}(\hat{u}_{t,x}^\varepsilon) &= f(u_{t,x}^{0,1}) + \sum_{\ell=1}^{n} \frac{\varepsilon^\ell}{\ell!} \frac{\partial^\ell}{\partial \varepsilon^\ell} (f(\hat{u}_{t,x}^\varepsilon))|_{\varepsilon=0} \\
& \quad + \varepsilon^{n+1} \int_0^1 \frac{(1-\eta)^n}{n!} \frac{\partial^n}{\partial \eta^n} \left( f(\hat{u}_{t,x}^\eta) \right) |_{\eta=\varepsilon \eta} d\eta \\
& = f(u_{t,x}^{0,1}) + \sum_{\ell=1}^{n} \frac{\varepsilon^\ell}{\ell!} \sum c_\ell(\beta_1, \ldots, \beta_k) f(k)(u_{t,x}^{0,1}) \\
& \quad \times \prod_{s=1}^{k} \frac{u_{t,x}^{0,\beta_s+1}}{\beta_s + 1} + \varepsilon^{n+1} \int_0^1 \frac{(1-\eta)^n}{n!} \sum c_{n+1}(\beta_1, \ldots, \beta_k) \\
& \quad \times f(k)(\hat{u}_{t,x}^{\eta \varepsilon}) \prod_{s=1}^{k} \hat{u}_{t,x}^{\eta \varepsilon,\beta_s} d\eta. 
\end{align*}
\]

Taking expectations and using (4.1) we have

\[
\begin{align*}
E \left[ \hat{f}(\hat{u}_{t,x}^\varepsilon) \right] &= E \left[ f(u_{t,x}^{0,1}) \right] + \sum_{\ell=1}^{n} \frac{\varepsilon^\ell}{\ell!} \sum c_\ell(\beta_1, \ldots, \beta_k) \\
& \quad \times E \left[ f(u_{t,x}^{0,1}) H_k \left( u_{t,x}^{0,1}, \prod_{s=1}^{k} \frac{u_{t,x}^{0,\beta_s+1}}{\beta_s + 1} \right) \right] \\
& \quad + \varepsilon^{n+1} \int_0^1 \frac{(1-\eta)^n}{n!} \sum c_{n+1}(\beta_1, \ldots, \beta_k) \\
& \quad \times E \left[ f(\hat{u}_{t,x}^{\eta \varepsilon}) H_k \left( \hat{u}_{t,x}^{\eta \varepsilon}, \prod_{s=1}^{k} \hat{u}_{t,x}^{\eta \varepsilon,\beta_s} \right) \right] d\eta. \quad (4.6)
\end{align*}
\]
Lemma 3.2 and the fact that the random variables $u_{t,x}^{0,1}$ and $\hat{u}_{t,x}^{\eta \varepsilon}$ are non-degenerate allow us to ensure that the Radon measure defined by the expectations of (4.6) have $C^\infty$ bounded densities. Now, using (4.3) we get

$$\hat{p}_{t,x}^\varepsilon(y) = p_{t,x}^{0,1}(y) + \sum_{\ell=1}^n E \left[ \frac{\mathbb{1}\{u_{t,x}^{0,1} > y\} M(\ell)}{\ell!} \right] \varepsilon^\ell + \varepsilon^{n+1} R_{n+1}(y, \varepsilon),$$

where $p_{t,x}^{0,1}(y)$ is the density of $u_{t,x}^{0,1}$, $M(\ell)$ is given in (4.5) and

$$R_{n+1}(y, \varepsilon) = \int_0^1 \frac{(1-\eta)^n}{n!} E \left[ \mathbb{1}\{\hat{u}_{t,x}^{\eta \varepsilon} > y\} \right] \times H_{k+1} \left( \sum_{c_{n+1}} \prod_{s=1}^k \hat{u}_{t,x}^{\eta \varepsilon, \beta_s} \right) d\eta.$$

In order to bound uniformly this remainder we use the estimate (4.2) together with (3.18) and Lemma 3.2. On the other hand,

$$u_{t,x}^{0,1} \sim \mathcal{N}(0, E [ (u_{t,x}^{0,1})^2 ]),$$

that means,

$$p_{t,x}^{0,1}(y) = \frac{1}{\sqrt{2\pi E [ (u_{t,x}^{0,1})^2 ]}} \exp \left\{ -\frac{y^2}{2E [ (u_{t,x}^{0,1})^2 ]} \right\}.$$

Finally, to obtain the Taylor expansion (4.4) we only have to take account that, for $y = \Lambda_{t,x}$,

$$p_{t,x}^\varepsilon(y) = \frac{1}{\varepsilon} \hat{p}_{t,x}^\varepsilon(0).$$

\[ \square \]

5 Applications

Here we observe two examples where Theorem 4.1 can be applied.

5.1 Stochastic heat equation

Consider

$$\begin{cases}
\frac{\partial u_{t,x}^\varepsilon}{\partial t} - \frac{1}{2} \Delta u_{t,x}^\varepsilon = \varepsilon a_{(t,x)}(\varepsilon) \hat{F}(t, x) + b(u_{t,x}^\varepsilon), \\
u_{0,x}^\varepsilon = 0,
\end{cases} \quad (5.1)$$

t \geq 0, \ x \in \mathbb{R}^d, \ d \geq 1, \ \varepsilon \in (0, 1] \text{ and } F \text{ defined in Section 1.}$$

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By a solution to (5.1) we understand
\[
\begin{align*}
\varepsilon \frac{\partial^2 u_{t,x}}{\partial t^2} - \Delta u_{t,x} &= \varepsilon a(u_{s,y}) F(ds,dy) \\
+ \int_0^t ds \int_{\mathbb{R}^d} dy \, S_{H,t-s}^d(x-y) \, b(u_{s,y}),
\end{align*}
\]
where
\[
S_{H,t}^d(x) = \exp\left(-\frac{|x|^2}{2}\right) (2\pi t)^{d/2}
\]
is the fundamental solution of the deterministic heat equation with \(d\)-dimensional spatial parameter.

In order to check that Theorem 4.1 can be applied we have to assume the conditions on the two coefficients \((h_{c1})\) and \((h_{c2})\) and the following hypothesis on the noise \(F\)
\[
(H_\theta) \quad \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|^2)^\theta} \mu(d\xi) < +\infty,
\]
for some \(\theta \in (0, 1]\) and where \(\mu\) is the spectral measure associated to \(\Gamma\). Indeed, on the one hand, supposing \((H_1)\), Dalang (1999) has showed that \((h_{s1})\) is satisfied by \(S_{H}^d\); on the other hand, Theorem 3.2 in Márquez-Carreras, Mellouk and Sarrà (2001) proves \((h_{s2})\) assuming \((hc_{1})\), \((hc_{2})\) and the existence of some \(\theta \in (0, \frac{1}{2})\) satisfying \((H_\theta)\).

We refer the reader to Márquez-Carreras, Mellouk and Sarrà (2001) for a formulation of \((H_\eta)\) in terms of integrability condition measure \(\Gamma\) what makes explicit the role of the dimension.

5.2 Stochastic wave equation
Consider
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial^2 u_{t,x}^\varepsilon}{\partial t^2} - \Delta u_{t,x}^\varepsilon &= \varepsilon a(u_{s,y}^\varepsilon) \, \dot{F}(t,x) + b(u_{t,x}^\varepsilon), \\
u_{0,x}^\varepsilon &= \frac{\partial u_{0,x}^\varepsilon}{\partial t} = 0,
\end{array} \right.
\end{align*}
\]
for some \(\theta \in (0, 1]\) and \(\mu\) is the spectral measure associated to \(\Gamma\). Indeed, on the one hand, supposing \((H_1)\), Dalang (1999) has showed that \((h_{s1})\) is satisfied by \(S_{H}^d\); on the other hand, Theorem 3.2 in Márquez-Carreras, Mellouk and Sarrà (2001) proves \((h_{s2})\) assuming \((hc_{1})\), \((hc_{2})\) and the existence of some \(\theta \in (0, \frac{1}{2})\) satisfying \((H_\theta)\).

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u_{0,x}^\varepsilon &= \frac{\partial u_{0,x}^\varepsilon}{\partial t} = 0,
\end{array} \right.
\end{align*}
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for some \(\theta \in (0, 1]\) and \(\mu\) is the spectral measure associated to \(\Gamma\). Indeed, on the one hand, supposing \((H_1)\), Dalang (1999) has showed that \((h_{s1})\) is satisfied by \(S_{H}^d\); on the other hand, Theorem 3.2 in Márquez-Carreras, Mellouk and Sarrà (2001) proves \((h_{s2})\) assuming \((hc_{1})\), \((hc_{2})\) and the existence of some \(\theta \in (0, \frac{1}{2})\) satisfying \((H_\theta)\).

We refer the reader to Márquez-Carreras, Mellouk and Sarrà (2001) for a formulation of \((H_\eta)\) in terms of integrability condition measure \(\Gamma\) what makes explicit the role of the dimension.
where $S_W^d$ is the fundamental solution of the deterministic wave equation with $d$-dimensional spatial parameter, that means

$$S_W^1(x,t) = \frac{1}{2} \mathbb{1}_{\{|x| \leq t\}}, \quad x \in \mathbb{R}, \quad t \geq 0,$$

$$S_W^2(x,t) = \frac{1}{2\pi} (t^2 - |x|^2)^{-1/2} \mathbb{1}_{\{|x| \leq t\}}, \quad x \in \mathbb{R}^2, \quad t \geq 0.$$

As before, Dalang (1999) and Theorem 3.5 in Márquez-Carreras, Mellouk and Sarrà (2001) proves (hs\(_1\)) and (hs\(_2\)) assuming (hc\(_1\)), (hc\(_2\)) and the existence of some $\theta \in (0, \frac{3}{4})$ satisfying (H\(_\theta\)).

We also refer the reader to Márquez-Carreras, Mellouk and Sarrà (2001) for a formulation of (H\(_\eta\)) in terms of $\Gamma$. In this same paper Márquez-Carreras, Mellouk and Sarrà (2001), the reader can observe several examples of correlated measures $\Gamma$ such that the corresponding spectral measure $\mu$ satisfies (H\(_\eta\)).

References


