Simple Regression Based Tests for Spatial Dependence

Benjamin Born
BGSE

Jörg Breitung*
University of Bonn

This version: May 2009

JEL-Classification: *C12, C21*
Keywords: *LM test, Moran I test, spatial correlation*

We propose two simple diagnostic tests for spatial error autocorrelation and spatial lag dependence. The idea is to reformulate the testing problem such that the test statistics are asymptotically equivalent to the familiar LM test statistics. Specifically, our version of the test is based on a simple auxiliary regression and an ordinary regression t-statistic can be used to test for spatial autocorrelation and lag dependence. We also propose a variant of the test that is robust to heteroskedasticity and non-normality. This approach gives practitioners an easy to implement and robust alternative to existing tests. Monte Carlo studies show that our variants of the spatial LM tests possess comparable size and power properties even in small samples.

*Address: University of Bonn, Institute of Econometrics, Adenauerallee 24-42, 53113 Bonn, Germany, email: bborn@uni-bonn.de, breitung@uni-bonn.de.

1 Introduction

In recent years, the use of spatial methods in empirical studies has been growing steadily with the increasing availability of regional datasets. One effect that researchers have to deal with is spatial dependence (or, in a weaker form, spatial correlation). Anselin (2001) lists a statistic developed by Moran (1948) as the most commonly used test for spatial correlation. A corresponding LM test against a spatial error alternative was proposed by Burridge (1980) and against a spatial lag alternative by Anselin (1988a).

We show that after a minor reformulation of the model, we can test for spatial dependencies by regressing the OLS residuals on their spatial lags and then testing the significance of the spatial coefficient by an asymptotic t-test. This provides us with an easily implementable test that can be generalized straightforwardly to accommodate heteroscedastic and non-normal disturbances.

Monte Carlo simulations demonstrate that under the standard assumptions our version of the test performs similarly to Moran's I and the LM test. However, if the errors are heteroscedastic the latter tests suffer from size distortions whereas the regression based test (using White's (1990) estimator of the standard errors) turns out to be robust against a heteroscedastic errors process. We can also confirm the results from other simulation experiments (e.g. Anselin and Florax 1995) showing that in small samples Moran's I has higher power than the LM test. To improve the power of the regression test we suggest a modification such that the test approaches the power of Moran's statistic.

The remainder of the paper is organized as follows. Section 2 discusses a t-test against a spatial error alternative and section 3 focuses on tests against a spatial lag alternative. Size and power of these tests are compared to Moran's I and LM tests via Monte Carlo simulations in section 4. Finally, section 5 concludes.

2 Testing against spatial error alternatives

We consider the linear spatial first order autoregressive model with first order autoregressive disturbances (see, e.g. Anselin (1988b)), which is given by

$$\begin{aligned} y &= \phi W_1 y + X\beta + u \\ u &= \rho W_2 u + \varepsilon, \end{aligned} \tag{2.1}$$

where y is an $N \times 1$ vector of observations on a dependent variable, X is an $N \times k$ matrix of exogenous regressors, β is the associated $k \times 1$ parameter vector and ϕ and ρ

are spatial autoregressive parameters with $|\rho| < 1$ and $|\phi| < 1$.

Following Kelejian and Prucha (1999, 2001), we make the following assumptions concerning model (2.1):

Assumption 1. *(i) The errors* $\varepsilon_1, \ldots, \varepsilon_N$ *are i.i.d. with zero mean,* $E(\varepsilon_i) = \sigma^2$*, and* $E(|\varepsilon_i|^{2+\delta}) < \infty$ *for some* $\delta > 0$*. (ii) The spatial weight matrices* W_1 *and* W_2 *are* $N \times N$ *matrices of known constants. The elements on the main diagonal of the matrices are zero and the matrices* $(I - \rho W_2)$ *and* $(I - \phi W_1)$ *are nonsingular for all* $|\rho| < 1$ *and* $|\phi| < 1$*. The row and column sums of the matrices* W_1*,* W_2 *are bounded uniformly in absolute value as* $N \to \infty$*. (iii) The matrix* X *has full column rank and is independent of* u*.*

The spatial error model is obtained by setting $\phi = 0$, yielding

$$\begin{aligned} y &= X\beta + u \\ u &= (I_N - \rho W)^{-1}\varepsilon \end{aligned} \qquad (2.2)$$

where W_2 is replaced by W to simplify the notation.

Moran's I-statistic is defined as

$$I = \frac{\widehat{u}'W\widehat{u}}{\widehat{u}'\widehat{u}} \qquad (2.3)$$

where $\widehat{u} = y - X\widehat{\beta}$ is the vector of OLS residuals. The standardized version $(I - \mu_I)/\sigma_I$ is standard normally distributed, where

$$\begin{aligned} \mu_I &= tr(MW)/(N-k) \\ \sigma_I^2 &= [tr(MWMW') + tr(MW)^2 + (tr(MW))^2]/d - \mu_I^2, \end{aligned}$$

$M = I - X(X'X)^{-1}X'$, and $d = (n-k)(n-k+2)$. For $N \to \infty$ we have

$$\mu_I \to 0 \qquad (2.4)$$

$$N^{-1}\sigma_I^2 \to \lim_{N\to\infty} \frac{\sigma^2}{N} tr(W^2 + W'W). \qquad (2.5)$$

Burridge (1980) showed that the LM statistic results as

$$LM_\rho = \frac{(\widehat{u}'W\widehat{u})^2}{\widehat{\sigma}^4 \, \mathrm{tr}(W^2 + W'W)}, \qquad (2.6)$$

where $\widehat{\sigma}^2 = \widehat{u}'\widehat{u}/N$. Using (2.4) and (2.5) it is easy to see that the square of Moran's I

and the LM statistic are asymptotically equivalent.

To motivate the regression version of the test, assume that β is known and consider the t-statistic for the null hypothesis $\rho = 0$ in the regression $u = \rho W u + \varepsilon$ which can be written as

$$t_\rho = \frac{u'Wu}{\widehat{\sigma}\sqrt{u'W'Wu}}, \qquad (2.7)$$

where $\widehat{\sigma}^2$ is the usual variance estimator of ε. It is easy to see that under the null hypothesis t_ρ is not asymptotically standard normally distributed. For the numerator of t_ρ we obtain

$$\begin{aligned} u'Wu &= \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \begin{pmatrix} 0 & w_{1,2} & \cdots & w_{1,n} \\ w_{2,1} & \ddots & & \\ \vdots & & \ddots & w_{n-1,n} \\ w_{n,1} & \cdots & w_{n,n-1} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \\ &= \sum_{i=1}^{N} \sum_{j \neq i} w_{ij} u_i u_j \\ &= \sum_{i=2}^{N} \sum_{j=1}^{i-1} (w_{ij} + w_{ji}) u_i u_j \\ &= \sum_{i=2}^{N} u_i \xi_i \ , \end{aligned}$$

where

$$\xi_i = \sum_{j=1}^{i-1} (w_{ij} + w_{ji}) u_j$$

Defining $\xi = (0, \xi_2, \ldots, \xi_N)'$ we can re-write the numerator as

$$u'Wu = u'(D_1 + D_2)u = u'\xi$$

where D_1 and D_2 are lower triangular matrices such that $W = D_1 + D_2'$ and $\xi = (D_1 + D_2)u$. Note that $u'(D_1 + D_2)u = u'(D_1 + D_2')u = u'Wu$. However, there is an important difference between the two formulations of the numerator. Whereas ξ_i is associated with an increasing σ-field generated by $\{u_1, \ldots, u_{i-1}\}$ this is not the case for the variable $z_i = \sum_{j \neq i} w_{ij} u_j$, as this variable depends on $\{u_j | j \neq i\}$. This has important consequences for the variance. Specifically, under the null hypothesis we have

$$var(u'\xi) = \sigma^2 E(\xi'\xi) \text{ but } var(u'z) \neq \sigma^2 E(z'z).$$

If W is symmetric, it is not difficult to show that $var(u'z) = 2\sigma^2 E(z'z)$. The factor 2 results from the fact that due to the symmetric nature of the sum the products $u_i u_j$ occur two times for each combination of i and j. We therefore suggest to use ξ instead of $z = Wu$ as the regressor in the test regression, where D_1 results from W by setting all elements above the main diagonal equal to zero. Analogously, D_2' is obtained from setting the elements below the main diagonal equal to zero.

If β is unknown, the errors u are replaced by $\widehat{u} = y - X\widehat{\beta}$ and the regression test for spatial error correlation is the t-statistic for the null hypothesis $\rho = 0$ in the regression

$$\widehat{u} = \rho\widehat{\xi} + e, \tag{2.8}$$

that is,

$$\widetilde{t}_\rho = \frac{\widehat{u}'\widehat{\xi}}{\widehat{\sigma}\sqrt{\widehat{\xi}'\widehat{\xi}}}. \tag{2.9}$$

where $\widehat{\xi} = (D_1 + D_2)\widehat{u}$ and $\widehat{\sigma}^2$ is the usual estimator for the variance of the errors e.

The following proposition considers the asymptotic properties of the test statistic.

Proposition 1. *(i) Under the null hypothesis $\rho = 0$ and $N \to \infty$ we have*

$$\widetilde{t}_\rho \xrightarrow{d} \mathcal{N}(0, 1). \tag{2.10}$$

(ii) The regression test is asymptotically equivalent to Moran's I and the LM statistic in the sense that under the null hypothesis $\widetilde{t}_\rho - I \xrightarrow{p} 0$ and $\widetilde{t}_\rho^{\,2} - LM \xrightarrow{p} 0$.

Proof. (i) Under the null hypothesis we have $\widehat{u} = \varepsilon - X(\widehat{\beta} - \beta)$. It follows that the numerator of $\widetilde{t}_\rho$ can be written as

$$\begin{aligned} \widehat{u}'\widehat{\xi} &= \widehat{u}'(D_1 + D_2)\widehat{u} \\ &= \varepsilon'(D_1 + D_2)\varepsilon - 2\varepsilon'(D_1 + D_2)'X(\widehat{\beta} - \beta) + (\widehat{\beta} - \beta)'X'(D_1 + D_2)X(\widehat{\beta} - \beta) \end{aligned}$$

Since ε_i is a martingale difference sequence with respect to ξ_i for $i = 1, 2, \ldots$ we obtain from the central limit theorem for martingale difference sequence (see e.g. White 2001)

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{n}\varepsilon_i\xi_i \overset{d}{\to} \mathcal{N}(0,V_1)$$

where

$$\begin{aligned} V_1 &= \lim_{N\to\infty} E\left(N^{-1}\sum_{i=1}^{N}\varepsilon_i^2\xi_i^2\right) \\ &= \sigma^4 \lim_{N\to\infty}\frac{1}{N}tr[(D_1+D_2)'(D_1+D_2)] \end{aligned}$$

Since D_1 and D_2 are lower triangular matrices with zeros on the leading diagonal we have $D_sD_t = 0$ for $s,t\in\{1,2\}$. Thus,

$$\begin{aligned} \mathrm{tr}(W^2+W'W) &= \mathrm{tr}[(D_1+D_2')(D_1+D_2')+(D_1'+D_2)(D_1+D_2')] \\ &= \mathrm{tr}(D_1^2+D_1D_2'+D_2'D_1+D_2'D_2' \\ &\quad + D_1'D_1+D_2D_1+D_1'D_2'+D_2D_2') \\ &= \mathrm{tr}(D_1'D_1)+\mathrm{tr}(D_2'D_2)+2\mathrm{tr}(D_2'D_1) \\ &= \mathrm{tr}[(D_1+D_2)'(D_1+D_2)], \end{aligned} \tag{2.11}$$

From Assumption 1 (ii) it follows that $0 < N^{-1}tr(W'W) < \infty$ and $0 < N^{-1}tr(W^2) < \infty$ for all N and therefore $0 < V_1 < \infty$.

Since, under Assumption 1, $\widehat{\beta}-\beta$ is $O_p(N^{-1/2})$ and X is independent of u, we obtain for the other two terms of the numerator

$$\begin{aligned} &\varepsilon'(D_1+D_2)'X(\widehat{\beta}-\beta) = O_p(N^{1/2})O_p(N^{-1/2}) = O_p(1) \\ &(\widehat{\beta}-\beta)'X'(D_1+D_2)X(\widehat{\beta}-\beta) = O_p(N^{-1/2})O_p(N)O_p(N^{-1/2}) = O_p(1) \end{aligned}$$

it follows that

$$\frac{1}{\sqrt{N}}\widehat{u}'\widehat{\xi} \overset{d}{\longrightarrow} \mathcal{N}(0,V_1).$$

To derive the asymptotic properties of the denominator we first consider

$$\widehat{\xi}'\widehat{\xi} = \xi'\xi - 2\xi'(D_1+D_2)X(\widehat{\beta}-\beta)+(\widehat{\beta}-\beta)'X'(D_1+D_2)'(D_1+D_2)X(\widehat{\beta}-\beta).$$

Using the law of large numbers we obtain

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}\xi_i^2 \;\xrightarrow{p}\; & \lim_{N\to\infty} E\left(\frac{1}{N}\sum_{i=1}^{N}\xi_i^2\right) \\ & \sigma^2 \lim_{N\to\infty} N^{-1} tr[(D_1+D_2)'(D_1+D_2)] = \frac{1}{\sigma^2}V_1.\end{aligned}$$

Furthermore,

$$\begin{aligned}2\xi'(D_1+D_2)X(\widehat{\beta}-\beta) &= O_p(1) \\ (\widehat{\beta}-\beta)'X'(D_1+D_2)'(D_1+D_2)X(\widehat{\beta}-\beta) &= O_p(1).\end{aligned}$$

and, hence,

$$\frac{1}{N}\widehat{\xi}'\widehat{\xi} \xrightarrow{p} V_1 \ .$$

In a similar manner it can be shown that $\widehat{\sigma}^2 \xrightarrow{p} \sigma^2$. From these results it follows that $\widetilde{t}_\rho$ has a standard normal limiting distribution.

(ii) Following Kelejian and Prucha (2001), Moran's I statistic can be written as

$$I = N^{-1/2}\widehat{u}'W\widehat{u}/\sqrt{\widehat{V}_1}$$

where $\widehat{V}_1$ is a consistent estimator of V_1. In particular,

$$\widehat{V}_1 = \frac{1}{N}\widehat{u}'\widehat{u}\, tr(W^2 + W'W)$$

in which case the LM statistic is the square of Moran's I. Note that the numerators of Moran's I and $\widetilde{t}_\rho$ are identical. The only difference is the denominator. However, since $\widehat{V}_1 \xrightarrow{p} V_1$ it follows that $\widetilde{t}_\rho - I \xrightarrow{p} 0$ as $N \to \infty$. Similarly, $\widetilde{t}_\rho^{\,2} - LM \xrightarrow{p} 0$. □

A difference between the regression test and the LM test is that the latter estimates the variance of the numerator by imposing the null hypothesis. To simplify the discussion assume that β is known. Under the null hypothesis we have

$$var(u'\xi) = \sigma^2 E(\xi'\xi) = \sigma^4 tr[(D_1+D_2)'(D_1+D_2)] = \sigma^4 tr(W^2 + W'W)$$

Under the alternative we have

$$E(\xi'\xi) = \sigma^2 tr[(I-W')^{-1}(D_1+D_2)'(D_1+D_2)(I-W)^{-1}]$$

Using the expansion $(I - W')^{-1} = I + W + W^2 + W^3 + \cdots$ it becomes clear that under the alternative $E(\xi'\xi)$ is larger than $\sigma^2 tr(W^2 + W'W)$ which is used for the LM statistic. It follows that under the alternative the regression statistic $\widetilde{t}_\rho$ is systematically smaller (in absolute value) than the LM statistic and, therefore, the power of the regression test tends to be smaller than the power of the LM statistic. This negative effect on the power of the test can be avoided by replacing $\widehat{u}$ in the denominator of $\widetilde{t}_\rho$ by the residual $\widehat{e} = \widehat{u} - \hat{\rho}\widehat{\xi}$, where $\hat{\rho}$ denotes the OLS estimator from a regression of $\widehat{u}$ on $\widehat{\xi}$. Thus, the modified regression statistic results as

$$\widetilde{t}_\rho^* = \frac{\widehat{u}'\widehat{\xi}}{\widehat{\sigma}\sqrt{\widehat{e}'(D_1 + D_2)'(D_1 + D_2)\widehat{e}}}. \tag{2.12}$$

Our Monte Carlos simulations presented in Section 4 suggest that this modification indeed yields a more powerful test statistic.

An important advantage of the regression test is that it can be made robust against heteroskedasticity by employing White's (1980) heteroskedasticity consistent covariance estimator. This yields the heteroskedasticity robust test statistic

$$\widetilde{t}_\rho = \frac{\widehat{u}'\widehat{\xi}}{\sqrt{\sum_{i=1}^N \widehat{u}_i^2 \widehat{\xi}_i^2}} \tag{2.13}$$

Note that we have imposed the null hypothesis $\rho = 0$ in White's variance estimator. An alternative is to replace the residuals $\widehat{u}_i$ in the denominator by the OLS residuals $\widehat{e}$ of the auxiliary regression $\widehat{u} = \rho\widehat{\xi} + e$. However, Monte Carlo simulations suggest that the former estimators yields superior size properties of the test.

3 Testing against spatial lag alternatives

Setting $\rho = 0$ the linear spatial autoregressive model (2.1) with first order autoregressive disturbances becomes the spatial lag model

$$y = \phi W y + X\beta + \varepsilon \tag{3.1}$$

where again we suppress the index for the weight matrix W_1. Anselin (1988a) derives the LM test statistic for the null hypothesis $\phi = 0$. The one-sided version of the test statistic results as

$$LM_\phi = \frac{y'MWy}{\sqrt{\widehat{\sigma}^4 tr(W^2 + W'W) + \widehat{\sigma}^2 \widehat{\beta}' X'W'MWX\widehat{\beta}}} \tag{3.2}$$

where $\widehat{\beta}$ is the OLS estimator from a regression of y on X and $\widehat{\sigma}^2$ is the usual variance estimator of the residuals.

The estimate of ϕ from (3.1) is identical to the estimate from the regression:

$$\widehat{\phi} = \frac{y'MWy}{y'W'MWy} \tag{3.3}$$

As in the case of the test for spatial autocorrelation the numerator of this estimator is identical to the numerator of the LM statistic. This suggests that a regression test can be constructed that is asymptotically equivalent to the LM statistic (3.2).

To derive this estimator we employ the same technique as in the previous section. First note that the numerator of the LM statistic can be re-written as

$$y'MWy = \widehat{u}'W\widehat{u} + \widehat{u}'W\widehat{y},$$

where $\widehat{y} = X\widehat{\beta}$. Using $W = D_1 + D_2'$ and $\widehat{u}'(D_1 + D_2')\widehat{u} = \widehat{u}'(D_1 + D_2)\widehat{u}$ we obtain

$$y'MWy = \widehat{u}'\widehat{\xi}^*,$$

where

$$\widehat{\xi}^* = (D_1 + D_2)\widehat{u} + MW\widehat{y}.$$

Note that we do not need to decompose W in the last expression of this equation as X is assumed to be exogenous.[1] In the proof of Proposition 2 it is shown that the asymptotic properties are not affected by using $\widehat{y} = X\widehat{\beta}$ instead of $X\beta$.

The regression test for a spatial lag results as the ordinary t-statistic for the hypothesis $\phi = 0$ in the regression

$$\widehat{u} = \phi\widehat{\xi}^* + \eta,$$

[1]Note further that we have introduced the matrix M in the last term. Due to the idempotency of M this matrix does not affect the product $y'MWy$. However, the matrix M affects the denominator of the test statistic and is required to derive the results presented in Proposition 2.

yielding the test statistic

$$\widetilde{t}_\phi = \frac{\widehat{u}'\widehat{\xi}^*}{\widehat{\sigma}\sqrt{\widehat{\xi}^{*\prime}\widehat{\xi}^*}} \tag{3.4}$$

In the following proposition the limiting distribution of the test statistic is presented.

Proposition 2. *Assume that y can be represented as in* (3.1). *Under $H_0 : \phi = 0$ and Assumption 1 it follows that*

$$\widetilde{t}_\phi \ \xrightarrow{d} \mathcal{N}(0,1)$$

and $\widetilde{t}_\phi - LM_\phi \xrightarrow{p} 0$.

Proof. As shown in the proof of Proposition 1 we have

$$\frac{1}{\sqrt{N}}\,\widehat{u}'W\widehat{u} = \frac{1}{\sqrt{N}}\,u'Wu + o_p(1) \xrightarrow{d} \mathcal{N}(0, V_1).$$

Furthermore we obtain

$$\widehat{u}'MW\widehat{y} \ = \ \widehat{u}'MWX\beta + \widehat{u}'MWX(\widehat{\beta} - \beta). \tag{3.5}$$

Using

$$\begin{aligned} X'W'M\widehat{u} &= X'W'Mu \\ &= X'W'u - X'W'X(X'X)^{-1}X'u \\ &= O_p(N^{1/2}) - O_p(N)O_p(N^{-1})O_p(N^{1/2}) \\ &= O_p(N^{1/2}) \end{aligned}$$

and $\widehat{\beta} - \beta = O_p(N^{-1/2})$ we obtain

$$\frac{1}{\sqrt{N}}\,\widehat{u}'MW\widehat{y} = \frac{1}{\sqrt{N}}\,u'WX\beta + O_p(N^{-1/2}).$$

Since X is exogenous it follows that

$$E\left(\frac{1}{\sqrt{N}}\widehat{u}'\widehat{\xi}^*\right) \ \rightarrow \ 0 \tag{3.6}$$

$$\text{Var}\left(\frac{1}{\sqrt{N}}\widehat{u}'\widehat{\xi}^*\right) \ \rightarrow \ \sigma^2 \lim_{N\to\infty} E\left(T^{-1}\xi^{*\prime}\xi^*\right) \tag{3.7}$$

where $\xi^* = (D_1 + D_2)u + MWX\beta$. Since

$$E(\xi^{*'}\xi^*) = \sigma^2 tr[(D_1 + D_2)'(D_1 + D_2)] + \beta' X' W' MWX\beta$$
$$= \sigma^2 tr(W^2 + W'W) + \beta' X' W' MWX\beta$$

and $\widehat{\sigma}^2 \xrightarrow{p} \sigma^2$ it follows that,

$$\widetilde{t}_\phi = \frac{\widehat{u}'W\widehat{u} + \widehat{u}'W\widehat{y}}{\sqrt{\sigma^4 tr(W^2 + W'W) + \sigma^2\beta' X' W' MWX\beta}} + o_p(1)$$

and, therefore, $\widetilde{t}_\phi$ is asymptotically equivalent to the LM_ϕ statistic and possesses a standard normal limiting distribution. □

Note that the only difference between the the test statistics $\widetilde{t}_\rho$ and $\widetilde{t}_\phi$ is that the the former test is based on the correlation between $\hat{u}$ and $\widehat{\xi}$, whereas the latter test considers the correlation between $\widehat{u}$ and $\widehat{\xi}^* = \widehat{\xi} + WX\widehat{\beta}$. Therefore, both statistics are highly correlated and the statistic $\widetilde{t}_\rho$ tends to reject the null hypothesis of no spatial autocorrelation if the data are generated by the spatial lag model (and vice versa). Therefore, these tests are not able to indicate the correct model specification. It is therefore desirable to construct a test for the null hypothesis that the specification with spatial error correlation is correct versus the alternative that the spatial lag model is correct. Such a test can be constructed as follows. Assume that (as in many empirical applications) the spatial weight matrices W_1 and W_2 are identical. In this case the "hybrid model" that encompasses both specifications can be written as

$$y = (I + \phi W)X\beta + (\phi + \rho)Wu + \varepsilon$$

The LM test for the hypothesis that the spatial autocorrelation model is correct is given by $\phi = 0$ whereas the parameter ρ is left unrestricted. Let $\widetilde{\rho}_{ml}$ and $\widetilde{\beta}_{ml}$ denote the restricted maximum likelihood estimators of ρ based the spatial autocorrelation model (2.2). Then the LM statistic is equivalent to a regression of the residuals $\widetilde{\varepsilon} = (I - \widetilde{\rho}_{ml}W)(y - X\widetilde{\beta}_{ml})$ on $WX\widetilde{\beta}_{ml}$.

4 Monte Carlo Simulations

In all experiments, we set the exogenous variable X to be a vector of ones with coefficient $\beta = 1$. The vector ε is obtained by drawing N independently and jointly standard normally distributed random numbers. In the case of the spatial error specification we generate the $N \times 1$ vector of observations as

$$y = X\beta + (I_N - \rho W_2)^{-1}\varepsilon. \tag{4.1}$$

With the spatial lag specification, y is generated by

$$y = (I_N - \phi W_1)^{-1}(X\beta + \varepsilon). \tag{4.2}$$

In the construction of the spatial weighting matrix, we follow Kelejian and Prucha (1999), Mutl (2005) and Kapoor, Kelejian, and Prucha (2007) and use a "3 ahead and 3 behind" specification. In this design, the matrix has in its i-th row, $3 < i < N - 3$, nonzero elements in positions $i-3, i-2, i-1, i+1, i+2, i+3$ so that each element of the weighting matrix is directly related to the three immediate neighbors in front of and behind it. The first three and last three rows are adjusted to create a circular world. Following common praxis in the spatial econometrics applications, we also row normalize the weighting matrix so that all nonzero entries in our specification have a value of $1/6$.

We use 1000 replications under the null hypothesis and count the number of times the test statistics are larger than the critical values corresponding to a 5% significance level. Table 1 shows the results for the spatial error specification with homoskedastic errors. All tests have approximately the correct size of 0.05 although the size of the Moran's I test is considerably smaller than that of the other ones. The two panels in figure 1 show that our regression test has lower power than both the LM test and Moran's I. But with the modified statistic $\widetilde{t}_\rho^*$ suggested in section 2, the power is similar as that of Moran's I. We can also see that the tests gain considerable power by the increase in the sample size.

To analyze the test properties under heteroskedasticity we divide the sample into two clusters with variances:

$$\sigma^2 = \begin{cases} 1 & \text{for } i = 1, \ldots, \lfloor n/2 \rfloor, \\ 5 & \text{for } i = \lfloor n/2 \rfloor + 1, \ldots, n, \end{cases} \tag{4.3}$$

where $\lfloor x \rfloor := \max_{k \in \mathbb{Z}, k \leq x}(k)$. The results are stated in table 2 where we can see that the sizes of the LM and the Moran's I test are distorted by the heteroskedasticity while the regression test with the White correction keeps its correct size. Figure 2 only shows the power for the regression test because we would not use the other tests with such large size distortions. The test has lower power due to the White correction but given that the other tests have large size distortions, it is still the better choice.

In the spatial lag specification, both our regression test and the LM test against a spatial lag alternative have the correct size until we introduce heteroskedasticity in the form of (4.3) which leads to large size distortions of the LM test.

5 Concluding Remarks

We propose two new test procedures for spatial dependence. A reformulation of the model allows us to test against a spatial error or spatial lag specification by simply regressing the OLS residuals on their spatial lags and testing the significance of the spatial coefficient by an asymptotic t-test. We show that these tests are asymptotically equivalent to the existing Moran's I and LM tests yet simpler to implement. Furthermore, using the approach of White (1980) it is straightforward to construct a test that is robust against heteroskedastic errors.

Monte Carlo simulations suggest that our new tests have good size properties, even under heteroskedasticity, where Moran's I and LM tests suffer from size distortions. One result of our simulations is that the regression test has slightly less power than the LM test against a spatial error alternative. We also confirm the results from other simulation experiments (e.g. Anselin and Florax 1995) that Moran's I has higher power than the LM test in small samples. However, after modifying the denominator of the regression t-statistic, the power approaches that of Moran's I even in small samples. In medium and large samples, the performance of all tests becomes very similar.

Hence, we believe that the proposed tests will give researchers an additional and easily implementable tool for their applied work.

References

Anselin, L. (1988a): "Lagrange multiplier test diagnostics for spatial dependence and spatial heterogeneity," *Geographical Analysis*, 20, 1–17.

——— (1988b): *Spatial Econometrics: Methods and Models*. Kluwer, Dordrecht.

——— (2001): "Spatial Econometrics," in *A Companion to Theoretical Econometrics*, ed. by B. Baltagi. Blackwell, Oxford.

Anselin, L., and R. Florax (1995): "Small sample properties of tests for spatial dependence in regression models: some further results," in *New Directions in Spatial Econometrics*, ed. by L. Anselin, and R. Florax, pp. 21–74. Springer-Verlag, Berlin.

Burridge, P. (1980): "On the Cliff-Ord test for spatial autocorrelation," *Journal of the Royal Statistical Society B*, 42, 107–108.

Kapoor, M., H. H. Kelejian, and I. R. Prucha (2007): "Panel data models with spatially correlated error components," *Journal of Econometrics*, 140, 97–130.

Kelejian, H. H., and I. R. Prucha (1999): "A Generalized Moments Estimator for the Autoregressive Parameter in a Spatial Model," *International Economic Review*, 40(2), 509–533.

——— (2001): "On the asymptotic distribution of the Moran I test statistic with applications," *Journal of Econometrics*, 104, 219–257.

Moran, P. (1948): "The interpretation of statistical maps," *Biometrika*, 35, 255–260.

Mutl, J. (2005): "Dynamic Panel Data Models with Spatially Correlated Innovations," Ph.D. thesis, University of Maryland.

White, H. (1980): "A Heteroscedasticity-Consistent Covariance Matrix Estimator and a Direct Test for Heteroscedasticity," *Econometrica*, 48, 817–838.

——— (2001): *Asymptotic Theory for Econometricians*. Academic Press, San Diego, revised edn.

	LM Test	Regr. Test	Mod. Regr. Test	Moran's I
$N = 50$	0.042	0.058	0.070	0.042
$N = 100$	0.051	0.057	0.063	0.044
$N = 150$	0.047	0.054	0.055	0.040
$N = 200$	0.047	0.044	0.050	0.037
$N = 300$	0.046	0.053	0.046	0.031
$N = 500$	0.047	0.051	0.047	0.029

Table 1: Errortest: Size comparison under homoskedasticity

	LM Test	Regr. Test	Moran's I
$N = 50$	0.084	0.052	0.077
$N = 100$	0.109	0.047	0.077
$N = 150$	0.144	0.057	0.081
$N = 200$	0.134	0.050	0.079
$N = 300$	0.132	0.052	0.077
$N = 500$	0.138	0.043	0.076

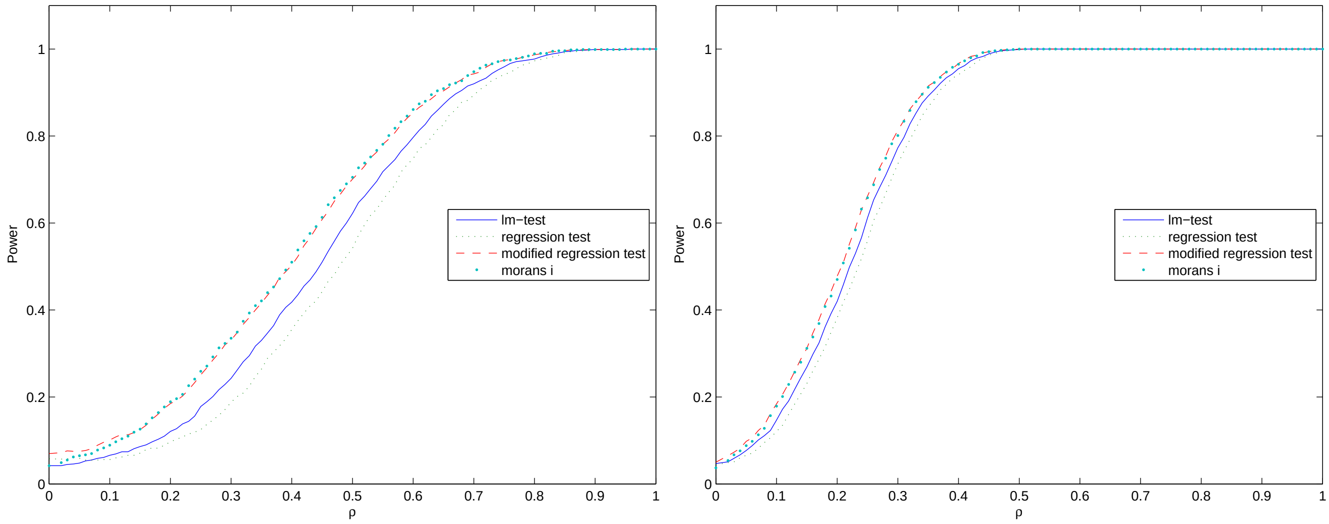
Table 2: Errortest: Size comparison under heteroskedasticity

	LM Test	Regr. Test
$N = 50$	0.044	0.049
$N = 100$	0.040	0.047
$N = 150$	0.047	0.048
$N = 200$	0.046	0.048
$N = 300$	0.054	0.051
$N = 500$	0.056	0.057

Table 3: Lagtest: Size comparison under homoskedasticity

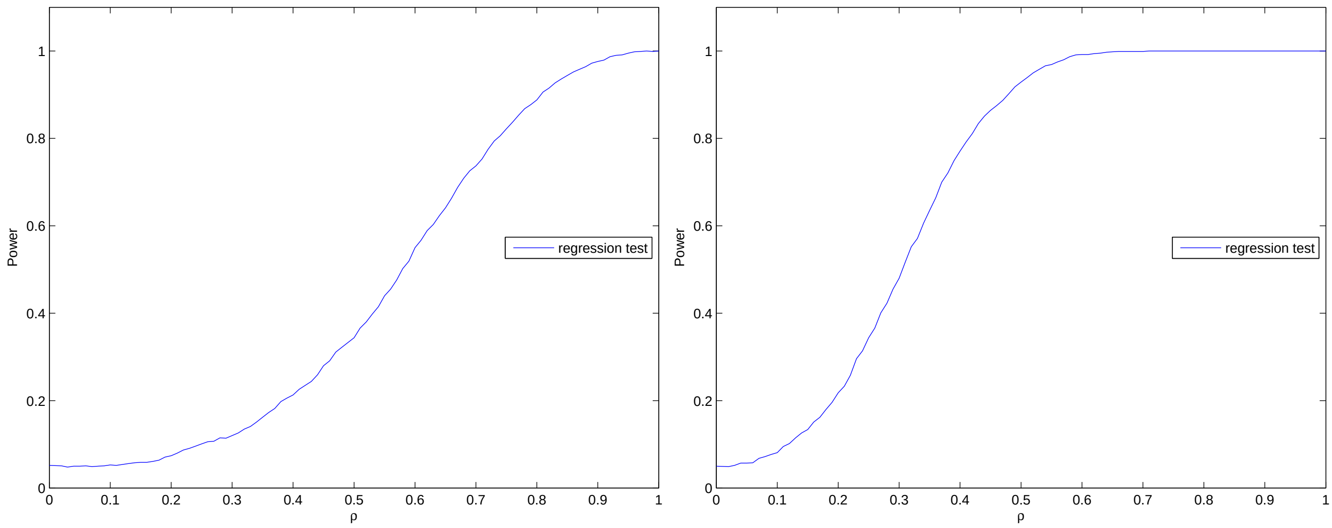
	LM Test	Regr. Test
$N = 50$	0.096	0.042
$N = 100$	0.113	0.047
$N = 150$	0.143	0.054
$N = 200$	0.140	0.049
$N = 300$	0.145	0.051
$N = 500$	0.133	0.044

Table 4: Lagtest: Size comparison under heterokedasticity



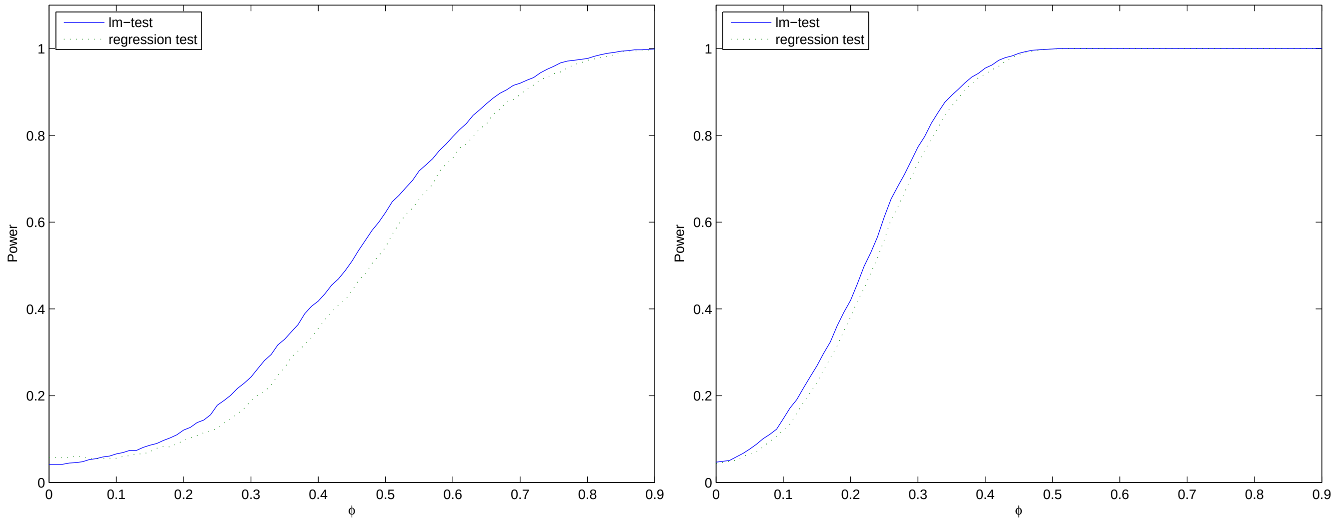
(a) $N = 50$ (b) $N = 200$

Figure 1: Errortest: Comparison of power functions under homoskedasticity



(a) $N = 50$ (b) $N = 200$

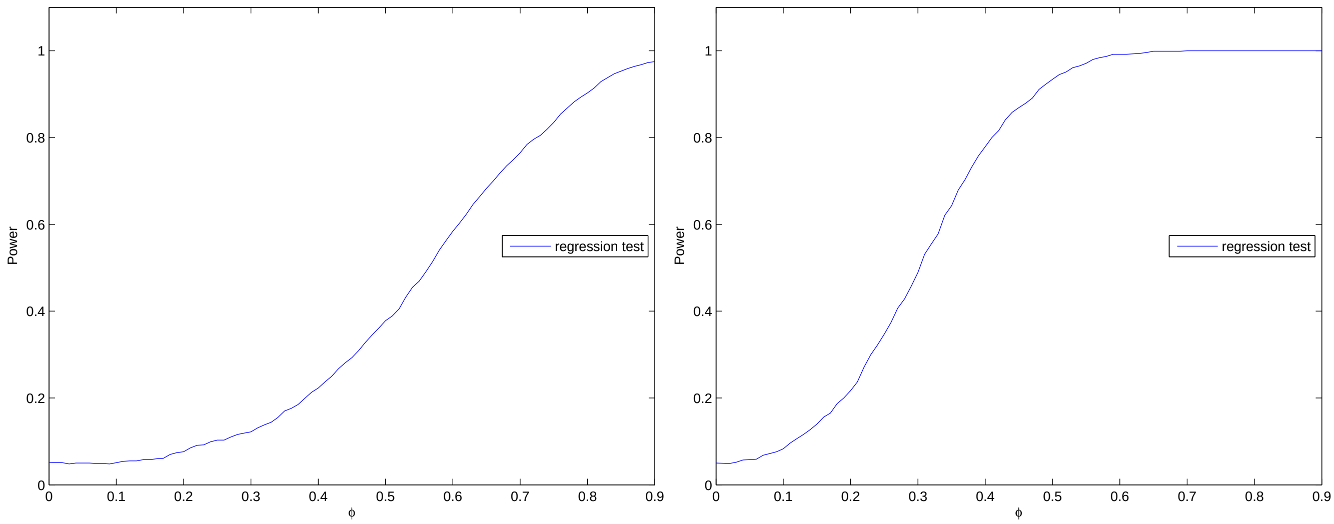
Figure 2: Errortest: Comparison of power functions under heteroskedasticity



(a) $N = 50$

(b) $N = 200$

Figure 3: Lagtest: Comparison of power functions under homoskedasticity



(a) $N = 50$

(b) $N = 200$

Figure 4: Lagtest: Comparison of power functions under heteroskedasticity