Many triangulated 3-spheres

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In 1988, Kalai proved via an explicit construction that for every $d \geq 4$, there are far more combinatorial types of simplicial PL d-dimensional spheres than types of simplicial convex $(d+1)$-dimensional polytopes. However, for $d = 3$ his construction only yields polytopal (i.e., convex) spheres, and the relation between the number of combinatorial types of 3-spheres and that of convex 4-polytopes remained open.

In joint work with Günter M. Ziegler, we put together a construction by Heffter (1898) using finite fields with modern ideas by Eppstein, Kuperberg & Ziegler (2002) to construct $2^{\Omega(n^{5/4})}$ combinatorial types of triangulated 3-spheres on n vertices. Since by a result of Goodman and Pollack (1986) there are no more than $2^{O(n \log n)}$ combinatorial types of simplicial 4-polytopes, this proves that asymptotically, there are far more combinatorial types of triangulated 3-spheres than of simplicial 4-polytopes on n vertices.