Künneth formula implies Poincaré duality and finiteness

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Let k be a perfect field, and K be a field of coefficients. We consider a presheaf E of commutative K-DGA over the category of smooth schemes over k. Define $H^*(X)$ to be the Nisnevich hypercohomology of X with coefficients in E for any smooth scheme X over k. We suppose that we have the following properties:

1) E has the Brown-Gersten Property (Nisnevich descent).

2) $H^*(Speck)$ is concentrated in degree 0 and $H^0(Speck) = K$.

3) $H^*(G_m)$ is concentrated in degrees 0 and 1 and $H^0(G_m) = H^1(G_m) = K$.

4) The cohomology theory H^* satisfies the Knneth formula. Then for any smooth scheme X over k, $H^*(X)$ is a finite dimensional K-vector space and the Poincaré duality holds for H^* (we also associate to H^* a version with compact support for the open varieties).

The strategy is to prove that E is representable by in a suitable triangulated category of motives $DM(k)$ (defined by the homotopy theory of schemes of Morel and Voevodsky) and that E induces a covariant monoidal realization of DM in the derived category of K-vector spaces. Then the result comes quite formally from the properties of $DM(k)$. The main examples that fit in this frame are the de Rham cohomology and the rigid cohomology.