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Small points on subvarieties of algebraic tori: results and methods

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The former Manin-Mumford conjecture predicts that the set of torsion points of a curve of genus ≥ 2 embedded in its jacobian is finite. More generally, let $\mathbb{G}$ be a semi-abelian variety and V an algebraic subvariety of $\mathbb{G}$, defined over some algebraically closed field K. We say that V is a *torsion subvariety* if V is a translate of a proper subtorus by a torsion point of $\mathbb{G}$. We also denote by V_{tors} the set of torsion points of $\mathbb{G}$ lying on V. Then we have the following generalization of the Manin-Mumford conjecture:

i) If V is not a torsion subvariety, then the set V_{tors} of torsion points of $\mathbb{G}$ lying on V is not Zariski dense.

ii) The Zariski closure of V_{tors} is a finite union of torsion subvarieties.

The two assertions are clearly equivalent and were proved by Raynaud (1983) when $\mathbb{G}$ is an abelian variety, by Laurent (1984) if $\mathbb{G} = \mathbb{G}_m^n$, and finally by Hindry (1988) in the general situation.
We assume from now on that all varieties are algebraic, defined over $\overline{\mathbb{Q}}$ and geometrically irreducible. Bogomolov (1981) gave the following generalization of the former Manin-Mumford conjecture: a curve $\mathcal{C}$ of genus ≥ 2 embedded in its jacobian is discrete for the metric induced by the Néron-Tate height. In

other words, Bogomolov conjecture asserts that the set of points of "sufficiently small" height on $\mathcal{C}$ is finite, while the former Manin-Mumford conjecture makes a similar assertion on the set of torsion points (which are precisely the points of zero height).

More generally, let $\mathbb{G}$ be a semi-abelian variety and let $\hat{h}$ be a normalized height on $\mathbb{G}(\overline{\mathbb{Q}})$. Hence, $\hat{h}$ is the Neron-Tate height if $\mathbb{G}$ is abelian, and it is the Weil height if $\mathbb{G} = \mathbb{G}_m^n \hookrightarrow \mathbb{P}_n$; in particular $\hat{h}$ is a non-negative function on $\mathbb{G}$ and $\hat{h}(P) = 0$ if and only if P is a torsion point. Given an algebraic subvariety of $\mathbb{G}$, we denote by V^* the complement in V of the Zariski closure of the set of torsion points of V. Therefore, by the former Manin-Mumford conjecture, $V \backslash V^* = \overline{V_{\rm tors}}$ is a finite union of torsion varieties. We have ("Generalized Bogomolov Conjecture"):

i) If V is not a torsion subvariety, then there exists $\theta > 0$ such that the set $V(\theta) = \{P \in V \text{ such that } \hat{h}(P) \leq \theta\}$ is not Zariski dense in V.

ii) V^* is discrete for the metric induced by $\hat{h}$, *i.e.*

$$\inf\{\hat{h}(P) \text{ such that } P \in V^*\} > 0.$$

Again, the two assertions are equivalent and were proved for $\mathbb{G} = \mathbb{G}_m^n$ by Zhang (1995). In the abelian case, Ullmo (1998) proved Bogomolov's original formulation for curves ($\dim(V) = 1$); immediately after Zhang (1998) prove the above result for a subvariety of arbitrary dimension of an abelian variety. The semi-abelian case was solved by David and Philippon (2000).

In this seminar we describe several quantitative versions of the Generalized Bogomolov Conjecture for a torus $\mathbb{G} = \mathbb{G}_m^n$ which are sharp "up to an ε".