HANKEL PLANES

Let k be an algebraically closed field of characteristic 0. A *Hankel matrix* $H \in k^{p+1,m+1}$ (sometimes called *catalecticant*) is a matrix of type

$$H = \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \dots & \lambda_m \\ \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_{m+1} \\ \lambda_2 & \lambda_3 & \lambda_4 & \dots & \lambda_{m+2} \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_p & \lambda_{p+1} & \lambda_{p+2} & \dots & \lambda_{m+p} \end{pmatrix}$$

Such matrices arise in many parts of mathematics and are stricly connected with the standard rational normal curve $X_m \subseteq \mathbb{P}^m$.

We consider the following problem: given a matrix (written by its rows)

$$A = \begin{pmatrix} A^0 \\ A^1 \\ \vdots \\ A^m \end{pmatrix} \in k^{m+1,n+1}$$

using these rows one can construct, for any integer $p \geq 0$, the block Toeplitz matrix

$$T_A(p) = \begin{pmatrix} A^0 & \mathbf{0} & \dots & \mathbf{0} \\ A^1 & A^0 & \dots & \mathbf{0} \\ A^2 & A^1 & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ A^m & A^{m-1} & \dots & \dots \\ \mathbf{0} & A^m & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & A^m \end{pmatrix} \in k^{m+p+1,(n+1)(p+1)}$$

where $\mathbf{0} = (0, 0, \dots, 0)$ is the null vector.

We want to determine the rank of $T_A(p)$ starting from the rank of A; more precisely, denoting by $R(p) \subseteq k^{m+p+1}$ the vector space of relations among the rows of $T_A(p)$, our problem is to determine $R(p)$ for any $p > 0$ when $R = R(0)$ is given.

The connection between the ranks of these matrices is not purely numerical and requires a deep analysis of the space R. As one can expect this problem strongly involves X_m and moreover allows to define new geometric objects of the projective space $\mathbb{P}^m$ – the *Hankel planes* – whose geometric properties are related with X_m.