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Discriminants of systems of equations

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A system of k complex algebraic equations in n variables is said to be *typical*, if the homeomorphism type of the set of its solutions in the n-dimensional complex torus does not change as we perturb its (non-zero) coefficients, or if its Euler characteristic does not change (these two assumtions are equivalent for non-homogeneous systems). Non-typical systems form a non-empty hypersurface B in the space of systems of equations, whose monomials are contained in given finite sets $A_1, \ldots, A_k$. This is in contrast to the set of systems with singular solutions: dual defect lattice subsets are still not classified.
Moreover, a generic system in an irreducible component $B_i \subset B$ differs from a typical system by the Euler characteristics of its set of solutions. Taking this non-zero difference e_i of the Euler characteristics as the multiplicity of B_i, we turn the hypersurface B into an effective divisor $\sum e_i B_i$, whose equaltion we call the *Euler discriminant* of a system of algebraic equations with given sets of monomials $A_1, \ldots, A_k$. Despite its topological definition, there is a simple linear-algebraic determinantal formula for it; in particular, its Newton polytope equals $\sum_{a_1+\ldots+a_k=n+1} A_1^{a_1} \ldots A_k^{a^k}$, where the monomial $A_0 \ldots A_n$ stands for the mixed secondary polytope of finite sets $A_0, \ldots, A_n$ in Z^n.
The Euler discriminant generalizes well known algebraic objects: for $k = n+1$, it equals the Sturmfels sparse resultant of $n+1$ equaltions of n variables (which is defined by vanishing on all consistent systems of equations); for $k = 1$, it a posteriori coincides with the Gelfand-Kapranov-Zelevinsky principal determinant of an equation (which is the resultant of the equation and its partial derivatives); however, this object seems to have been overlooked for $1 < k \leq n$ so far. We study the Euler discriminant in a more general setting, which covers a number of noteworthy special cases besides the discriminant of a system of equations, described above. One such special case is the bifurcation set of a polynomial map $f : C^m \to C^n$. Z. Jelonek proved that it is a hypersurface for $m = n$ and estimated its degree. We prove that it is a hypersurface for arbitrary $m > n$, provided that the components of f are sufficiently generic with respect to their Newton diagrams, and find the Newton polytope of this hypersurface.