Singular fibers of the classical Gelfand–Cetlin system on $\mathfrak{u}(n)^*$

joint work with
E. Miranda (UPC Barcelona) and N. T. Zung (UPS Toulouse)

Damien Bouloc
Université Toulouse III – Paul Sabatier

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Why singular fibers?

(M, ω) symplectic manifold of dimension $2n$

Definition

An integrable Hamiltonian system on (M, ω) is the data of n functions $f_1, \ldots, f_n : M \to \mathbb{R}$ such that:

- $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \neq 0$ on an open dense subset of M,
- $\{f_i, f_j\} = 0$ for any $1 \leqslant i, j \leqslant n$.

The momentum map of the system is the map:

$$F = (f_1, \ldots, f_n) : M \to \mathbb{R}^n.$$

A point $x \in M$ is singular if $\mathrm{d}F(x) = 0$.

Why singular fibers?

$(q_1, \ldots, q_n, p_1, \ldots, p_n)$ coordinates on $\mathbb{T}^n \times D^n$

$$\omega_{can} = \sum \mathrm{d}p_i \wedge \mathrm{d}q_i$$

Theorem (Arnold–Liouville–Mineur)

If $L = F^{-1}(c)$ is a compact connected regular fiber of the system, then

$$L \approx \mathbb{T}^n.$$

Moreover, there exists a tubular neighborhood $\mathcal{U}$ of L st

$$(\mathcal{U}, \omega) \approx (\mathbb{T}^n \times D^n, \omega_{can})$$

and such that F depends only on $p_1, \ldots, p_n$.

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$\longrightarrow$ What about the singular fibers?

Coadjoint orbits of $U(n)$

- Let $U(n)$ be the Lie group of unitary matrices of size n:

$$U(n) = \{M \in \mathcal{M}_n(\mathbb{C}) \mid M^* M = I_n = MM^*\}$$

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- We identify its dual $\mathfrak{u}(n)^*$ with the space of Hermitian matrices:

$$\mathcal{H}(n) = \{A \in \mathcal{M}_n(\mathbb{C}) \mid A^* = A\}$$

via the Killing form + the multiplication $H \mapsto iH$.

Coadjoint orbits of $U(n)$

Under the identification $\mathfrak{u}(n)^* \approx \mathcal{H}(n)$, the coadjoint representation becomes:

$$\mathrm{Ad}^* : \left| \begin{array}{rcl} U(n) & \longrightarrow & \mathrm{Aut}(\mathcal{H}(n)) \\ P & \longmapsto & \mathrm{Ad}^*_P : (A \mapsto PAP^*) \end{array} \right.$$

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Claim

The set of Hermitian matrices with eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ is exactly the coadjoint orbit $\mathcal{O}_\lambda$ of

$$D_\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in \mathcal{H}(n)$$

Coadjoint orbits of $U(n)$

Fix $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ a set of real eigenvalues.

Denote by U_λ the subgroup of unitary matrices commuting with

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In particular, generic orbits have dimension $n(n-1)$

The Gelfand–Cetlin system on a generic orbit $\mathcal{O}_\lambda$

Denote by A_k the upper-left submatrix of size k of A

A_k is also Hermitian $\to$ k real eigenvalues

$$\begin{array}{ll}
A_{n-1}: & \gamma_1^{n-1}(A) \geqslant \gamma_2^{n-1}(A) \geqslant \gamma_3^{n-1}(A) \geqslant \cdots \geqslant \gamma_{n-1}^{n-1}(A) \\
A_{n-2}: & \gamma_1^{n-2}(A) \geqslant \gamma_2^{n-2}(A) \geqslant \cdots \geqslant \gamma_{n-2}^{n-2}(A) \\
\vdots & \cdots \\
A_2: & \gamma_1^{2}(A) \geqslant \gamma_2^{2}(A) \\
A_1: & \gamma_1^{1}(A)
\end{array}$$

The Gelfand–Cetlin system on a generic orbit $\mathcal{O}_\lambda$

Definition (Guillemin, Sternberg)

The Gelfand–Cetlin system on a generic coadjoint orbit $\mathcal{O}_\lambda$ of $\mathfrak{u}(n)^* \approx \mathcal{H}(n)$ is the integrable Hamiltonian system defined by the family of functions

$$\{\gamma_j^k \mid 1 \leqslant j \leqslant k < n\}$$

Goal: Describe the fibers of the momentum map :

$$F = (\gamma_i^j)_{1 \leqslant i \leqslant j < n} : \mathcal{O}_\lambda \longrightarrow \mathbb{R}^{\frac{n(n-1)}{2}}$$

The Gelfand–Cetlin system on a generic orbit $\mathcal{O}_\lambda$

The image of the momentum map of the Gelfand–Cetlin system is given by the Gelfand–Cetlin diagram:

$$
\begin{array}{ccccccccccccccc}
\lambda_1 & & > & & \lambda_2 & & > & & \lambda_3 & & > & \cdots & > & \lambda_{n-1} & > & & \lambda_n \\
& \geqslant & & \leqslant & & \geqslant & & \leqslant & & \geqslant & \leqslant & & & & \geqslant & \leqslant & \\
& & \gamma_1^{n-1} & & & & \gamma_2^{n-1} & & & \gamma_3^{n-1} & & & \cdots & & & \gamma_{n-1}^{n-1} & \\
& & & \geqslant & \leqslant & & & \geqslant & \leqslant & & & & & \geqslant & \leqslant & & \\
& & & & \gamma_1^{n-2} & & & & \gamma_2^{n-2} & & & \cdots & & \gamma_{n-2}^{n-2} & & & \\
& & & & & \ddots & & & & & & & \iddots & & & & \\
& & & & & & \geqslant & \leqslant & & \geqslant & \leqslant & & & & & & \\
& & & & & & \gamma_1^2 & & & \gamma_2^2 & & & & & & & \\
& & & & & & & \geqslant & \leqslant & & & & & & & & \\
& & & & & & & & \gamma_1^1 & & & & & & & &
\end{array}
$$

The Gelfand–Cetlin system on a generic orbit $\mathcal{O}_\lambda$

Claim

$A \in \mathcal{O}_\lambda$ is a singular point of the Gelfand–Cetlin system if and only if there is at least one equality in the Gelfand–Cetlin diagram

Geometric interpretation

In what follows we always assume $\lambda_1 > \cdots > \lambda_n > 0$.

To $A \in \mathcal{O}_\lambda$ associate the set $E_A = \{x \in \mathbb{C}^n \mid \langle Ax \mid x \rangle = 1\}$.

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If $A = C\, D_\lambda C^*$, then $\varphi_{C^*} = \varphi_C^{-1}$ maps E_A to

$$E_{D_\lambda} = \{y \in \mathbb{C}^n \mid \lambda_1 |y_1|^2 + \cdots + \lambda_n |y_n|^2 = 1\}$$

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Definition

We say that E_A is a n-dimensional complex ellipsoid with axes $v_1, \ldots, v_n$ (the eigenvectors of A) and radii $\frac{1}{\sqrt{\lambda_1}}, \ldots, \frac{1}{\sqrt{\lambda_n}}$.

Geometric interpretation

For $1 \leqslant k \leqslant n$, consider the canonical inclusion

$$i_k : (x_1, \ldots, x_k) \in \mathbb{C}^k \mapsto (x_1, \ldots, x_k, 0, \ldots, 0) \in \mathbb{C}^n$$

and denote by $\langle \cdot \mid \cdot \rangle_k$ the canonical Hermitian product on $\mathbb{C}^k$.

Then

$$\forall\, x \in \mathbb{C}^k, \quad \langle A_k x \mid x \rangle_k = \langle A\, i_k(x) \mid i_k(x) \rangle$$

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Claim

The intersection of E_A with $V_{\mathrm{std}}^k = \mathbb{C}^k \times \{0\}^{n-k}$ is the image by i_k of the k-dimensional ellipsoid E_{A_k}

Geometric interpretation

E_A complex ellipsoid associated to A

$+$ standard complete flag $V^0_{\mathrm{std}} \subset V^1_{\mathrm{std}} \subset \cdots \subset V^n_{\mathrm{std}} = \mathbb{C}^n$

$=$ sequence of ellipsoids $E_{A_1} \subset E_{A_2} \subset \cdots \subset E_{A_n} = E_A$

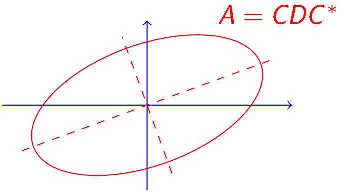
$$\downarrow \varphi_C^{-1}$$

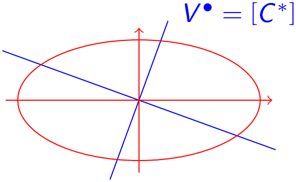
E_{D_λ} "standard" ellipsoid associated to λ

$+$ complete flag $V^0_C \subset V^1_C \subset \cdots \subset V^n_C = \mathbb{C}^n$

$=$ sequence of ellipsoids $E^1 \subset E^2 \subset \cdots \subset E^n = E_{D_\lambda}$

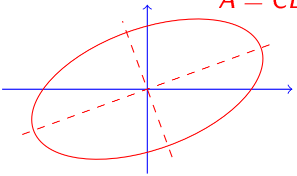
Geometric interpretation



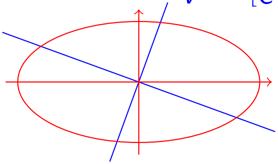


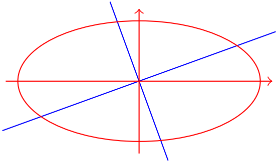
Geometric interpretation

$A = CDC^*$



$V^\bullet = [C^*]$





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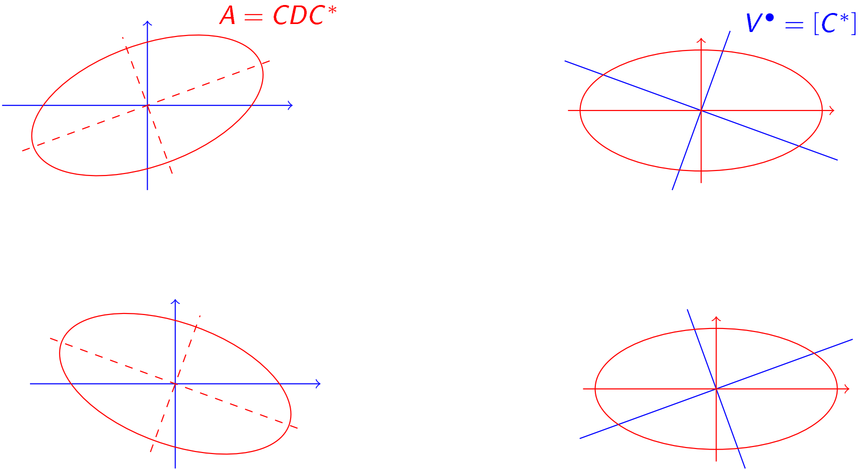

Symmetry group of ellipsoid flags

Fix $C \in U(n)$ a basis for some flag $V^\bullet$ and set:

- G_k the set of all symmetries of the ellipsoid $E^k = E_{D_\lambda} \cap V^k$

$\rightarrow$ one can define new basis $\phi_* C$ for any

$$\phi = (\phi_1, \ldots, \phi_n) \in G_1 \times \cdots \times G_n.$$

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Claim

C, C' give ellipsoids with same eigenvalues if and only if

$$C' = \phi_* C, \qquad \phi \in G_1 \times \cdots \times G_n.$$

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Fix $C \in U(n)$ a basis for some flag $V^\bullet$ and set:

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  - fix also E^{k-1},
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Claim

$$\phi_* C = \psi_* C \quad \Longleftrightarrow \quad \psi = \phi \cdot h, \quad h \in H_1 \times \cdots \times H_n$$

Consequently,

$$q \circ \Gamma_\lambda^{-1}(c) \approx (G_1 \times \cdots \times G_n)/(H_1 \times \cdots \times H_n)$$

Symmetry group of ellipsoid flags

Theorem (B, Miranda, Zung)

The fiber $F^{-1}(c)$ of the Gelfand–Cetlin system is a submanifold diffeomorphic to

$$((G_1 \times \cdots \times G_n)/(H_1 \times \cdots \times H_n))/U_\lambda$$

Examples:

- if c is regular we find $F^{-1}(c) = \mathbb{T}^{\frac{n(n-1)}{2}}$
- on a generic coadjoint orbit $\mathcal{O}_{(a,b,c)}$ of $U(3)$

$$F^{-1}(b, b, b) \approx U(2)/\mathbb{T}^1 \approx S^3$$

$\longrightarrow$ Lagrangian fiber!

Some remarks

- The geometric model of the singular fiber is uniquely determined by the equalities in the Gelfand–Cetlin diagram

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Some remarks

- The geometric model of the singular fiber is uniquely determined by the equalities in the Gelfand–Cetlin diagram
- Our results hold when $\mathcal{O}_\lambda$ is not a generic orbit (but one needs to define more carefully the Hamiltonian system on $\mathcal{O}_\lambda$)
- This system has characteristics similar to other integrable Hamiltonian systems (Kapovich–Millson bending flows, Nohara–Ueda on $\mathrm{Gr}(2, \mathbb{C}^n)$)