Proper Lie groupoids are real analytic

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Theorem

Any Lie proper groupoid $\mathcal{G}$ admits a compatible (real) analytic structure.

$$\mathcal{G} \xrightarrow{\cong(C^\infty)} \mathcal{H}, \qquad \mathcal{H} \text{ is } C^\omega.$$

Theorem

A generalization of Zung's non-linear average iteration method.

In particular, compatibility of Zung's method with C^ω-data defined on open (saturated) subsets.

[arXiv:161209012]

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$$C^0 \rightsquigarrow C^{1,1}$$

▷ **Mnfds:** $\begin{cases} C^0 \not\Rightarrow C^1, z_1^5 + z_2^3 + z_3^2 + z_4^2 + z_5^2 + \sum_{j=1}^5 e^{j-1} z_j^6 [Kuiper] \\ \end{cases}$

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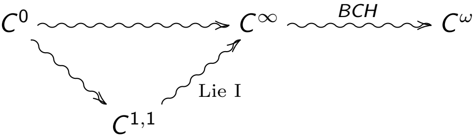

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▷ **Hilbert-Smith conjecture ($G \ltimes M$)**: G loc. compact top. group, $G \ltimes M$, $G \hookrightarrow \mathrm{Homeo}(M) \Longrightarrow G$ is a Lie group.

- True if $M = M^3$ [Pardon]
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▷ Analogy with manifold story:

$$\begin{cases} G \to \mathrm{Homeo}(M) \nRightarrow G \to \mathrm{Diff}^k(M), k \geq 1 : \mathbb{Z}_2 \ltimes S^3 \quad \text{[Bing]} \\ \\ \end{cases}$$

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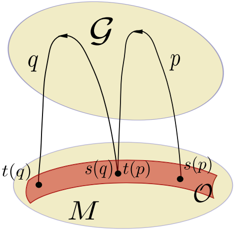
▷ **Lie groupoid** $\mathcal{G} \overset{s,t}{\rightrightarrows} M$ formalizes (smooth) partial symmetries:



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- Each $x \in M$ has $G_x \subset \mathcal{G}$ isotropy group.
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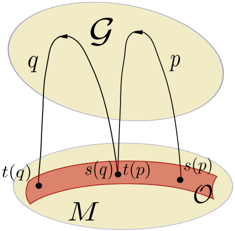

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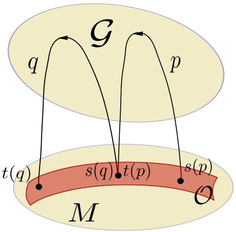

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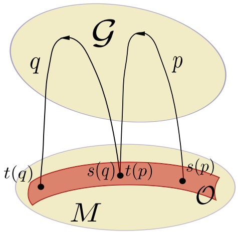

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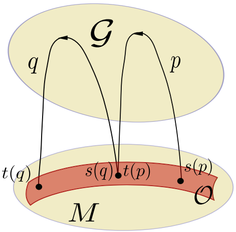

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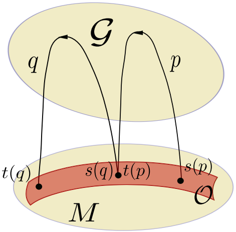

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$$\begin{array}{c} P \times_M P \\ \pi_1 \downarrow\downarrow \pi_2 \\ P \end{array}$$

4.Hol. & homotopy oid of $\mathcal{F}$:

$$\begin{array}{ccc} \mathrm{Hol}(\mathcal{F}) & \longleftarrow & \Pi_1(\mathcal{F}) \\ \downarrow\downarrow & & \downarrow\downarrow \\ M & & M \end{array}$$

5.Action groupoid:

$$G \times M \overset{\ltimes}{\underset{\pi_2}{\rightrightarrows}} M$$

6.Action oid for submersion $G \ltimes (P \xrightarrow{p} M)$:

$$P \times_{G \ltimes M} P \overset{[\pi_2]}{\underset{[\pi_1]}{\rightrightarrows}} P/G$$

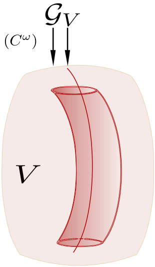
7.(Integrable) Poisson mnfld. (M, π):

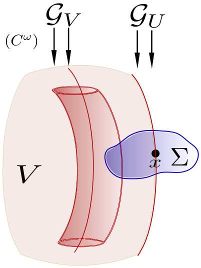
$$\Sigma(\pi) \rightrightarrows M$$

Definition ("Compact type conditions")

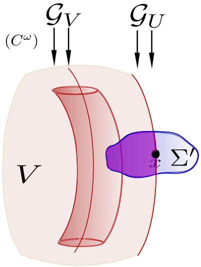
$\mathcal{G} \rightrightarrows M$ is proper when $s \times t : \mathcal{G} \to M \times M$ is proper.

$\mathcal{G} \rightrightarrows M$ s-proper when $s : \mathcal{G} \to M$ is proper.


$\mathcal{G}_V$
(C^ω)
V



Grp: $G \ltimes U \overset{[\text{Palais}]}{\cong} G \times_{G_x} \Sigma \overset{[\text{Bochner}]}{\cong} G \times_{G_x} E^r$



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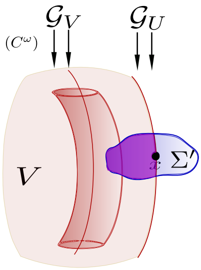

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Oid: $\mathcal{G}_U \overset{P=s^{-1}(\Sigma)}{\cong} P \times_{\mathcal{G}_\Sigma} P + \mathcal{G}_\Sigma \overset{\text{[Zung]}}{\cong} G_x \ltimes E^r(\mathcal{G}_\Sigma \xrightarrow{\Phi} G_x)$

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Oid: $\mathcal{G}_\Sigma \xrightarrow{\Phi} G_x \rightsquigarrow \mathcal{G}_\Sigma \xrightarrow{\Phi'} G_x$, $\Phi'|_{\mathcal{G}_{\Sigma' \cap V}}$ is C^ω.

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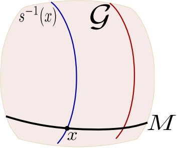

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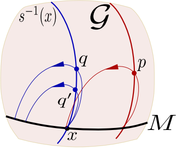

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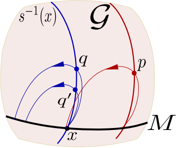

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▷ : $\mathcal{G} \xrightarrow{\phi} G \rightsquigarrow \psi(q,p) = \phi^{-1}(p)\psi^{-1}(q)\phi(qp) \rightsquigarrow \Delta(\phi) := d_G(\psi, e)$.

$$A(\phi) = \exp\left(\int_{s^{-1}(t(p))} \log(\psi(q,p))\mu\right), \quad \hat{\phi} := A(\phi)\phi.$$

Theorem (Zung)

Given $(G, \langle\cdot,\cdot\rangle)$ compact+bi-invariant, $\exists C > 0$ s.t $\forall \mathcal{G} \rightrightarrows M$ s-proper + μ Haar density, $\forall \phi \in C^\infty(\mathcal{G}, G)$ with $\Delta(\phi) < C$, if $\phi_{n+1} := \hat{\phi}_n$ then

$$\phi_n \xrightarrow{C^0(C^\infty)} \Phi, \quad \Phi : \mathcal{G} \to G \text{ morphism.}$$

▷ If $\mathcal{G} \rightrightarrows M$ s-proper $x \in M$ fixed point, $W \subset M$ saturated, $\phi|_{\mathcal{G}_W}$, $\mu|_{\mathcal{G}_W}$ are C^ω, then $\phi_n|_{\mathcal{G}_W} \in C^\omega$.

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▷ : How do we know if $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is C^ω?:

Quant: $||D^r f||_K \leq R(M^r)r!$, $\forall r$ (BCH formula.......)

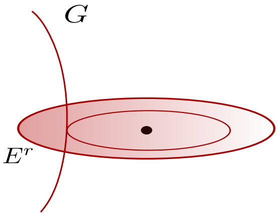
Qual: $\exists F : \mathcal{U} \subset \mathbb{C}^n \to \mathbb{C}^m$ holomorphic with $F|_U = f$.

Idea: "Complexify" the averaging (Weyl unitary trick of sorts).

$$\begin{cases} G \hookrightarrow \mathbb{G}, \\ \mathcal{G} \hookrightarrow \mathcal{H} \text{ \underline{local} holomorphic groupoid,} \\ \text{not just } \phi_n \rightsquigarrow \phi_{n\mathbb{C}} \text{ (commom domain of } \phi_{n\mathbb{C}} \text{must be } \mathcal{H}). \end{cases}$$

- **Weyl's unitary trick from the Lie groupoid perspective:**

  $G \ltimes E \rightsquigarrow \mathbb{G} \ltimes \mathbb{E}$, $f \in \mathrm{Hol}(\mathbb{E}) \Rightarrow \int_G f \mu_G \in \mathrm{Hol}(\mathbb{E})$.



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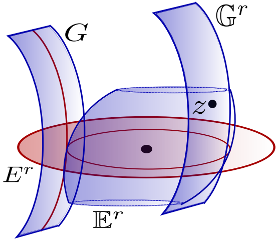

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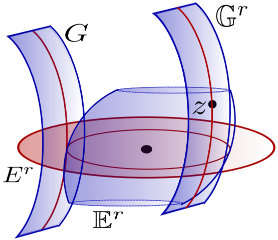

(i) $\mathcal{K} := G \times \mathbb{E}^r$ is an **s-proper core** for $\mathbb{G}^r \ltimes \mathbb{E}^r$:

- $m(\mathcal{K} \times_{s,t} \mathbb{G}^r \ltimes \mathbb{E}^r) \subset \mathbb{G}^r \ltimes \mathbb{E}^r$.
- $\mathcal{K}$ submanifold $\pitchfork$ s; $s_{\mathcal{K}} : \mathcal{K} \to \mathbb{E}^r$ s-proper.
- $\mathcal{K} \circlearrowleft \mathcal{K}$ (where defined) identifies $s_{\mathcal{K}}$-fibers.

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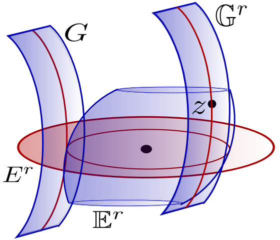

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(ii) $\mu_G \in \Gamma(\mathcal{D}^2_s) \rightsquigarrow \mu_{\mathbb{G}} \in \mathrm{Hol}(K_s) \rightsquigarrow \mu_{\mathbb{G}}|_{\mathcal{K}}$ is a **Holom. Haar density**:

- $\int_{\mathcal{K}} \mu_{\mathbb{G}} : \Gamma^\omega(\mathbb{G}^r \ltimes \mathbb{E}^r, \mathbb{C}) \to \Gamma^\omega(\mathbb{E}^r, \mathbb{C})$.

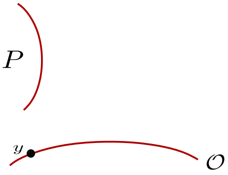
  Locally $(\mathbb{G}^r_z, \mathcal{K}_z) \cong (\mathbb{C}^n, \mathbb{R}^n) \Rightarrow K_s|_{\mathcal{K}} \cong \mathcal{D}^2_s \otimes \underline{\mathbb{C}}$.
- $\mathrm{Hol}(\mathbb{G}^r \ltimes \mathbb{E}^r) \to \mathrm{Hol}(\mathbb{E}^r)$.

Theorem

Given $(H, ||\cdot||)$ *cx. Lie group +(normalized) norm* $+W \subset H$ *rel. compact,* $\exists C > 0$ *s.t*
$\forall \mathcal{H} \rightrightarrows M$ *loc. hol. oid* $+ \mathcal{K} \subset \mathcal{H}$ *s-proper core* $+ \mu$ *Haar density,*
$\forall \phi \in C^\infty(\mathcal{H}, H)$ *with* $\phi(\mathcal{H}) \subset W$ *and* $\Delta(\phi) < C$*,*
if $\phi_{n+1} := \hat{\phi}_n$ *then*

$$\phi_n \xrightarrow{C^0} \Phi, \quad \Phi : \mathcal{H} \to H,\ (\mathcal{K}) - \text{morphism}.$$

- $\mathcal{G} \rightrightarrows M$ s-proper, $y \in \mathcal{O} \subset M \rightsquigarrow \mathcal{G}_{\mathcal{O}} \cong P \times_G P \rightrightarrows \mathcal{O}$ $(P = s^{-1}(y))$.



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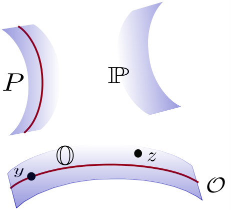

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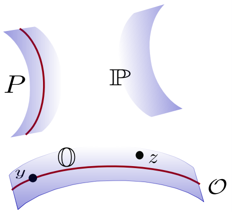

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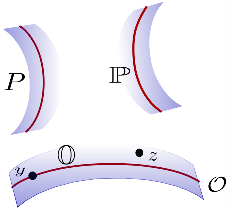

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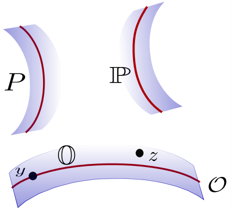
- Fix G-inv. analytic metric on P bi-invariant on fibers

$$\overset{\text{unique}}{\rightsquigarrow} (TP^r, J), \quad \mathrm{d}_P \text{ solves Monge} - \text{Ampere}\,[\mathit{Guillemin} - \mathit{Stenzel}]$$

$$(TP^r, J) \subset \mathbb{P}$$

$$\mathcal{KN} = \begin{cases} \mu^{-1}(0) \\ \min \mathrm{d}_P|_{\text{fibers}} \end{cases}$$

▷ $\mathcal{G} \rightrightarrows M$ s-proper, $y \in \mathcal{O} \subset M \rightsquigarrow \mathcal{G}_{\mathcal{O}} \cong P \times_G P \rightrightarrows \mathcal{O}$ $(P = s^{-1}(y))$.



- $P \times_G P \hookrightarrow \mathbb{P} \times_{\mathbb{G}^r} \mathbb{P}$

$$\mathcal{K} = P \times_G (\mathcal{KN}), \quad \mathcal{KN} \subset \mathbb{P}$$

▷ Fix G-inv. analytic metric on P bi-invariant on fibers

$$\overset{\text{unique}}{\rightsquigarrow} (TP^r, J), \quad \mathrm{d}_P \text{ solves Monge} - \text{Ampere}\,[\mathit{Guillemin} - \mathit{Stenzel}]$$

$$(TP^r, J) \subset \mathbb{P}$$

$$\mathcal{KN} = \begin{cases} \mu^{-1}(0) \\ \min \mathrm{d}_P|_{\text{fibers}} \end{cases}$$