Beyond the SHGH Conjecture

Algebraic Geometry Seminar
University of Barcelona

October 7, 2016

Juan C. Migliore

University of Notre Dame

Introduction

Joint work with:

David Cook II

Brian Harbourne

Uwe Nagel

Introduction

Joint work with:

David Cook II

Brian Harbourne

Uwe Nagel

(Motivated by a paper of Di Gennaro - Ilardi - Vallès.)

Let

- $R = K[x_1, \dots, x_n]$ with K an infinite field.

Let

- $R = K[x_1, \ldots, x_n]$ with K an infinite field.
- R/I be a standard graded artinian algebra

Let

- $R = K[x_1, \ldots, x_n]$ with K an infinite field.
- R/I be a standard graded artinian algebra
- ℓ be a general linear form.

Let

- $R = K[x_1, \ldots, x_n]$ with K an infinite field.
- R/I be a standard graded artinian algebra
- ℓ be a general linear form.

Definition. We say R/I has the Weak Lefschetz Property (WLP) if, for every positive integer i, we have that

$$\times \ell : [R/I]_{i-1} \to [R/I]_i$$

has maximal rank (i.e. is either injective or surjective).

Let

- $R = K[x_1, \dots, x_n]$ with K an infinite field.
- R/I be a standard graded artinian algebra
- ℓ be a general linear form.

Definition. We say R/I has the Weak Lefschetz Property (WLP) if, for every positive integer i, we have that

$$\times\ell : [R/I]_{i-1} \to [R/I]_i$$

has maximal rank (i.e. is either injective or surjective).

We say that R/I has the Strong Lefschetz Property (SLP) if, for all positive integers i and k, we have

$$\times\ell^k : [R/I]_{i-k} \to [R/I]_i$$

has maximal rank.

Let

- $R = K[x_1, \ldots, x_n]$ with K an infinite field.
- R/I be a standard graded artinian algebra
- ℓ be a general linear form.

Definition. We say R/I has the Weak Lefschetz Property (WLP) if, for every positive integer i, we have that

$$\times\ell : [R/I]_{i-1} \to [R/I]_i$$

has maximal rank (i.e. is either injective or surjective).

We say that R/I has the Strong Lefschetz Property (SLP) if, for all positive integers i and k, we have

$$\times\ell^k : [R/I]_{i-k} \to [R/I]_i$$

has maximal rank. We also can consider $\times\ell^k$ for specific values of k and/or i.

Informally, the Lefschetz properties measure the following.

Informally, the Lefschetz properties measure the following.

Given certain dimensions

(specifically $\dim[R/I]_i$ and $\dim[R/I]_{i-1}$ for WLP),

Informally, the Lefschetz properties measure the following.

Given certain dimensions

(specifically $\dim[R/I]_i$ and $\dim[R/I]_{i-1}$ for WLP),

there is some general choice to make

(specifically the general linear form ℓ)

Informally, the Lefschetz properties measure the following.

Given certain dimensions

(specifically $\dim[R/I]_i$ and $\dim[R/I]_{i-1}$ for WLP),

there is some general choice to make

(specifically the general linear form ℓ)

leading to a new dimension

(specifically $\dim \operatorname{coker}(\times \ell)$)

Informally, the Lefschetz properties measure the following.

Given certain dimensions

(specifically $\dim[R/I]_i$ and $\dim[R/I]_{i-1}$ for WLP),

there is some general choice to make

(specifically the general linear form ℓ)

leading to a new dimension

(specifically $\dim \operatorname{coker}(\times \ell)$)

and we want to know when this is larger than "expected."

Informally, the Lefschetz properties measure the following.

Given certain dimensions

(specifically $\dim[R/I]_i$ and $\dim[R/I]_{i-1}$ for WLP),

there is some general choice to make

(specifically the general linear form ℓ)

leading to a new dimension

(specifically $\dim \operatorname{coker}(\times \ell)$)

and we want to know when this is larger than "expected."

In this sense, the main topic of this talk shares this Lefschetz philosophy. There will be a direct connection at the end.

Expected number of independent conditions

Let K be an infinite field of arbitrary characteristic and let $R = K[x, y, z]$.

Expected number of independent conditions

Let K be an infinite field of arbitrary characteristic and let $R = K[x, y, z]$.

Easy Question 1. Consider the complete linear system $\mathcal{L}_j$ of plane curves of degree j.

Recall $\dim \mathcal{L}_j = \dim_K [R]_j - 1 = \binom{j+2}{2} - 1$.

Expected number of independent conditions

Let K be an infinite field of arbitrary characteristic and let $R = K[x, y, z]$.

Easy Question 1. Consider the complete linear system $\mathcal{L}_j$ of plane curves of degree j.

Recall $\dim \mathcal{L}_j = \dim_K [R]_j - 1 = \binom{j+2}{2} - 1$.

Let $P \in \mathbb{P}^2$. What is the dimension of the linear system of plane curves of degree j passing through P?

Expected number of independent conditions

Let K be an infinite field of arbitrary characteristic and let $R = K[x, y, z]$.

Easy Question 1. Consider the complete linear system $\mathcal{L}_j$ of plane curves of degree j.

Recall $\dim \mathcal{L}_j = \dim_K [R]_j - 1 = \binom{j+2}{2} - 1$.

Let $P \in \mathbb{P}^2$. What is the dimension of the linear system of plane curves of degree j passing through P?

Answer. Regardless of the choice of P, the dimension is

$$\dim \mathcal{L}_j - 1 = \binom{j+2}{2} - 2.$$

That is, P *imposes one independent condition* on $\mathcal{L}_j$.

Easy Question 2. Consider the complete linear system $\mathcal{L}_j$ of plane curves of degree j.

Let $\{P_1, \ldots, P_d\} \subset \mathbb{P}^2$ be an arbitrary set of distinct points. How many independent conditions do $P_1, \ldots, P_d$ impose on $\mathcal{L}_j$?

Easy Question 2. Consider the complete linear system $\mathcal{L}_j$ of plane curves of degree j.

Let $\{P_1, \ldots, P_d\} \subset \mathbb{P}^2$ be an arbitrary set of distinct points. How many independent conditions do $P_1, \ldots, P_d$ impose on $\mathcal{L}_j$?

Answer. There's no single answer. It depends on the choice of the points $P_1, \ldots, P_d$ and on j. (Specifically on the Hilbert function.)

Easy Question 2. Consider the complete linear system $\mathcal{L}_j$ of plane curves of degree j.

Let $\{P_1, \ldots, P_d\} \subset \mathbb{P}^2$ be an arbitrary set of distinct points. How many independent conditions do $P_1, \ldots, P_d$ impose on $\mathcal{L}_j$?

Answer. There's no single answer. It depends on the choice of the points $P_1, \ldots, P_d$ and on j. (Specifically on the Hilbert function.)

Easy Question 3. Assume that $P_1, \ldots, P_d$ are chosen generally. Then how many independent conditions do they impose on $\mathcal{L}_j$?

Easy Question 2. Consider the complete linear system $\mathcal{L}_j$ of plane curves of degree j.

Let $\{P_1, \ldots, P_d\} \subset \mathbb{P}^2$ be an arbitrary set of distinct points. How many independent conditions do $P_1, \ldots, P_d$ impose on $\mathcal{L}_j$?

Answer. There's no single answer. It depends on the choice of the points $P_1, \ldots, P_d$ and on j. (Specifically on the Hilbert function.)

Easy Question 3. Assume that $P_1, \ldots, P_d$ are chosen generally. Then how many independent conditions do they impose on $\mathcal{L}_j$?

Answer. As many as possible. If there aren't too many points, they impose independent conditions.

More precisely, they impose $\min\left\{\binom{j+2}{2}, d\right\}$ independent conditions.

Slightly less easy question. Let $P \in \mathbb{P}^2$ be an arbitrary point. Let $m \geq 1$. How many conditions are imposed on plane curves of degree j if we require them to have multiplicity at least m at P?

Slightly less easy question. Let $P \in \mathbb{P}^2$ be an arbitrary point. Let $m \geq 1$. How many conditions are imposed on plane curves of degree j if we require them to have multiplicity at least m at P?

Answer. The number of independent conditions is

$$\min \left\{ \binom{j+2}{2}, \binom{m+1}{2} \right\}.$$

(Think about partial derivatives.)

Slightly less easy question. Let $P \in \mathbb{P}^2$ be an arbitrary point. Let $m \geq 1$. How many conditions are imposed on plane curves of degree j if we require them to have multiplicity at least m at P?

Answer. The number of independent conditions is

$$\min\left\{\binom{j+2}{2}, \binom{m+1}{2}\right\}.$$

(Think about partial derivatives.)

Notation/Terminology. Let mP be the scheme defined by the (saturated) ideal I_P^m.

Slightly less easy question. Let $P \in \mathbb{P}^2$ be an arbitrary point. Let $m \geq 1$. How many conditions are imposed on plane curves of degree j if we require them to have multiplicity at least m at P?

Answer. The number of independent conditions is

$$\min\left\{\binom{j+2}{2}, \binom{m+1}{2}\right\}.$$

(Think about partial derivatives.)

Notation/Terminology. Let mP be the scheme defined by the (saturated) ideal I_P^m.

If $\binom{m+1}{2} \leq \binom{j+2}{2}$, we'll say that "the fat point mP imposes $\binom{m+1}{2}$ independent conditions on $\mathcal{L}_j$."

Very hard question. ("Fat" analog of earlier question.) Let $\{P_1, \dots, P_d\} \subset \mathbb{P}^2$ be a general set of points. Let $m_1, \dots, m_d$ be positive integers.

Very hard question. ("Fat" analog of earlier question.) Let $\{P_1, \ldots, P_d\} \subset \mathbb{P}^2$ be a general set of points. Let $m_1, \ldots, m_d$ be positive integers.

How many conditions do $\{m_1 P_1, \ldots, m_d P\}$ impose on $\mathcal{L}_j$?

Very hard question. ("Fat" analog of earlier question.) Let $\{P_1, \ldots, P_d\} \subset \mathbb{P}^2$ be a general set of points. Let $m_1, \ldots, m_d$ be positive integers.

How many conditions do $\{m_1 P_1, \ldots, m_d P\}$ impose on $\mathcal{L}_j$?

Notation. Denote by $X = m_1 P_1 + \cdots + m_d P_d$ the above union of fat points.

Very hard question. ("Fat" analog of earlier question.) Let $\{P_1, \ldots, P_d\} \subset \mathbb{P}^2$ be a general set of points. Let $m_1, \ldots, m_d$ be positive integers.

How many conditions do $\{m_1 P_1, \ldots, m_d P\}$ impose on $\mathcal{L}_j$?

Notation. Denote by $X = m_1 P_1 + \cdots + m_d P_d$ the above union of fat points.

Naive guess: in analogy with the case where $m_i = 1$ for all i, they should impose

$$\min\left\{ \binom{j+2}{2}, \binom{m_1+1}{2} + \binom{m_2+1}{2} + \cdots + \binom{m_d+1}{2} \right\}$$

independent conditions.

Very hard question. ("Fat" analog of earlier question.) Let $\{P_1, \ldots, P_d\} \subset \mathbb{P}^2$ be a general set of points. Let $m_1, \ldots, m_d$ be positive integers.

How many conditions do $\{m_1 P_1, \ldots, m_d P\}$ impose on $\mathcal{L}_j$?

Notation. Denote by $X = m_1 P_1 + \cdots + m_d P_d$ the above union of fat points.

Naive guess: in analogy with the case where $m_i = 1$ for all i, they should impose

$$\min \left\{ \binom{j+2}{2}, \binom{m_1+1}{2} + \binom{m_2+1}{2} + \cdots + \binom{m_d+1}{2} \right\}$$

independent conditions. Let's call this the expected number of conditions.

When $d = 1$ the naive guess is true! What about $d \geq 2$?

When $d = 1$ the naive guess is true! What about $d \geq 2$?

Example. Let $d = 2$ and $m_1 = m_2 = 2$.

When $d = 1$ the naive guess is true! What about $d \geq 2$?

Example. Let $d = 2$ and $m_1 = m_2 = 2$.

The “prediction” is that the scheme $2P_1 + 2P_2$ imposes

$$\binom{1+2}{2} + \binom{1+2}{2} = 6$$

independent conditions on forms of any degree j, or else the linear system is empty.

When $d = 1$ the naive guess is true! What about $d \geq 2$?

Example. Let $d = 2$ and $m_1 = m_2 = 2$.

The "prediction" is that the scheme $2P_1 + 2P_2$ imposes

$$\binom{1+2}{2} + \binom{1+2}{2} = 6$$

independent conditions on forms of any degree j, or else the linear system is empty.

But consider $j = 2$. Since $\dim K[x, y, z]_2 = 6$, this means the "prediction" is that there is no curve of degree 2 double at both points.

When $d = 1$ the naive guess is true! What about $d \geq 2$?

Example. Let $d = 2$ and $m_1 = m_2 = 2$.

The "prediction" is that the scheme $2P_1 + 2P_2$ imposes

$$\binom{1+2}{2} + \binom{1+2}{2} = 6$$

independent conditions on forms of any degree j, or else the linear system is empty.

But consider $j = 2$. Since $\dim K[x, y, z]_2 = 6$, this means the "prediction" is that there is no curve of degree 2 double at both points. Is this true?

When $d = 1$ the naive guess is true! What about $d \geq 2$?

Example. Let $d = 2$ and $m_1 = m_2 = 2$.

The "prediction" is that the scheme $2P_1 + 2P_2$ imposes

$$\binom{1+2}{2} + \binom{1+2}{2} = 6$$

independent conditions on forms of any degree j, or else the linear system is empty.

But consider $j = 2$. Since $\dim K[x, y, z]_2 = 6$, this means the "prediction" is that there is no curve of degree 2 double at both points. Is this true?

No! There is a unique line containing P_1, P_2. Its square is double at both points!

But “most” of the time it is true that the (general) fat points impose the expected number of conditions.

But “most” of the time it is true that the (general) fat points impose the expected number of conditions.

Open Question. How do you prove this, and what are the exceptions?

But “most” of the time it is true that the (general) fat points impose the expected number of conditions.

Open Question. How do you prove this, and what are the exceptions?

There is a long history of research on this problem, culminating in the so-called SHGH conjecture.

Note: SHGH = Segre-Harbourne-Gimigliano-Hirschowitz.

But “most” of the time it is true that the (general) fat points impose the expected number of conditions.

Open Question. How do you prove this, and what are the exceptions?

There is a long history of research on this problem, culminating in the so-called SHGH conjecture.

Note: SHGH = Segre-Harbourne-Gimigliano-Hirschowitz.

SHGH gives a complete conjectural answer to the question. They describe precisely when you do not get the expected number of conditions (as happened in our example).

But “most” of the time it is true that the (general) fat points impose the expected number of conditions.

Open Question. How do you prove this, and what are the exceptions?

There is a long history of research on this problem, culminating in the so-called SHGH conjecture.

Note: SHGH = Segre-Harbourne-Gimigliano-Hirschowitz.

SHGH gives a complete conjectural answer to the question. They describe precisely when you do not get the expected number of conditions (as happened in our example).

When $m_1 = m_2 = \cdots = m_d = 2$, the conjecture agrees with the Alexander-Hirschowitz theorem. (See also K. Chandler’s work.)

Question. How can we move beyond the SHGH conjecture?

Question. How can we move beyond the SHGH conjecture?

1. One could ask exactly the same question for a general set of fat points in $\mathbb{P}^n$. So far there isn't even a conjectured complete answer.

Question. How can we move beyond the SHGH conjecture?

1. One could ask exactly the same question for a general set of fat points in $\mathbb{P}^n$. So far there isn't even a conjectured complete answer.

2. Staying in $\mathbb{P}^2$, start with a linear system $\mathcal{L}$ that is not complete!

Question. How can we move beyond the SHGH conjecture?

1. One could ask exactly the same question for a general set of fat points in $\mathbb{P}^n$. So far there isn't even a conjectured complete answer.

2. Staying in $\mathbb{P}^2$, start with a linear system $\mathcal{L}$ that is not complete!

   Specifically, let

   $$\mathcal{L} = \left|[I_Z]_{j+1}\right|,$$

   the linear system of curves of degree $j+1$ passing through a fixed (reduced?) set of points Z.

   Note: The number of conditions that Z imposes on curves of degree $j+1$ is irrelevant here.

Initial Questions.

- Does $X = m_1 P_1 + \cdots + m_d P_d$ (where $P_1, \ldots, P_d$ are general) impose the expected number of conditions on $\mathcal{L}$?

Initial Questions.

- Does $X = m_1P_1 + \cdots + m_dP_d$ (where $P_1, \ldots, P_d$ are general) impose the expected number of conditions on $\mathcal{L}$? (Yes when all the m_i are 1.)

Initial Questions.

- Does $X = m_1 P_1 + \cdots + m_d P_d$ (where $P_1, \ldots, P_d$ are general) impose the expected number of conditions on $\mathcal{L}$? (Yes when all the m_i are 1.)
- If not, can we predict when they do not?

Initial Questions.

- Does $X = m_1P_1 + \cdots + m_dP_d$ (where $P_1, \ldots, P_d$ are general) impose the expected number of conditions on $\mathcal{L}$? (Yes when all the m_i are 1.)
- If not, can we predict when they do not?
- How does the geometry of Z relate to this question?

Initial Questions.

- Does $X = m_1P_1 + \cdots + m_dP_d$ (where $P_1, \ldots, P_d$ are general) impose the expected number of conditions on $\mathcal{L}$? (Yes when all the m_i are 1.)

- If not, can we predict when they do not?

- How does the geometry of Z relate to this question?

- Are there connections between this and other interesting questions?

Initial Questions.

- Does $X = m_1P_1 + \cdots + m_dP_d$ (where $P_1, \ldots, P_d$ are general) impose the expected number of conditions on $\mathcal{L}$? (Yes when all the m_i are 1.)
- If not, can we predict when they do not?
- How does the geometry of Z relate to this question?
- Are there connections between this and other interesting questions?
- Clearly this question is intractable as stated. What is the first non-trivial special case? Even $d = 1$ is interesting!

A seemingly unrelated topic!

A seemingly unrelated topic!

Let K be an arbitrary (!!!) algebraically closed field.

The next part of the talk is mostly standard. It describes the amazing (to me) fact that even in positive characteristic, things work better than I would have expected.

Let K be an arbitrary (!!!) algebraically closed field.

The next part of the talk is mostly standard. It describes the amazing (to me) fact that even in positive characteristic, things work better than I would have expected.

Let $R = K[x, y, z]$. Let $f = \ell_1 \cdots \ell_d$ be a product of d linear forms, none a scalar multiple of any other.

Let K be an arbitrary (!!!) algebraically closed field.

The next part of the talk is mostly standard. It describes the amazing (to me) fact that even in positive characteristic, things work better than I would have expected.

Let $R = K[x, y, z]$. Let $f = \ell_1 \cdots \ell_d$ be a product of d linear forms, none a scalar multiple of any other.

Let $\mathcal{A}_f$ be the line arrangement in $\mathbb{P}^2$ defined by f. Note the lines of $\mathcal{A}_f$ are dual to a reduced set of d distinct points Z.

Let K be an arbitrary (!!!) algebraically closed field.

The next part of the talk is mostly standard. It describes the amazing (to me) fact that even in positive characteristic, things work better than I would have expected.

Let $R = K[x, y, z]$. Let $f = \ell_1 \cdots \ell_d$ be a product of d linear forms, none a scalar multiple of any other.

Let $\mathcal{A}_f$ be the line arrangement in $\mathbb{P}^2$ defined by f. Note the lines of $\mathcal{A}_f$ are dual to a reduced set of d distinct points Z.

We'll consider two ideals:

- $J = (f_x, f_y, f_z, f)$

Let K be an arbitrary (!!!) algebraically closed field.

The next part of the talk is mostly standard. It describes the amazing (to me) fact that even in positive characteristic, things work better than I would have expected.

Let $R = K[x, y, z]$. Let $f = \ell_1 \cdots \ell_d$ be a product of d linear forms, none a scalar multiple of any other.

Let $\mathcal{A}_f$ be the line arrangement in $\mathbb{P}^2$ defined by f. Note the lines of $\mathcal{A}_f$ are dual to a reduced set of d distinct points Z.

We'll consider two ideals:

- $J = (f_x, f_y, f_z, f)$
- $J' = (f_x, f_y, f_z)$.

Recall Euler's theorem:

$$xf_x + yf_y + zf_z = df.$$

Recall Euler's theorem:

$$xf_x + yf_y + zf_z = df.$$

- If char(K) does not divide $d = \deg(f)$, then

  $$xf_x + yf_y + zf_z = df$$

  is a non-zero scalar multiple of f, so $J = J'$ in this case.

Recall Euler's theorem:

$$xf_x + yf_y + zf_z = df.$$

- If char(K) does not divide $d = \deg(f)$, then

$$xf_x + yf_y + zf_z = df$$

  is a non-zero scalar multiple of f, so $J = J'$ in this case.

- If char(K) does divide deg(f) then

$$xf_x + yf_y + zf_z = 0.$$

  This is a degree one syzygy on f_x, f_y, f_z.

Recall Euler's theorem:

$$xf_x + yf_y + zf_z = df.$$

- If $\mathrm{char}(K)$ does not divide $d = \deg(f)$, then

  $$xf_x + yf_y + zf_z = df$$

  is a non-zero scalar multiple of f, so $J = J'$ in this case.

- If $\mathrm{char}(K)$ does divide $\deg(f)$ then

  $$xf_x + yf_y + zf_z = 0.$$

  This is a degree one syzygy on f_x, f_y, f_z.

  In this case it is not necessarily true that f is in the ideal generated by its first partial derivatives, although it can happen.

Example. Let

$$f = xyz(x+y) = (x^2y + xy^2)z \text{ with } \text{char}(K) = 2.$$

So

$$J' = (f_x, f_y, f_z) = (y^2z, x^2z, x^2y + xy^2)$$

and $f = z \cdot f_z \in J' = J$.

Example. Let

$$f = xyz(x+y)(x+z) \text{ with } \text{char}(K) = 5.$$

One can check that $f \notin J'$ so $J' \subsetneq J$.

Define the submodule

$$D(Z) \subset R\frac{\partial}{\partial x} \oplus R\frac{\partial}{\partial y} \oplus R\frac{\partial}{\partial z} \cong R^3$$

to be the K-linear derivations δ such that $\delta(f) \in Rf$.

Define the submodule

$$D(Z) \subset R\frac{\partial}{\partial x} \oplus R\frac{\partial}{\partial y} \oplus R\frac{\partial}{\partial z} \cong R^3$$

to be the K-linear derivations δ such that $\delta(f) \in Rf$.

In particular,

- $D(Z)$ contains the Euler derivation $\delta_E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$;

Define the submodule

$$D(Z) \subset R\frac{\partial}{\partial x} \oplus R\frac{\partial}{\partial y} \oplus R\frac{\partial}{\partial z} \cong R^3$$

to be the K-linear derivations δ such that $\delta(f) \in Rf$.

In particular,

- $D(Z)$ contains the Euler derivation $\delta_E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$;
- δ_E generates a submodule $R\delta_E \cong R(-1)$ of $D(Z)$.

Define the submodule

$$D(Z) \subset R\frac{\partial}{\partial x} \oplus R\frac{\partial}{\partial y} \oplus R\frac{\partial}{\partial z} \cong R^3$$

to be the K-linear derivations δ such that $\delta(f) \in Rf$.

In particular,

- $D(Z)$ contains the Euler derivation $\delta_E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$;
- δ_E generates a submodule $R\delta_E \cong R(-1)$ of $D(Z)$.

We define the quotient $D_0(Z) = D(Z)/R\delta_E$.

Define the submodule

$$D(Z) \subset R\frac{\partial}{\partial x} \oplus R\frac{\partial}{\partial y} \oplus R\frac{\partial}{\partial z} \cong R^3$$

to be the K-linear derivations δ such that $\delta(f) \in Rf$.

In particular,

- $D(Z)$ contains the Euler derivation $\delta_E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$;
- δ_E generates a submodule $R\delta_E \cong R(-1)$ of $D(Z)$.

We define the quotient $D_0(Z) = D(Z)/R\delta_E$.

Let $\mathcal{D}_Z$, $\widetilde{D(Z)}$ be the sheafifications of $D_0(Z)$ and $D(Z)$ resp. What can we say about $\mathcal{D}_Z$ and about $\widetilde{D(Z)}$?

We have the following facts (omitting proofs):

- $\mathcal{D}_Z$ is locally free of rank 2.

We have the following facts (omitting proofs):

- $\mathcal{D}_Z$ is locally free of rank 2.
- When char(K) does not divide d, $\mathcal{D}_Z$ is isomorphic to the syzygy bundle (suitably twisted) of J'.

We have the following facts (omitting proofs):

- $\mathcal{D}_Z$ is locally free of rank 2.
- When char(K) does not divide d, $\mathcal{D}_Z$ is isomorphic to the syzygy bundle (suitably twisted) of J'.
- When char(K) does divide d, $\widetilde{D(Z)}$ is isomorphic to the syzygy bundle (suitably twisted) of J.

We have the following facts (omitting proofs):

- $\mathcal{D}_Z$ is locally free of rank 2.
- When char(K) does not divide d, $\mathcal{D}_Z$ is isomorphic to the syzygy bundle (suitably twisted) of J'.
- When char(K) does divide d, $\widetilde{D(Z)}$ is isomorphic to the syzygy bundle (suitably twisted) of J.
- The restriction of $\mathcal{D}_Z$ to a general line $\ell \cong \mathbb{P}^1$ splits as a direct sum
$$\mathcal{O}_{\mathbb{P}^1}(-a_Z) \oplus \mathcal{O}_{\mathbb{P}^1}(-b_Z)$$
for positive integers $a_Z \leq b_Z$ satisfying
$$a_Z + b_Z = \deg f - 1 = d - 1.$$

We have the following facts (omitting proofs):

- $\mathcal{D}_Z$ is locally free of rank 2.
- When char(K) does not divide d, $\mathcal{D}_Z$ is isomorphic to the syzygy bundle (suitably twisted) of J'.
- When char(K) does divide d, $\widetilde{D(Z)}$ is isomorphic to the syzygy bundle (suitably twisted) of J.
- The restriction of $\mathcal{D}_Z$ to a general line $\ell \cong \mathbb{P}^1$ splits as a direct sum

  $$\mathcal{O}_{\mathbb{P}^1}(-a_Z) \oplus \mathcal{O}_{\mathbb{P}^1}(-b_Z)$$

  for positive integers $a_Z \leq b_Z$ satisfying

  $$a_Z + b_Z = \deg f - 1 = d - 1.$$

  The ordered pair (a_Z, b_Z) is the splitting type of $\mathcal{D}_Z$ (or Z).

Unexpected curves

Fix a set of points, $Z \subset \mathbb{P}^2$. Fix an integer $j \geq 1$.

Let $\mathcal{L} = \left|[I_Z]_{j+1}\right|$. (Incomplete linear system.) Let $P \in \mathbb{P}^2$ be a general point.

Unexpected curves

Fix a set of points, $Z \subset \mathbb{P}^2$. Fix an integer $j \geq 1$.

Let $\mathcal{L} = \big|[I_Z]_{j+1}\big|$. (Incomplete linear system.) Let $P \in \mathbb{P}^2$ be a general point.

First interesting case of our general problem. Consider the fat point jP.

How many conditions does jP impose on $\mathcal{L}$?

Unexpected curves

Fix a set of points, $Z \subset \mathbb{P}^2$. Fix an integer $j \geq 1$.

Let $\mathcal{L} = \left|[I_Z]_{j+1}\right|$. (Incomplete linear system.) Let $P \in \mathbb{P}^2$ be a general point.

First interesting case of our general problem. Consider the fat point jP.

How many conditions does jP impose on $\mathcal{L}$?

We expect that jP will impose

$$\min\left\{\binom{j+1}{2}, \dim[I_Z]_{j+1}\right\}$$

independent conditions on $\mathcal{L}$.

For which Z and for which values of j do we get unexpected behavior?

For which Z and for which values of j do we get unexpected behavior?

Definition. Let Z, P and $\mathcal{L} = |[I_Z]_{j+1}|$ be as above.

For which Z and for which values of j do we get unexpected behavior?

Definition. Let Z, P and $\mathcal{L} = |[I_Z]_{j+1}|$ be as above. Then

Z admits an unexpected curve of degree $j+1$

if, for a general point P, the general fat point jP fails to impose the expected number of conditions on $\mathcal{L}$.

For which Z and for which values of j do we get unexpected behavior?

Definition. Let Z, P and $\mathcal{L} = |[I_Z]_{j+1}|$ be as above. Then

Z admits an unexpected curve of degree $j+1$

if, for a general point P, the general fat point jP fails to impose the expected number of conditions on $\mathcal{L}$.

That is, Z admits an unexpected curve of degree $j+1$ if

$$\dim[I_Z \cap I_P^j]_{j+1} := \dim[I_{Z+jP}]_{j+1} > \max\left\{\dim[I_Z]_{j+1} - \binom{j+1}{2}, 0\right\}.$$

For which Z and for which values of j do we get unexpected behavior?

Definition. Let Z, P and $\mathcal{L} = |[I_Z]_{j+1}|$ be as above. Then

Z admits an unexpected curve of degree $j+1$

if, for a general point P, the general fat point jP fails to impose the expected number of conditions on $\mathcal{L}$.

That is, Z admits an unexpected curve of degree $j+1$ if

$$\dim[I_Z \cap I_P^j]_{j+1} := \dim[I_{Z+jP}]_{j+1} > \max\left\{\dim[I_Z]_{j+1} - \binom{j+1}{2}, 0\right\}.$$

We say that Z *admits an unexpected curve* if such a j exists.

For which Z and for which values of j do we get unexpected behavior?

Definition. Let Z, P and $\mathcal{L} = |[I_Z]_{j+1}|$ be as above. Then

Z admits an unexpected curve of degree $j+1$

if, for a general point P, the general fat point jP fails to impose the expected number of conditions on $\mathcal{L}$.

That is, Z admits an unexpected curve of degree $j+1$ if

$$\dim[I_Z \cap I_P^j]_{j+1} := \dim[I_{Z+jP}]_{j+1} > \max\left\{\dim[I_Z]_{j+1} - \binom{j+1}{2}, 0\right\}.$$

We say that Z *admits an unexpected curve* if such a j exists.

Note: (1) Z might have unexpected curves in more than one degree.

For which Z and for which values of j do we get unexpected behavior?

Definition. Let Z, P and $\mathcal{L} = |[I_Z]_{j+1}|$ be as above. Then

Z admits an unexpected curve of degree $j+1$

if, for a general point P, the general fat point jP fails to impose the expected number of conditions on $\mathcal{L}$.

That is, Z admits an unexpected curve of degree $j+1$ if

$$\dim[I_Z \cap I_P^j]_{j+1} := \dim[I_{Z+jP}]_{j+1} > \max\left\{\dim[I_Z]_{j+1} - \binom{j+1}{2}, 0\right\}.$$

We say that Z *admits an unexpected curve* if such a j exists.

Note: (1) Z might have unexpected curves in more than one degree. (2) Lefschetz philosophy.

For which Z and for which values of j do we get unexpected behavior?

Definition. Let Z, P and $\mathcal{L} = |[I_Z]_{j+1}|$ be as above. Then

Z admits an unexpected curve of degree $j+1$

if, for a general point P, the general fat point jP fails to impose the expected number of conditions on $\mathcal{L}$.

That is, Z admits an unexpected curve of degree $j+1$ if

$$\dim[I_Z \cap I_P^j]_{j+1} := \dim[I_{Z+jP}]_{j+1} > \max\left\{\dim[I_Z]_{j+1} - \binom{j+1}{2}, 0\right\}.$$

We say that Z *admits an unexpected curve* if such a j exists.

Note: (1) Z might have unexpected curves in more than one degree. (2) Lefschetz philosophy. (3) Only one general point.

Some of the questions answered in our paper.

1. What properties of Z force the existence of an unexpected curve? (Necessary and sufficient conditions.)

Some of the questions answered in our paper.

1. What properties of Z force the existence of an unexpected curve? (Necessary and sufficient conditions.)

2. The definition of unexpected curves allowed for their existence in more than one degree. If there are unexpected curves at all, in what degrees do they exist?

Some of the questions answered in our paper.

1. What properties of Z force the existence of an unexpected curve? (Necessary and sufficient conditions.)

2. The definition of unexpected curves allowed for their existence in more than one degree. If there are unexpected curves at all, in what degrees do they exist?

3. Describe the unexpected curves:

Some of the questions answered in our paper.

1. What properties of Z force the existence of an unexpected curve? (Necessary and sufficient conditions.)

2. The definition of unexpected curves allowed for their existence in more than one degree. If there are unexpected curves at all, in what degrees do they exist?

3. Describe the unexpected curves:

    - When are they irreducible?

Some of the questions answered in our paper.

1. What properties of Z force the existence of an unexpected curve? (Necessary and sufficient conditions.)

2. The definition of unexpected curves allowed for their existence in more than one degree. If there are unexpected curves at all, in what degrees do they exist?

3. Describe the unexpected curves:

   - When are they irreducible?
   - If they are reducible, what do the irreducible components look like?

Some of the questions answered in our paper.

1. What properties of Z force the existence of an unexpected curve? (Necessary and sufficient conditions.)

2. The definition of unexpected curves allowed for their existence in more than one degree. If there are unexpected curves at all, in what degrees do they exist?

3. Describe the unexpected curves:

   - When are they irreducible?
   - If they are reducible, what do the irreducible components look like?

4. What are some examples of sets of points with unexpected curves?

Merging the two topics: Main results

So what does this have to do with arrangements?

Let Z be a reduced set of points of $\mathbb{P}^2$.

Merging the two topics: Main results

So what does this have to do with arrangements?

Let Z be a reduced set of points of $\mathbb{P}^2$.

The points of Z are dual to a line arrangement $\mathcal{A}_f$, where f is a product of distinct linear forms.

Merging the two topics: Main results

So what does this have to do with arrangements?

Let Z be a reduced set of points of $\mathbb{P}^2$.

The points of Z are dual to a line arrangement $\mathcal{A}_f$, where f is a product of distinct linear forms.

Let P be a general point of $\mathbb{P}^2$. Recall:

Notation. $Z + jP$ is the scheme defined by the ideal $I_Z \cap I_P^j \subset R$.

Merging the two topics: Main results

So what does this have to do with arrangements?

Let Z be a reduced set of points of $\mathbb{P}^2$.

The points of Z are dual to a line arrangement $\mathcal{A}_f$, where f is a product of distinct linear forms.

Let P be a general point of $\mathbb{P}^2$. Recall:

Notation. $Z + jP$ is the scheme defined by the ideal $I_Z \cap I_P^j \subset R$.

Let $\mathcal{I}_Z$ and $\mathcal{I}_{Z+jP}$ be the corresponding ideal sheaves.

Merging the two topics: Main results

So what does this have to do with arrangements?

Let Z be a reduced set of points of $\mathbb{P}^2$.

The points of Z are dual to a line arrangement $\mathcal{A}_f$, where f is a product of distinct linear forms.

Let P be a general point of $\mathbb{P}^2$. Recall:

Notation. $Z + jP$ is the scheme defined by the ideal $I_Z \cap I_P^j \subset R$.

Let $\mathcal{I}_Z$ and $\mathcal{I}_{Z+jP}$ be the corresponding ideal sheaves.

Recall that the splitting type of $\mathcal{D}_Z$ is (a_Z, b_Z) with $a_Z \leq b_Z$ and $a_Z + b_Z = \deg Z - 1$.

The following is a (non-trivial) consequence of a result of Faenzi and Vallès.

Lemma.

$$\dim[I_{Z+jP}]_{j+1} = \max\{0, j - a_Z + 1\} + \max\{0, j - b_Z + 1\}.$$

The following is a (non-trivial) consequence of a result of Faenzi and Vallès.

Lemma.

$$\dim[I_{Z+jP}]_{j+1} = \max\{0, j - a_Z + 1\} + \max\{0, j - b_Z + 1\}.$$

This combines information about Z with information about the dual line arrangement $\mathcal{A}_f$.

The following is a (non-trivial) consequence of a result of Faenzi and Vallès.

Lemma.

$$\dim[I_{Z+jP}]_{j+1} = \max\{0, j - a_Z + 1\} + \max\{0, j - b_Z + 1\}.$$

This combines information about Z with information about the dual line arrangement $\mathcal{A}_f$.

Remark. From this lemma it follows immediately that

- $\dim[I_{Z+a_Z P}]_{a_Z+1}$ is either equal to 1 or to 2;

The following is a (non-trivial) consequence of a result of Faenzi and Vallès.

Lemma.

$$\dim[I_{Z+jP}]_{j+1} = \max\{0, j - a_Z + 1\} + \max\{0, j - b_Z + 1\}.$$

This combines information about Z with information about the dual line arrangement $\mathcal{A}_f$.

Remark. From this lemma it follows immediately that

- $\dim[I_{Z+a_Z P}]_{a_Z+1}$ is either equal to 1 or to 2;
- $\dim[I_{Z+a_Z P}]_{a_Z+1} = 2$ if and only if $a_Z = b_Z$.

Definition. Let Z be a reduced 0-dimensional subscheme of $\mathbb{P}^2$.

(a) The multiplicity index is

$$m_Z = \min\{j \in \mathbb{Z} \mid \dim[I_{Z+jP}]_{j+1} > 0\}$$

Definition. Let Z be a reduced 0-dimensional subscheme of $\mathbb{P}^2$.

(a) The multiplicity index is

$$\begin{aligned} m_Z &= \min\{j \in \mathbb{Z} \mid \dim[I_{Z+jP}]_{j+1} > 0\} \\ &= a_Z \text{ (by the Faenzi-Vallès lemma)} \end{aligned}$$

Definition. Let Z be a reduced 0-dimensional subscheme of $\mathbb{P}^2$.

(a) The multiplicity index is

$$\begin{aligned} m_Z &= \min\{j \in \mathbb{Z} \mid \dim[I_{Z+jP}]_{j+1} > 0\} \\ &= a_Z \text{ (by the Faenzi-Vallès lemma)} \end{aligned}$$

(b) $t_Z = \min\{j \mid \dim[I_Z]_{j+1} > \binom{j+1}{2}\}$. (Depends only on the Hilbert function of Z.)

Definition. Let Z be a reduced 0-dimensional subscheme of $\mathbb{P}^2$.

(a) The multiplicity index is

$$m_Z = \min\{j \in \mathbb{Z} \mid \dim[I_{Z+jP}]_{j+1} > 0\}$$
$$= a_Z \text{ (by the Faenzi-Vallès lemma)}$$

(b) $t_Z = \min\{j \mid \dim[I_Z]_{j+1} > \binom{j+1}{2}\}$. (Depends only on the Hilbert function of Z.)

(c) The speciality index is

$$u_Z = \min\{j \mid Z + jP \text{ imposes independent conditions on plane curves of degree } j+1\}$$

Definition. Let Z be a reduced 0-dimensional subscheme of $\mathbb{P}^2$.

(a) The multiplicity index is

$$m_Z = \min\{j \in \mathbb{Z} \mid \dim[I_{Z+jP}]_{j+1} > 0\}$$
$$= a_Z \text{ (by the Faenzi-Vallès lemma)}$$

(b) $t_Z = \min\{j \mid \dim[I_Z]_{j+1} > \binom{j+1}{2}\}$. (Depends only on the Hilbert function of Z.)

(c) The speciality index is

$$u_Z = \min\{j \mid Z + jP \text{ imposes independent conditions on plane curves of degree } j+1\}$$

$$= \min\{j \mid \dim[I_{Z+jP}]_{j+1} = \binom{j+3}{2} - \binom{j+1}{2} - |Z|\}$$

Definition. Let Z be a reduced 0-dimensional subscheme of $\mathbb{P}^2$.

(a) The multiplicity index is

$$\begin{aligned} m_Z &= \min\{j \in \mathbb{Z} \mid \dim[I_{Z+jP}]_{j+1} > 0\} \\ &= a_Z \ \text{(by the Faenzi-Vallès lemma)} \end{aligned}$$

(b) $t_Z = \min\{j \mid \dim[I_Z]_{j+1} > \binom{j+1}{2}\}$. (Depends only on the Hilbert function of Z.)

(c) The speciality index is

$$\begin{aligned} u_Z &= \min\{j \mid Z + jP \text{ imposes independent conditions on plane curves of degree } j+1\} \\ &= \min\{j \mid \dim[I_{Z+jP}]_{j+1} = \binom{j+3}{2} - \binom{j+1}{2} - |Z|\} \\ &= \min\{j \mid h^1(\mathcal{I}_{Z+jP}(j+1)) = 0\}. \end{aligned}$$

Theorem. *Regardless of whether Z has an unexpected curve or not, we have:*

$$m_Z + u_Z = |Z| - 2.$$

Theorem. *Regardless of whether Z has an unexpected curve or not, we have:*

$$m_Z + u_Z = |Z| - 2.$$

Corollary *For the splitting type (a_Z, b_Z) of $\mathcal{D}_Z$ we have $a_Z = m_Z$ and $b_Z = u_Z + 1$.*

Theorem. *Regardless of whether* Z *has an unexpected curve or not, we have:*

$$m_Z + u_Z = |Z| - 2.$$

Corollary *For the splitting type* (a_Z, b_Z) *of* $\mathcal{D}_Z$ *we have* $a_Z = m_Z$ *and* $b_Z = u_Z + 1$.

Corollary. *Regardless of whether* Z *has an unexpected curve or not, we have:*

(a)

Theorem. *Regardless of whether* Z *has an unexpected curve or not, we have:*

$$m_Z + u_Z = |Z| - 2.$$

Corollary *For the splitting type* (a_Z, b_Z) *of* $\mathcal{D}_Z$ *we have* $a_Z = m_Z$ *and* $b_Z = u_Z + 1$.

Corollary. *Regardless of whether* Z *has an unexpected curve or not, we have:*

(a) $m_Z < u_Z \quad \Leftrightarrow \quad |Z| \geq 2m_Z + 3$

Theorem. *Regardless of whether Z has an unexpected curve or not, we have:*

$$m_Z + u_Z = |Z| - 2.$$

Corollary *For the splitting type* (a_Z, b_Z) *of* $\mathcal{D}_Z$ *we have* $a_Z = m_Z$ *and* $b_Z = u_Z + 1$.

Corollary. *Regardless of whether Z has an unexpected curve or not, we have:*

(a) $m_Z < u_Z \quad \Leftrightarrow \quad |Z| \geq 2m_Z + 3$

(b) $m_Z = u_Z \quad \Leftrightarrow \quad |Z| = 2m_Z + 2$

Theorem. *Regardless of whether Z has an unexpected curve or not, we have:*

$$m_Z + u_Z = |Z| - 2.$$

Corollary *For the splitting type (a_Z, b_Z) of $\mathcal{D}_Z$ we have $a_Z = m_Z$ and $b_Z = u_Z + 1$.*

Corollary. *Regardless of whether Z has an unexpected curve or not, we have:*

(a) $m_Z < u_Z \quad \Leftrightarrow \quad |Z| \geq 2m_Z + 3$

(b) $m_Z = u_Z \quad \Leftrightarrow \quad |Z| = 2m_Z + 2$

(c) $m_Z > u_Z \quad \Leftrightarrow \quad |Z| = 2m_Z + 1$

Theorem. *Regardless of whether Z has an unexpected curve or not, we have:*

$$m_Z + u_Z = |Z| - 2.$$

Corollary *For the splitting type* (a_Z, b_Z) *of* $\mathcal{D}_Z$ *we have* $a_Z = m_Z$ *and* $b_Z = u_Z + 1$.

Corollary. *Regardless of whether Z has an unexpected curve or not, we have:*

$$\left.\begin{array}{lllll} \text{(a)} & m_Z < u_Z & \Leftrightarrow & |Z| \geq 2m_Z + 3 \\ \text{(b)} & m_Z = u_Z & \Leftrightarrow & |Z| = 2m_Z + 2 \end{array}\right\} \Leftrightarrow \dim[I_{Z+m_Z P}]_{m_Z+1} = 1$$

$$\begin{array}{llll} \text{(c)} & m_Z > u_Z & \Leftrightarrow & |Z| = 2m_Z + 1 \end{array}$$

Theorem. *Regardless of whether Z has an unexpected curve or not, we have:*

$$m_Z + u_Z = |Z| - 2.$$

Corollary *For the splitting type (a_Z, b_Z) of $\mathcal{D}_Z$ we have $a_Z = m_Z$ and $b_Z = u_Z + 1$.*

Corollary. *Regardless of whether Z has an unexpected curve or not, we have:*

$$\left.\begin{array}{llll} \text{(a)} & m_Z < u_Z & \Leftrightarrow & |Z| \geq 2m_Z + 3 \\ \text{(b)} & m_Z = u_Z & \Leftrightarrow & |Z| = 2m_Z + 2 \end{array}\right\} \Leftrightarrow \dim[I_{Z+m_Z P}]_{m_Z+1} = 1$$

$$\text{(c)} \quad m_Z > u_Z \quad \Leftrightarrow \quad |Z| = 2m_Z + 1 \quad \Leftrightarrow \dim[I_{Z+m_Z P}]_{m_Z+1} = 2$$

Theorem. *An unexpected curve exists if and only if*

$$m_Z < t_Z,$$

Theorem. *An unexpected curve exists if and only if*

$$m_Z < t_Z,$$

in which case $t_Z \leq u_Z$.

Theorem. *An unexpected curve exists if and only if*

$$m_Z < t_Z,$$

in which case $t_Z \leq u_Z$.

In this situation Z *has an unexpected curve of degree* k *if and only if*

$$a_Z + 1 = m_Z + 1 \leq k < u_Z + 1 = b_z.$$

Theorem. *An unexpected curve exists if and only if*

$$m_Z < t_Z,$$

in which case $t_Z \leq u_Z$.

In this situation Z *has an unexpected curve of degree* k *if and only if*

$$a_Z + 1 = m_Z + 1 \leq k < u_Z + 1 = b_z.$$

Corollary. *If* Z *admits an unexpected curve then* $b_Z - a_Z \geq 2$.

Theorem. *An unexpected curve exists if and only if*

$$m_Z < t_Z,$$

in which case $t_Z \leq u_Z$.

In this situation Z *has an unexpected curve of degree* k *if and only if*

$$a_Z + 1 = m_Z + 1 \leq k < u_Z + 1 = b_z.$$

Corollary. *If* Z *admits an unexpected curve then* $b_Z - a_Z \geq 2$.

Is the converse true?

Theorem.

$$\textit{An unexpected curve exists} \Leftrightarrow \begin{cases} \text{(a) } b_Z - a_Z \geq 2 \\ \\ \end{cases}$$

Theorem.

$$\textit{An unexpected curve exists} \Leftrightarrow \begin{cases} \text{(a) } b_Z - a_Z \geq 2 \\ \\ \text{(b) } h_Z(t_Z) = |Z|. \end{cases}$$

Theorem.

$$\textit{An unexpected curve exists} \Leftrightarrow \begin{cases} \text{(a) } b_Z - a_Z \geq 2 \\ \\ \text{(b) } h_Z(t_Z) = |Z|. \end{cases}$$

Note

$$(b) \quad \Leftrightarrow \quad h^1(\mathcal{I}_Z(t_Z)) = 0$$

$\Leftrightarrow$ Z imposes independent conditions on curves of degree t_Z.

Irreducibility

- An unexpected curve can only hope to be irreducible in degree $m_Z + 1$.

Irreducibility

- An unexpected curve can only hope to be irreducible in degree $m_Z + 1$.
- We give a necessary and sufficient condition for the existence of irreducible curves in the linear system $|[I_{Z+m_Z P}]_{m_Z+1}|$, assuming $m_Z \leq u_Z$.

Structure of unexpected curves

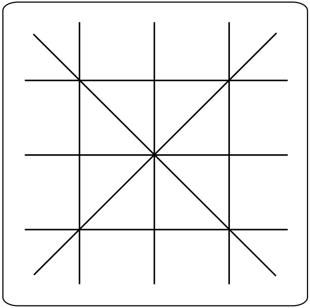
We give a careful description. Briefly, an unexpected curve consists of the union of

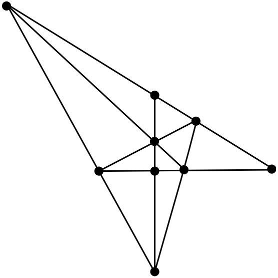
- an irreducible rational curve of some degree e having a point of multiplicity $e-1$ and

- certain lines.

Some Examples/results

Example. [Di Gennaro, Ilardi and Vallès] (This motivated our paper!)

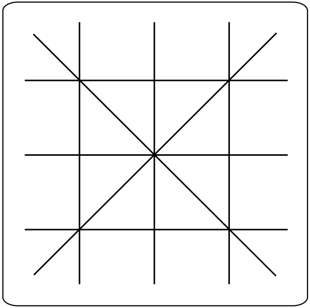


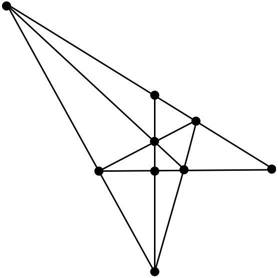


The points dual to the B3 configuration admit an unexpected curve of degree 4.

Some Examples/results

Example. [Di Gennaro, Ilardi and Vallès] (This motivated our paper!)





The points dual to the B3 configuration admit an unexpected curve of degree 4. (This admits a direct generalization, omitted here.)

Example. For this example, for simplicity we assume our ground field has characteristic 0.

Example. For this example, for simplicity we assume our ground field has characteristic 0.

Consider the line configuration $\mathcal{A}_f$ given by the lines defined by

$$\begin{aligned} f &= xyz(x+y)(x-y)(2x+y)(2x-y)(x+z)(x-z) \\ &\quad (y+z)(y-z)(x+2z)(x-2z)(y+2z)(y-2z) \\ &\quad (x-y+z)(x-y-z)(x-y+2z)(x-y-2z). \end{aligned}$$

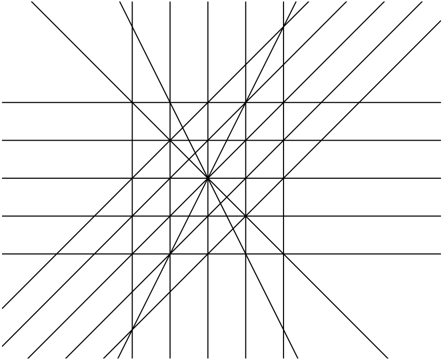
Example. For this example, for simplicity we assume our ground field has characteristic 0.

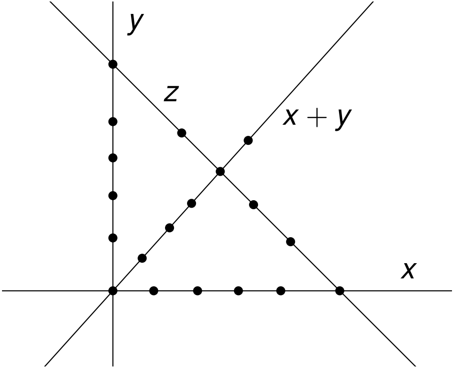
Consider the line configuration $\mathcal{A}_f$ given by the lines defined by

$$\begin{aligned} f &= xyz(x+y)(x-y)(2x+y)(2x-y)(x+z)(x-z) \\ &\quad (y+z)(y-z)(x+2z)(x-2z)(y+2z)(y-2z) \\ &\quad (x-y+z)(x-y-z)(x-y+2z)(x-y-2z). \end{aligned}$$

Note $d = 19$. Let Z be the corresponding reduced scheme consisting of the 19 points that are dual to these lines.

The following figures show $\mathcal{A}_f$ and Z.




y
z
x + y
x

It is not hard to verify that the first difference of the Hilbert function of Z is

$$\Delta h_Z = (1, 2, 3, 4, 4, 4, 1),$$

from which we find (after a calculation) that $t_Z = 9$.

It is not hard to verify that the first difference of the Hilbert function of Z is

$$\Delta h_Z = (1, 2, 3, 4, 4, 4, 1),$$

from which we find (after a calculation) that $t_Z = 9$.

Picking a random point P, one employs various methods to prove that $a_Z = m_Z = 8$.

It is not hard to verify that the first difference of the Hilbert function of Z is

$$\Delta h_Z = (1, 2, 3, 4, 4, 4, 1),$$

from which we find (after a calculation) that $t_Z = 9$.

Picking a random point P, one employs various methods to prove that $a_Z = m_Z = 8$.

Since $m_Z < t_Z$, Z admits an unexpected curve of degree $m_Z + 1 = 9$.

It is not hard to verify that the first difference of the Hilbert function of Z is

$$\Delta h_Z = (1, 2, 3, 4, 4, 4, 1),$$

from which we find (after a calculation) that $t_Z = 9$.

Picking a random point P, one employs various methods to prove that $a_Z = m_Z = 8$.

Since $m_Z < t_Z$, Z admits an unexpected curve of degree $m_Z + 1 = 9$.

Since $|Z| = 19$, the splitting type is $(8, 10)$, and $u_Z = 10 - 1 = 9$.

So we have $m_Z = 8$, $t_Z = 9$ and $u_Z = 9$.

So we have $m_Z = 8$, $t_Z = 9$ and $u_Z = 9$. Recall our theorem:

An unexpected curve exists if and only if $m_Z < t_Z$.

In this situation Z has an unexpected curve of degree k

if and only if

$$m_Z + 1 \leq k < u_Z + 1.$$

So we have $m_Z = 8$, $t_Z = 9$ and $u_Z = 9$. Recall our theorem:

An unexpected curve exists if and only if $m_Z < t_Z$.

In this situation Z has an unexpected curve of degree k

if and only if

$$m_Z + 1 \leq k < u_Z + 1.$$

Thus in our example there is an unexpected curve for each degree k with

$$8 + 1 \leq k < 9 + 1.$$

That is, 9 is the only degree in which Z admits an unexpected curve. We have verified (using our criterion for irreducibility) that this curve is not irreducible.

Remark. Assume for convenience that K has characteristic zero.

Recall a line arrangement $\mathcal{A}_f$ in $\mathbb{P}^2$ is free if $\mathcal{D}_Z$ is free, i.e. if $J = J' = (f_x, f_y, f_z)$ is a saturated ideal.

Remark. Assume for convenience that K has characteristic zero.

Recall a line arrangement $\mathcal{A}_f$ in $\mathbb{P}^2$ is free if $\mathcal{D}_Z$ is free, i.e. if $J = J' = (f_x, f_y, f_z)$ is a saturated ideal.

In the previous example, $\mathcal{A}_f$ is not free. Freeness does not affect the above results, but it's a little more work to determine the splitting type for a particular non-free arrangement.

Remark. Assume for convenience that K has characteristic zero.

Recall a line arrangement $\mathcal{A}_f$ in $\mathbb{P}^2$ is free if $\mathcal{D}_Z$ is free, i.e. if $J = J' = (f_x, f_y, f_z)$ is a saturated ideal.

In the previous example, $\mathcal{A}_f$ is not free. Freeness does not affect the above results, but it's a little more work to determine the splitting type for a particular non-free arrangement.

The following result used the Grauert-Mülich theorem for the proof, so we assume characteristic zero also for this.

Theorem. *If Z is in linear general position then Z does not admit an unexpected curve.*

Remark. Assume for convenience that K has characteristic zero.

Recall a line arrangement $\mathcal{A}_f$ in $\mathbb{P}^2$ is free if $\mathcal{D}_Z$ is free, i.e. if $J = J' = (f_x, f_y, f_z)$ is a saturated ideal.

In the previous example, $\mathcal{A}_f$ is not free. Freeness does not affect the above results, but it's a little more work to determine the splitting type for a particular non-free arrangement.

The following result used the Grauert-Mülich theorem for the proof, so we assume characteristic zero also for this.

Theorem. *If Z is in linear general position then Z does not admit an unexpected curve.*

(This is far from talking about a general set of points.)

Example.
Assume $\text{char}(K) = 2$.

Let Z be the 7 points of the Fano plane (embedded in $\mathbb{P}^2_K$).

Example.
Assume $\text{char}(K) = 2$.

Let Z be the 7 points of the Fano plane (embedded in $\mathbb{P}^2_K$).

Then $\dim[I_Z]_3 = 3$ and $2P$ should impose 3 conditions, so we expect there not to be a cubic containing Z and singular at a general point $P = [\alpha, \beta, \gamma]$.

Example.
Assume $\text{char}(K) = 2$.

Let Z be the 7 points of the Fano plane (embedded in $\mathbb{P}^2_K$).

Then $\dim[I_Z]_3 = 3$ and $2P$ should impose 3 conditions, so we expect there not to be a cubic containing Z and singular at a general point $P = [\alpha, \beta, \gamma]$.

But in fact there is one. One can easily check that

$$f = \alpha^2 yz(y+z) + \beta^2 xz(x+z) + \gamma^2 xy(x+y)$$

defines a curve C (reduced and irreducible in fact) which is singular at P, and hence C is an unexpected curve of degree 3 for Z.

Close the circle

Finally, we give a connection between unexpected curves and Lefschetz properties. (There are actually several such connections.)

Close the circle

Finally, we give a connection between unexpected curves and Lefschetz properties. (There are actually several such connections.)

- Recall WLP involves the rank of

$$\times L : [R/I]_i \to [R/I]_{i+1}$$

for all i.

Close the circle

Finally, we give a connection between unexpected curves and Lefschetz properties. (There are actually several such connections.)

- Recall WLP involves the rank of

$$\times L : [R/I]_i \rightarrow [R/I]_{i+1}$$

for all i.

- SLP involves the rank of

$$\times L^k : [R/I]_i \rightarrow [R/I]_{i+k}$$

for all i and all k.

Intermediate question:

- When does

$$\times L^2 : [R/I]_i \to [R/I]_{i+2}$$

have maximal rank?

Intermediate question:

- When does

$$\times L^2 : [R/I]_i \to [R/I]_{i+2}$$

have maximal rank?

Here is an interesting class of ideals:

$$\mathcal{C} = \{I = (L_1^{a_1}, \ldots, L_k^{a_k})\}$$

where $k \geq 3$, $a_1, \ldots, a_k \geq 2$ and $L_1, \ldots, L_k$ are linear forms in $K[x, y, z]$ (not necessarily general).

Intermediate question:

- When does

$$\times L^2 : [R/I]_i \to [R/I]_{i+2}$$

have maximal rank?

Here is an interesting class of ideals:

$$\mathcal{C} = \{I = (L_1^{a_1}, \ldots, L_k^{a_k})\}$$

where $k \geq 3$, $a_1, \ldots, a_k \geq 2$ and $L_1, \ldots, L_k$ are linear forms in $K[x, y, z]$ (not necessarily general).

Schenck-Seceleanu: Any such ideal has the WLP (3 variables)!

Intermediate question:

- When does

$$\times L^2 : [R/I]_i \to [R/I]_{i+2}$$

have maximal rank?

Here is an interesting class of ideals:

$$\mathcal{C} = \{I = (L_1^{a_1}, \dots, L_k^{a_k})\}$$

where $k \geq 3$, $a_1, \dots, a_k \geq 2$ and $L_1, \dots, L_k$ are linear forms in $K[x, y, z]$ (not necessarily general).

Schenck-Seceleanu: Any such ideal has the WLP (3 variables)!

But the above question about $\times L^2$ is meaningful. The following result was motivated by DIV.

Theorem. *Let*

- $\mathcal{A}(f)$ *be a line arrangement in* $\mathbb{P}^2$*, where* $f = L_1 \cdots L_d$*.*
- Z *be the set of points in* $\mathbb{P}^2$ *dual to these lines.*
- $I = (L_1^{j+1}, \ldots, L_d^{j+1})$*.*

Theorem. *Let*

- $\mathcal{A}(f)$ *be a line arrangement in* $\mathbb{P}^2$*, where* $f = L_1 \cdots L_d$*.*
- Z *be the set of points in* $\mathbb{P}^2$ *dual to these lines.*
- $I = (L_1^{j+1}, \ldots, L_d^{j+1})$*.*

Then

$$Z \text{ has an unexpected curve of degree } j+1$$

$$\text{if and only if}$$

$$\times L^2 : [R/I]_{j-1} \to [R/I]_{j+1} \text{ does not have maximal rank.}$$

Theorem. *Let*

- $\mathcal{A}(f)$ *be a line arrangement in* $\mathbb{P}^2$, *where* $f = L_1 \cdots L_d$.
- Z *be the set of points in* $\mathbb{P}^2$ *dual to these lines.*
- $I = (L_1^{j+1}, \ldots, L_d^{j+1})$.

Then

Z has an unexpected curve of degree $j+1$

if and only if

$\times L^2 : [R/I]_{j-1} \to [R/I]_{j+1}$ does not have maximal rank.

There is one additional ingredient to prove this.

Theorem. (J. Emsalem and A. Iarrobino)

Let $\wp_1, \ldots, \wp_m$ be the ideals of m distinct points in $\mathbb{P}^{n-1}$.

Theorem. (J. Emsalem and A. Iarrobino)

Let $\wp_1, \ldots, \wp_m$ *be the ideals of m distinct points in* $\mathbb{P}^{n-1}$*.*

Let $L_1, \ldots, L_m$ *be the dual linear forms.*

Theorem. (J. Emsalem and A. Iarrobino)

Let $\wp_1, \ldots, \wp_m$ be the ideals of m distinct points in $\mathbb{P}^{n-1}$.

Let $L_1, \ldots, L_m$ be the dual linear forms.

Choose positive integers $a_1, \ldots, a_m$.

Theorem. (J. Emsalem and A. Iarrobino)

Let $\wp_1, \ldots, \wp_m$ be the ideals of m distinct points in $\mathbb{P}^{n-1}$.

Let $L_1, \ldots, L_m$ be the dual linear forms.

Choose positive integers $a_1, \ldots, a_m$.

Then for any integer $k \geq \max\{a_i\}$,

$$\dim_K \left[R/(L_1^{a_1}, \ldots, L_m^{a_m})\right]_k = \dim_K \left[\wp_1^{k-a_1+1} \cap \cdots \cap \wp_m^{k-a_m+1}\right]_k .$$

Theorem. (J. Emsalem and A. Iarrobino)

Let $\wp_1, \ldots, \wp_m$ be the ideals of m distinct points in $\mathbb{P}^{n-1}$.

Let $L_1, \ldots, L_m$ be the dual linear forms.

Choose positive integers $a_1, \ldots, a_m$.

Then for any integer $k \geq \max\{a_i\}$,

$$\dim_K \left[R/(L_1^{a_1}, \ldots, L_m^{a_m})\right]_k = \dim_K \left[\wp_1^{k-a_1+1} \cap \cdots \cap \wp_m^{k-a_m+1}\right]_k.$$

In particular, for a general point P with defining ideal $\wp$ and dual linear form L, we have

$$\dim_K \left[R/(L_1^{j+1}, \ldots, L_d^{j+1}, L^2)\right]_{j+1} = \dim_K [\underbrace{\wp_1^1 \cap \cdots \cap \wp_n^1}_{I_Z} \cap \wp^j]_{j+1}.$$

Thank you.