3264 CONICS IN A SECOND

Paul Breiding
Bernd Sturmfels
Sascha Timme

MAX-PLANCK-GESELLSCHAFT

A **conic** in the plane $\mathbb{R}^2$ is the set of solutions

to a quadratic equation $A(x, y) = 0$, where

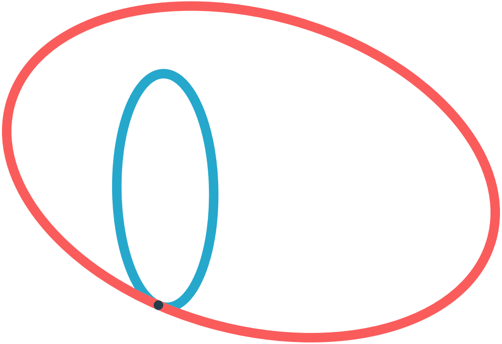
$$A(x, y) \ = \ a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + 1.$$

A second conic

$$U(x,y) \;=\; u_1x^2 + u_2xy + u_3y^2 + u_4x + u_5y + 1,$$

is **tangent** to A if there exists (x,y) such that

$$A(x,y) = 0, \quad U(x,y) = 0 \quad \text{and} \quad \det \begin{bmatrix} \frac{\partial A}{\partial x} & \frac{\partial U}{\partial x} \\ \frac{\partial A}{\partial y} & \frac{\partial U}{\partial y} \end{bmatrix} = 0.$$



Eliminating (x, y) from

$$A(x,y) = 0, \quad U(x,y) = 0 \quad \text{and} \quad \det \begin{bmatrix} \frac{\partial A}{\partial x} & \frac{\partial U}{\partial x} \\ \frac{\partial A}{\partial y} & \frac{\partial U}{\partial y} \end{bmatrix} = 0$$

we get the **tact invariant**:

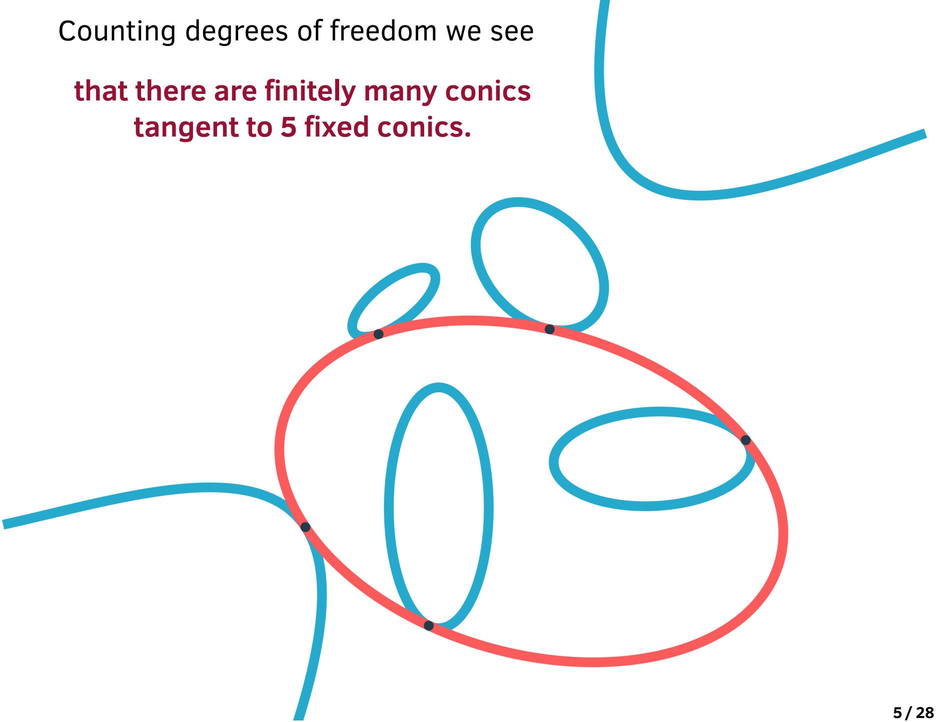
$$\begin{aligned} \mathcal{T}(A,U) = {} & 256a_1^4a_3^2u_3^2 - 128a_1^4a_3^2u_3u_5^2 + 16a_1^4a_3^2u_5^4 \\ & - 256a_1^4a_3a_5u_3^2u_5 + 128a_1^4a_3a_5u_3u_5^3 - 16a_1^4a_3a_5u_5^5 \\ & - 512a_1^4a_31u_3^3 + \cdots + a_5^4u_1^2u_2^4. \end{aligned}$$

For fixed $u_1 \ldots, u_5$ the tact invariant $\mathcal{T}(A,U)$ is a polynomial

of **degree 6** in the **5 variables** $a_1, \ldots, a_5$.

Counting degrees of freedom we see

that there are finitely many conics tangent to 5 fixed conics.



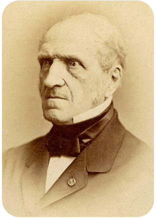
How many conics?

The question **How many?** started the modern development of enumerative geometry.

Recall: $\mathcal{T}(A, U)$ is a polynomial of **degree 6** in **5 variables**.

Based on this, **Jakob Steiner** claimed in 1848 that there are $6^5 = 7776$ (complex) conics tangent to five conics.

In 1864 **Michel Chasles** gave the correct answer of 3264.



Why was Steiner wrong?

He missed that there is a **Veronese surface** of **extraneous** solutions, namely the conics that are **squares of linear forms**.

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To answer **enumerative geometry questions** we work in the **Chow ring** of the space of complete conics.

Applying intersection theory

This Chow ring contains three special classes $\mathbf{P}$ and $\mathbf{L}$:

1. $\mathbf{P}$ encodes the conics passing through a fixed point
2. $\mathbf{L}$ encodes the conics tangent to a fixed line
3. $\mathbf{C}$ encodes the conics tangent to a given conic

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The following relations hold:

$$P^5 = L^5 = 1\,,\ \ P^4L = PL^4 = 2 \ \text{ and }\ P^3L^2 = P^2L^3 = 4.$$

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The desired intersection number is now obtained from the Binomial Theorem:

$$\begin{aligned} C^5 &= 2^5 \cdot (L+P)^5 \\ &= 2^5 \cdot (L^5 + 5L^4P + 10L^3P^2 + 10L^2P^3 + 5LP^4 + P^5) \\ &= 2^5 \cdot (1 + 5\cdot 2 + 10\cdot 4 + 10\cdot 4 + 5\cdot 2 + 1) = 32 \cdot 102 = \mathbf{3264}. \end{aligned}$$

What about real solutions?

This yields the question:

> *Is there an instance of Steiner's problem whose 3264 solutions are **all real**?*

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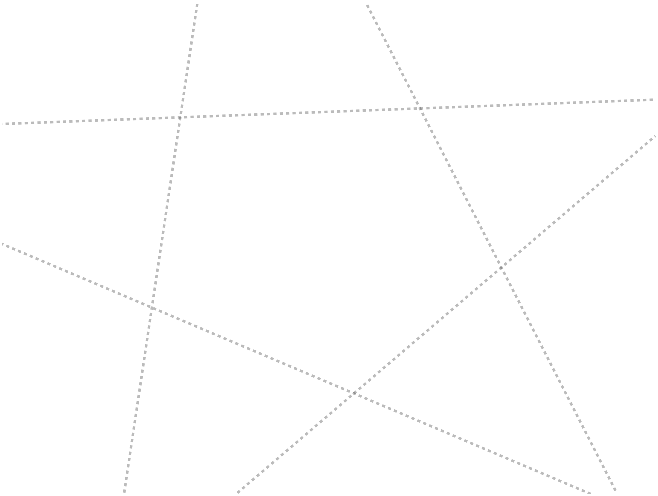
This yields the question:

> *Is there an instance of Steiner's problem whose 3264 solutions are **all real**?*

The answer is **YES!**

This was first observed by Fulton, and worked out in detail by Ronga, Tognoli and Vust, and Sottile.

Constructing 3264 real solutions



We start with 5 double lines, forming a pentagon.

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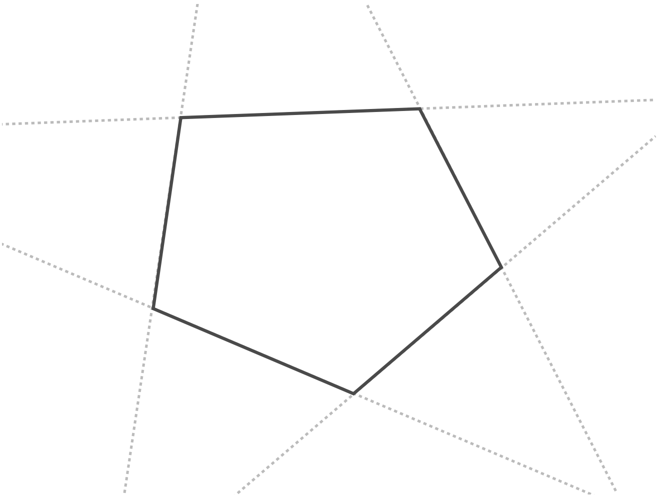

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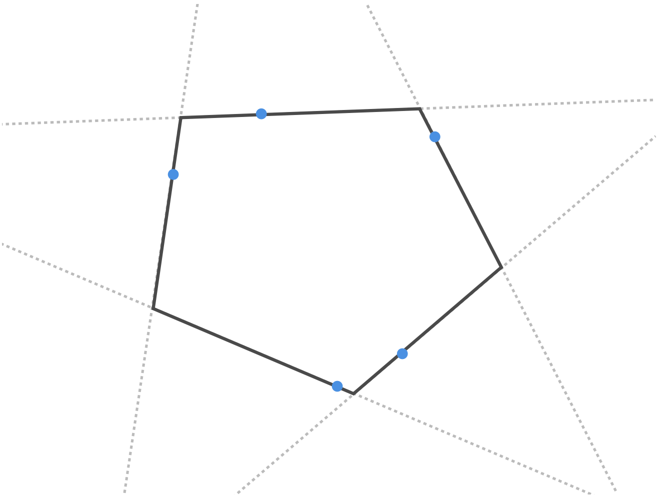

Mark a **special point** on each edge.

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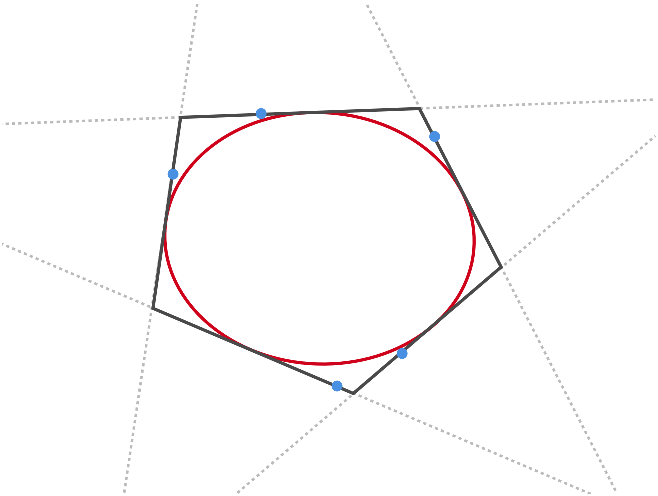

There are $\mathbf{102} = (\mathbf{L} + \mathbf{P})^{\mathbf{5}}$ conics which are tangent to a subset of the lines and going through the other special points.

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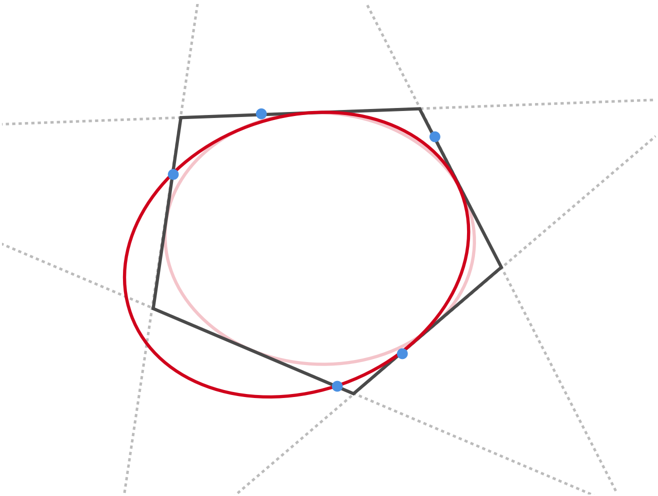

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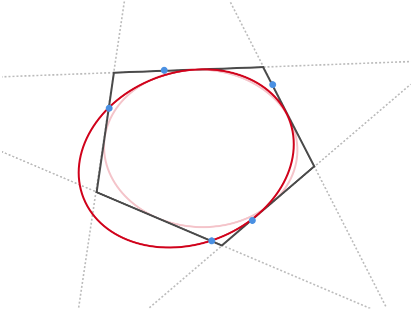
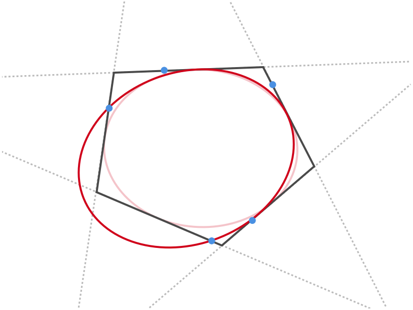


By a **small perturbation** each of the **102** conic splits into **32** solutions.



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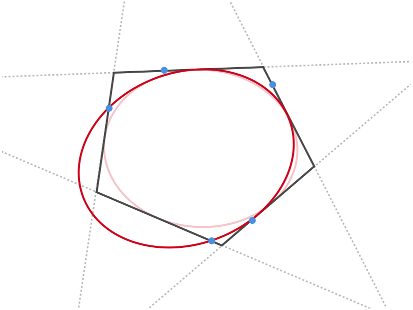
Constructing 3264 real solutions



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But what is **small**?

Can we find a **concrete instance** and give a **proof** that it has 3264 real solutions?

Numerical Algebraic Geometry

Homotopy Continuation

Homotopy continuation is a technique for **5 numerically** solving systems of polynomial equations.

Essentially, for solving a system

$$F = (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n))$$

it does the following: take another system

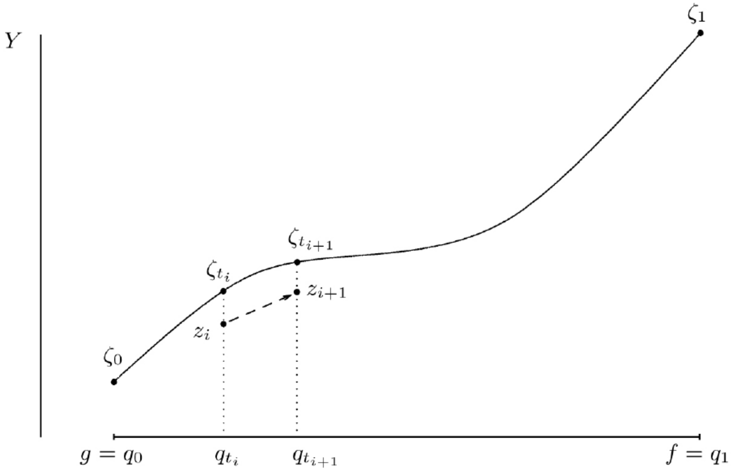
$$G = (g_1(x_1, \dots, x_n), \dots, g_n(x_1, \dots, x_n)),$$

of which we know all zeros. Then track the zeros along a path

$$H(x,t) \quad \text{with} \quad H(x,1) = G(x),\ H(x,0) = F(x)$$

towards F.

Homotopy Continuation Cartoon



X-axis = space of polynomial systems.
Y-axis = space of zeros.

We have to track the zeros $x(t)$ from $t = 1$ towards $t = 0$.

From

$$H(x(t), t) = 0 \text{ for } t \in [0, 1]$$

follows that $x(t)$ can be described by the **Davidenko differential equation**

$$H_x(x(t), t)\dot{x}(t) + H_t(x(t), t) = 0$$

Given a solution $x_1 = x(1)$ this is an **initial value problem**.

Predictor Corrector Scheme

Given some discretization $1 = t_0 > t_1 > \ldots > t_K = 0$ we can follow a path numerically using a **predictor- corrector scheme**.

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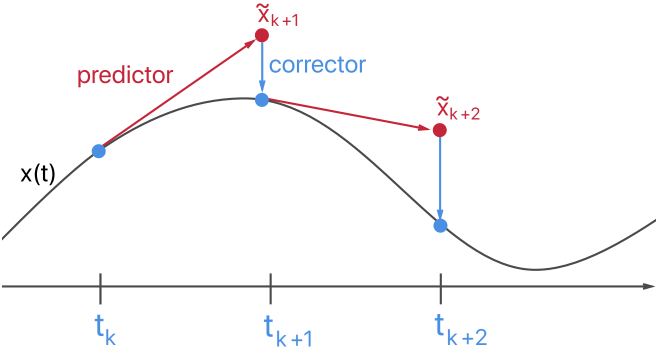

The corrector is usually Newton's method.

Computing all zeros

We can compute **all** zeros of a polynomial system F by embedding it in a **larger** family of polynomial systems where we can compute all solutions.

Examples:

- Bezout's theorem (total degree homotopy)
- Bernstein-Kushnirenko theorem (polyhedral homotopy)

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Another method for polynomial systems with **parametric coefficients** is based on the **monodromy** action induced by the fundamental group of the regular locus of the parameter space.

Monodromy - Theory

Assume F_p is a polynomial system in n variables with **parametric coefficients** depending on $p = (p_1, \ldots, p_m)$.

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Consider the variety

$$Y := \{\, (x,p) \in \mathbb{C}^n \times \mathbb{C}^m \mid F_p(x) = 0\}$$

and assume there exists an open set $Q \subset \mathbb{C}^m$ such that

$$\pi : Y \to Q, \quad (x,p) \to p$$

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A **loop** in Q based at q has D lifts to $\pi^{-1}(Q)$, one for each point in the fiber $\pi^{-1}(q)$.
Associating a point in the fiber $\pi^{-1}(q)$ to the endpoint of the corresponding lift gives a **permutation** in S_D.

Monodromy - Application

If F is a polynomial system with **parametric coefficients** and we know **one** solution for a **generic** parameter q we can use the **monodromy** action to populate the fiber $\pi^{-1}(q)$.

Example:

- The conics tangent to five given conics

Reformulating Steiner's problem

For the five conics A, B, C, D, E we solve the following system of 15 equations:

$$f_{A,B,C,D,E}(x)$$

$$= \begin{bmatrix} A(x_1, y_1) & U(x_1, y_1) & (\frac{\partial A}{\partial x} \cdot \frac{\partial U}{\partial y} - \frac{\partial A}{\partial y} \cdot \frac{\partial U}{\partial x})(x_1, y_1) \\ B(x_2, y_2) & U(x_2, y_2) & (\frac{\partial B}{\partial x} \cdot \frac{\partial U}{\partial y} - \frac{\partial B}{\partial y} \cdot \frac{\partial U}{\partial x})(x_2, y_2) \\ C(x_3, y_3) & U(x_3, y_3) & (\frac{\partial C}{\partial x} \cdot \frac{\partial U}{\partial y} - \frac{\partial C}{\partial y} \cdot \frac{\partial U}{\partial x})(x_3, y_3) \\ D(x_4, y_4) & U(x_4, y_4) & (\frac{\partial D}{\partial x} \cdot \frac{\partial U}{\partial y} - \frac{\partial D}{\partial y} \cdot \frac{\partial U}{\partial x})(x_4, y_4) \\ E(x_5, y_5) & U(x_5, y_5) & (\frac{\partial E}{\partial x} \cdot \frac{\partial U}{\partial y} - \frac{\partial E}{\partial y} \cdot \frac{\partial U}{\partial x})(x_5, y_5) \end{bmatrix}.$$

in the 15 variables

$$x = (u_1, u_2, u_3, u_4, u_5, x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, x_5, y_5).$$

For homotopy continuation, this formulation is better than using the tact invariant.

Solving instances

Assume we have computed **3264** solutions for a generic instance

$$A_0, B_0, C_0, D_0, E_0$$

To find the solutions to a particular instance we can use the parameter homotopy

$$H(x,t) = f_{tA+(1-t)A_0,\dots,tE+(1-t)E_0}(x).$$

The power of numerical computations

Numerically solving for the **3264** conics outperforms symbolic computations in terms of speed.

We get solutions in **floating point representation**.
They are not exact!

Good news: we can rigorously certify the outcome of the numerical computation, if the approximations the true solutions are good enough.

Definition

Let $f(x) = (f_1(x), \ldots, f_n(x))$ be a system of n polynomials in n variables and $J(x)$ its $n \times n$ Jacobian matrix.

A point $z \in \mathbb{C}^n$ is an **approximate zero** of f if there exists a zero $\zeta \in \mathbb{C}^n$ of f such that the sequence of Newton iterates

$$z_{k+1} = x - J(x)^{-1} f(x)$$

starting at $z_0 = z$ satisfies

$$\|z_{k+1} - \zeta\|_2 \leq \frac{1}{2}\|z_k - \zeta\|_2^2 \quad \text{for all } k = 1, 2, 3, \ldots.$$

If this holds, then we call ζ the **associated zero** of z. Here, the zero ζ is assumed to be nonsingular, i.e. $\det(J(\zeta)) \neq 0$.

Consider

$$\alpha(f,z) \;=\; \beta(f,z)\cdot\gamma(f,z).$$

where

$$\beta(f,z) = \|J(z)^{-1}f(z)\|$$
$$\gamma(f,z) = \max_{k\geq 2}\left\|\frac{1}{k!}\,J(z)^{-1}J^{(k)}(z)\right\|^{\frac{1}{k-1}},$$

Theorem (Smale's α-theorem)

1. *If $\alpha(f,z) < 0.03$, then the point z is an approximate zero of the system f.*
2. *If $y \in \mathbb{C}^n$ is any point with $\|y-z\| < (20\,\gamma(f,z))^{-1}$, then y is also an approximate zero of f with the same associated zero ζ as z.*

The theorem can also be used to verify if z is a **real** solution.

Five conics that have 3264 real conics

Using numerical homotopy continuation we found the following instance of conics:

$$
\begin{array}{llllll}
\frac{10124547}{662488724}x^2 & +\frac{8554609}{755781377}xy & +\frac{5860508}{2798943247}y^2 & -\frac{251402893}{1016797750}x & -\frac{25443962}{277938473}y & +1 \\
\frac{520811}{1788018449}x^2 & +\frac{2183697}{542440933}xy & +\frac{9030222}{652429049}y^2 & -\frac{12680955}{370629407}x & -\frac{24872323}{105706890}y & +1 \\
\frac{6537193}{241535591}x^2 & -\frac{7424602}{363844915}xy & +\frac{6264373}{1630169777}y^2 & +\frac{13097677}{39806827}x & -\frac{29825861}{240478169}y & +1 \\
\frac{13173269}{2284890206}x^2 & +\frac{4510030}{483147459}xy & +\frac{2224435}{588965799}y^2 & +\frac{33318719}{219393000}x & +\frac{92891037}{755709662}y & +1 \\
\frac{8275097}{452566634}x^2 & -\frac{19174153}{408565940}xy & +\frac{5184916}{172253855}y^2 & -\frac{23713234}{87670601}x & +\frac{28246737}{81404569}y & +1
\end{array}
$$

With Smale's α-theorem and using exact arithmetic we prove that the **3264** conics of this instance are all real.

(Thanks to an implementation by Hauenstein and Sottile!)

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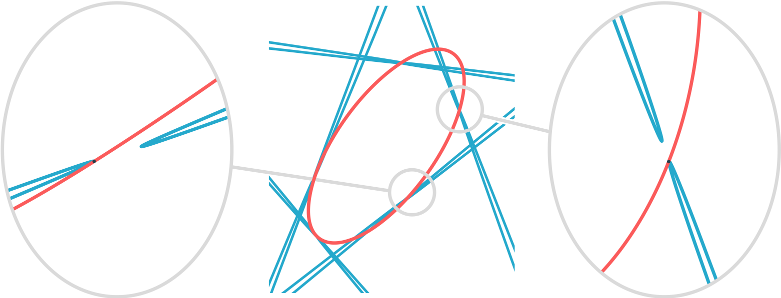


All of the **3264** real conics are animated here:

juliahomotopycontinuation.org/3264/

Which conics?

In 2019 we ask **which** conics are tangent to **your** five conics.

For answering this question, we have designed a web interface:

juliahomotopycontinuation.org/do-it-yourself/

Homotopy Continuation.jl

Take away story

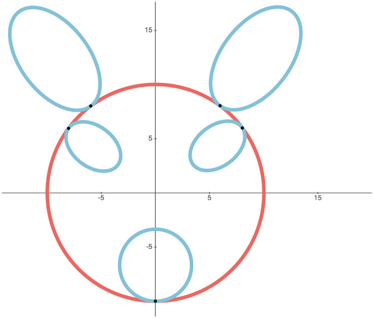
$$0.03x^2 + 0xy + 0.03y^2 + 0x + 0.4y + 1$$
$$2.56x^2 - 2.16xy + 3.19y^2 - 20x - 15y + 75$$
$$2.56x^2 + 2.16xy + 3.19y^2 + 20x - 15y + 75$$
$$22.96x^2 - 19.44xy + 17.29y^2 - 186x - 248y + 2100$$
$$22.96x^2 + 19.44xy + 17.29y^2 + 186x - 248y + 2100$$

Compute tangent conics

3264 complex solutions found in **1.29** seconds.

44 solutions are real: **6** ellipses and **38** hyperbolas.

$$-0.01x^2 + 0xy - 0.01y^2 + 0x + 0y + 1$$



Numerical computer algebra

- is fast and reliable,
- can be used for mathematical proofs.

Thank you for your attention!