A Pascal's Theorem for rational normal curves

Alessio Caminata[1]

Università degli Studi di Genova

January 15, 2021

[1] joint work with L. Schaffler.

Outline of the Talk

1. *Classical* Pascal's Theorem;
2. Bracket algebra and Grassmann–Cayley algebra;
3. The parameter space $V_{d,n}$;
4. Pascal's Theorem for rational normal curves.

Part I

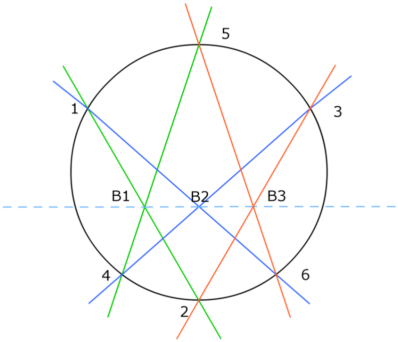
Classical Pascal's Theorem

Classical Pascal's Theorem

Work over $\mathbb{k}$, algebraically closed field (no assumption on char $\mathbb{k}$).
Often denote projective points $p_1, \ldots, p_n \in \mathbb{P}^d$ by numbers $1, \ldots, n$ only.

Pascal's Theorem or Mystic Hexagon Theorem

If $1, \ldots, 6 \in \mathbb{P}^2$ lie on a smooth conic $\Rightarrow$ points $\overline{12} \cap \overline{45}$, $\overline{23} \cap \overline{56}$, and $\overline{34} \cap \overline{61}$ are collinear



Picture from Wikipedia

Historical Remarks

Pascal's Theorem

If $1, \dots, 6 \in \mathbb{P}^2$ lie on a smooth conic $\Rightarrow$ points $\overline{12} \cap \overline{45}$, $\overline{23} \cap \overline{56}$, and $\overline{34} \cap \overline{61}$ are collinear.

- Blaise Pascal (1640) - *Essay pour les coniques*.

Historical Remarks

Pascal's Theorem

If $1, \dots, 6 \in \mathbb{P}^2$ lie on a smooth or **degenerate** conic $\Rightarrow$ points $\overline{12} \cap \overline{45}$, $\overline{23} \cap \overline{56}$, and $\overline{34} \cap \overline{61}$ are collinear.

- Pappus of Alexandria ($\simeq$ AD 300) - true for two lines as well, that is $1, 3, 5 \in \ell_1$, $2, 4, 6 \in \ell_2$.
- Blaise Pascal (1640) - *Essay pour les coniques*.

Historical Remarks

Pascal's Theorem

If $1, \ldots, 6 \in \mathbb{P}^2$ lie on a smooth or degenerate conic $\iff$ points $\overline{12} \cap \overline{45}$, $\overline{23} \cap \overline{56}$, and $\overline{34} \cap \overline{61}$ are collinear.

- Pappus of Alexandria ($\simeq$ AD 300) - true for two lines as well, that is $1, 3, 5 \in \ell_1$, $2, 4, 6 \in \ell_2$.
- Blaise Pascal (1640) - *Essay pour les coniques*.
- Braikenridge and MacLaurin (1733–35) - the converse holds.

Historical Remarks

Pascal's Theorem

If $1, \ldots, 6 \in \mathbb{P}^2$ lie on a smooth or degenerate conic $\Longleftrightarrow$ points $\overline{12} \cap \overline{45}$, $\overline{23} \cap \overline{56}$, and $\overline{34} \cap \overline{61}$ are collinear.

- Pappus of Alexandria ($\simeq$ AD 300) - true for two lines as well, that is $1, 3, 5 \in \ell_1$, $2, 4, 6 \in \ell_2$.
- Blaise Pascal (1640) - *Essay pour les coniques*.
- Braikenridge and MacLaurin (1733–35) - the converse holds.

Some generalizations:

- Möbius (1848) - polygon with $4n + 2$ sides inscribed in a conic.
- Chasles (1885) - *"Given C_1, C_2 planar cubics meeting at 9 distinct points, if a third (smooth) cubic passes through 8 of them, then passes also through the ninth."* $\longrightarrow$ Cayley and Bacharach...

Rational Normal Curves

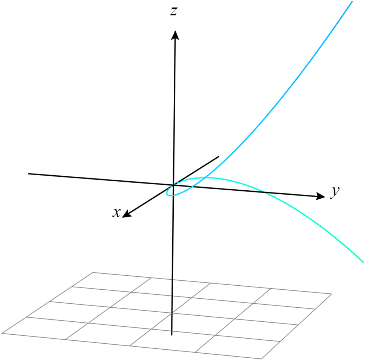
A **rational normal curve** (r.n.c.) $\mathcal{C}_d$ of degree d in $\mathbb{P}^d$ is a smooth rational curve of degree d.

Up to projectivity, $\mathcal{C}_d$ is the image of the degree d Veronese map

$$v_d : \mathbb{P}^1 \to \mathbb{P}^d$$
$$[u : v] \mapsto [u^d : u^{d-1}v : \cdots : v^d]$$

Examples:

- $\mathcal{C}_1$ is a line.
- $\mathcal{C}_2$ is a smooth conic.
- $\mathcal{C}_3$ is a twisted cubic.



Main Question

Castelnuovo's Lemma

Given $d + 3$ points in $\mathbb{P}^d$ in general linear position then there exists a (unique) rational normal curve passing through them.

Example ($d = 2$): There is always a conic through $d + 3 = 5$ points in g.l.p. in $\mathbb{P}^2$. Pascal's Theorem gives a synthetic linear condition for $d + 4 = 6$ points to lie on a conic.

Main Question

Castelnuovo's Lemma

Given $d+3$ points in $\mathbb{P}^d$ in general linear position then there exists a (unique) rational normal curve passing through them.

Example ($d = 2$): There is always a conic through $d+3=5$ points in g.l.p. in $\mathbb{P}^2$. Pascal's Theorem gives a synthetic linear condition for $d+4=6$ points to lie on a conic.

Question

Is there a coordinate-free synthetic linear condition for $d+4$ points in $\mathbb{P}^d$ to lie on a degree d rational normal curve?

Part II

Bracket Algebra and Grassmann–Cayley Algebra

Coble's Trick

$1, \dots, 6 \in \mathbb{P}^2$ lie on a (eventually degenerate) conic if and only if their images under the Veronese map $v_2 : \mathbb{P}^2 \to \mathbb{P}^5$ lie on a hyperplane, i.e.,

$$\phi = \det \begin{pmatrix} x_{0,1}^2 & x_{0,2}^2 & \cdots & x_{0,6}^2 \\ x_{1,1}^2 & & \cdots & x_{1,6}^2 \\ x_{2,1}^2 & & \cdots & \vdots \\ \vdots & & & \\ x_{1,1}x_{2,1} & & & x_{1,6}x_{2,6} \end{pmatrix} = 0$$

where $[x_{0,1} : x_{1,1} : x_{2,1}] \dots [x_{0,6} : x_{1,6} : x_{2,6}]$ are coordinates of the points.

Coble's Trick

$1, \ldots, 6 \in \mathbb{P}^2$ lie on a (eventually degenerate) conic if and only if their images under the Veronese map $v_2 : \mathbb{P}^2 \to \mathbb{P}^5$ lie on a hyperplane, i.e.,

$$\phi = \det \begin{pmatrix} x_{0,1}^2 & x_{0,2}^2 & \cdots & x_{0,6}^2 \\ x_{1,1}^2 & & \cdots & x_{1,6}^2 \\ x_{2,1}^2 & & \cdots & \vdots \\ \vdots & & & \\ x_{1,1}x_{2,1} & & & x_{1,6}x_{2,6} \end{pmatrix} = 0$$

where $[x_{0,1} : x_{1,1} : x_{2,1}] \ldots [x_{0,6} : x_{1,6} : x_{2,6}]$ are coordinates of the points. ϕ can be written as algebraic combination of maximal minors of

$$\begin{pmatrix} x_{0,1} & \cdots & x_{0,6} \\ x_{1,1} & \cdots & x_{1,6} \\ x_{2,1} & \cdots & x_{2,6} \end{pmatrix}$$

In fact, $\phi = [123][145][246][356] - [124][135][236][456]$.

Bracket Algebra

Theorem (Coble)

$1, \ldots, 6 \in \mathbb{P}^2$ lie on a conic if and only if $\phi = 0$, where

$$\phi = [123][145][246][356] - [124][135][236][456]$$

$\phi \in \Bbbk[\Lambda(6,3)]$ i.e. a polynomial in the brackets.

A **bracket** in $\Lambda(n,d)$ is a formal expression $[\lambda_1 \ldots \lambda_d]$ where $\lambda_1 < \ldots < \lambda_d$, $\lambda_1, \ldots, \lambda_d \in [n] = \{1, \ldots, n\}$.

Bracket Algebra

Theorem (Coble)

$1, \ldots, 6 \in \mathbb{P}^2$ lie on a conic if and only if $\phi = 0$, where

$$\phi = [123][145][246][356] - [124][135][236][456]$$

A **bracket** in $\Lambda(n, d)$ is a formal expression $[\lambda_1 \ldots \lambda_d]$ where $\lambda_1 < \ldots < \lambda_d$, $\lambda_1, \ldots, \lambda_d \in [n] = \{1, \ldots, n\}$.

Remark: There is an algebra homomorphism

$$\varepsilon : \mathbb{k}[\Lambda(n, d)] \to \mathbb{k}[x_{i,j} : \ 1 \leq i \leq d, \ 1 \leq j \leq n]$$

defined by extending $[\lambda_1 \cdots \lambda_d] \mapsto \det(x_{i,\lambda_i})$. The kernel gives syzygies between the brackets.

Example ($d = 2$, $n = 4$): We have the Plücker relation $[12][34] - [13][24] + [14][23] = 0$

Bracket Algebra

A **bracket** in $\Lambda(n, d)$ is a formal expression $[\lambda_1 \dots \lambda_d]$ where $\lambda_1 < \dots < \lambda_d$, $\lambda_1, \dots, \lambda_d \in [n] = \{1, \dots, n\}$.

Remark: There is an algebra homomorphism

$$\varepsilon : \mathbb{k}[\Lambda(n, d)] \to \mathbb{k}[x_{i,j} : \ 1 \leq i \leq d, \ 1 \leq j \leq n]$$

defined by extending $[\lambda_1 \cdots \lambda_d] \mapsto \det(x_{i,\lambda_i})$. The kernel gives syzygies between the brackets.

Example ($d = 2$, $n = 4$): We have the Plücker relation $[12][34] - [13][24] + [14][23] = 0$

We define the **bracket algebra** as

$$\mathcal{B}_{n,d} = \mathbb{k}[\Lambda(n, d)] / \ker \varepsilon.$$

Grassmann–Cayley Algebra

V d-dimensional $\mathbb{k}$-vector space. The **Grassmann–Cayley algebra** of V is the exterior algebra $\Lambda(V)$ with two operations:

- $\vee$ **join** is the standard exterior product. For $v_1, \ldots, v_k \in V$ we write also

  $$v_1 \cdots v_k = v_1 \vee \cdots \vee v_k$$

  and call it **extensor of step** k.

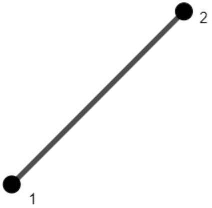
Grassmann–Cayley Algebra

V d-dimensional $\mathbb{k}$-vector space. The **Grassmann–Cayley algebra** of V is the exterior algebra $\Lambda(V)$ with two operations:

- $\vee$ **join** is the standard exterior product. For $v_1, \ldots, v_k \in V$ we write also
$$v_1 \cdots v_k = v_1 \vee \cdots \vee v_k$$
and call it **extensor of step** k.
- $\wedge$ **meet** given by the formula
$$(a_1 \ldots a_j) \wedge (b_1 \ldots b_k) = \sum_{\sigma} \operatorname{sign}(\sigma)[a_{\sigma(1)} \ldots a_{\sigma(d-k)} b_1 \ldots b_k] a_{\sigma(d-k+1)} \ldots a_{\sigma(j)},$$
where $a_1 \ldots a_j$ and $b_1 \ldots b_k$ are extensors of steps j and k with $j + k \geq d$ and the sum is taken over all permutations σ of $\{1, \ldots, j\}$ such that $\sigma(1) < \ldots < \sigma(d-k)$ and $\sigma(d-k+1) < \ldots < \sigma(j)$.

Geometric Interpretation

The **join ($\vee$)** refers to the line passing through points:

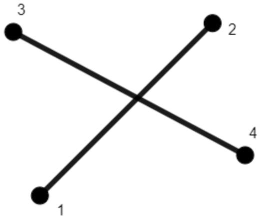


The line joining 1 and 2 is $1 \vee 2$ or 12.

When $d = \dim V = 3$ (i.e. over $\mathbb{P}^2$) three points joined makes a bracket, and three collinear points make the bracket vanish.

Geometric Interpretation

The **meet (∧)** refers to the intersection of two spaces:



The meet of the lines $1 \vee 2$ and $3 \vee 4$ is $(1 \vee 2) \wedge (3 \vee 4)$ or $12 \wedge 34$.

Expanding the meet using its definition and the distributivity of join and meet we also obtain expressions in the brackets.

Geometric Interpretation

More formally, we have a correspondence

$$\{\text{extensors of step } j\} \longleftrightarrow \{j - \text{dim. v. subspaces of } V\}$$
$$A \longmapsto \overline{A} = \{v \in V : \ A \vee v = 0\}$$

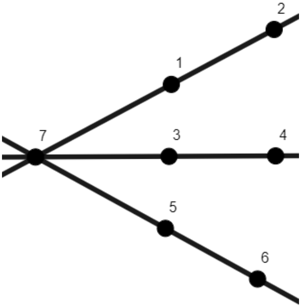
which extends to $\mathbb{P}(V)$.

Proposition

Let V be a $\Bbbk$-vector space of dimension d. Let $A = a_1 \vee \cdots \vee a_j$ and $B = b_1 \vee \cdots \vee b_k$ be two extensors of steps j and k respectively. Then

- $A \vee B \neq 0$ if and only if $a_1, \ldots, a_j, b_1, \ldots, b_k$ are linearly independent. In this case $\boldsymbol{\overline{A} + \overline{B} = \overline{A \vee B}} = \mathrm{span}\{a_1, \ldots, a_j, b_1, \ldots, b_k\}$.
- Assume $j + k \geq d$. Then $A \wedge B \neq 0$ if and only if $\overline{A} + \overline{B} = V$. In this case, $\boldsymbol{\overline{A} \cap \overline{B} = \overline{A \wedge B}}$. In particular, $A \wedge B$ can be represented by an appropriate extensor.

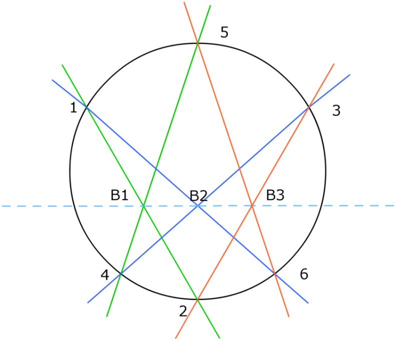
An Example



We have $\big((1 \vee 2) \wedge (3 \vee 4)\big) \vee 5 \vee 6 = 0$ because these three points are collinear.
Expanding using the definition of meet one obtains

$$\begin{aligned} 0 = \big(12 \wedge 34\big) \vee 56 &= \big([134]2 - [234]1\big) \vee 56 \\ &= [134][256] - [234][156] \end{aligned}$$

A Proof of Pascal's Theorem

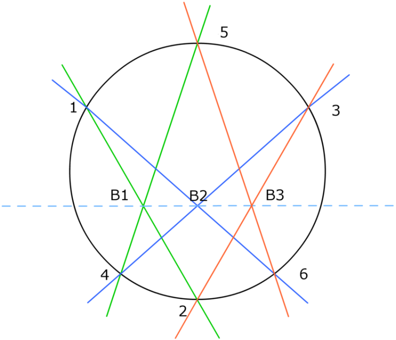


The collinearity of the three points can be written in GC algebra as

$$(12 \wedge 45) \vee (23 \wedge 56) \vee (34 \wedge 61) = 0$$

Picture from Wikipedia

A Proof of Pascal's Theorem



The collinearity of the three points can be written in GC algebra as

$$(12 \wedge 45) \vee (23 \wedge 56) \vee (34 \wedge 61) = 0$$

which can be expanded in bracket algebra obtaining

$$[123][145][246][356] - [124][135][236][456] = 0$$

The latter is the equation $\phi = 0$, which we know being equivalent to ask for the points $1, \ldots, 6$ to lie on a conic.

Picture from Wikipedia

The Path towards Rational Normal Curves

Grassmann–Cayley algebra	$(12 \wedge 45) \vee (23 \wedge 56) \vee (34 \wedge 61) = 0$
$\updownarrow$	$\updownarrow$
bracket algebra	$[123][145][246][356] - [124][135][236][456] = 0$

We attempt to generalize both parts of the previous proof of Pascal's Theorem to rational normal curves. Namely, our strategy is the following:

1. Find bracket equations that express the condition for $d + 4$ points in $\mathbb{P}^d$ to lie on a r.n.c.
2. Find the corresponding equations in GC algebra.

Part III

The Parameter Space $V_{d,n}$

The Parameter Space $V_{d,n}$

Let $d, n \in \mathbb{Z}_+$, we define the **Veronese compactification**

$$V_{d,n} := \overline{\{\mathbf{p} = (p_1, \ldots, p_n) \in (\mathbb{P}^d)^n : \ p_1, \ldots, p_n \in \mathcal{C}_d \text{ r.n.c.}\}} \subseteq (\mathbb{P}^d)^n$$

The Parameter Space $V_{d,n}$

Let $d, n \in \mathbb{Z}_+$, we define the **Veronese compactification**

$$V_{d,n} := \overline{\{\mathbf{p} = (p_1, \ldots, p_n) \in (\mathbb{P}^d)^n : \ p_1, \ldots, p_n \in \mathcal{C}_d \text{ r.n.c.}\}} \subseteq (\mathbb{P}^d)^n$$

- $d = 1$ or $n \leq d + 3$ then $V_{d,n} = (\mathbb{P}^d)^n$ (Castelnuovo's Lemma);
- $d \geq 2$ and $n \geq d + 4$ then $V_{d,n}$ is irreducible and $\dim V_{d,n} = d^2 + 2d + n - 3$.

The Parameter Space $V_{d,n}$

Let $d, n \in \mathbb{Z}_+$, we define the **Veronese compactification**

$$V_{d,n} := \overline{\{\mathbf{p} = (p_1, \dots, p_n) \in (\mathbb{P}^d)^n : \ p_1, \dots, p_n \in \mathcal{C}_d \text{ r.n.c.}\}} \subseteq (\mathbb{P}^d)^n$$

- $d = 1$ or $n \leq d + 3$ then $V_{d,n} = (\mathbb{P}^d)^n$ (Castelnuovo's Lemma);
- $d \geq 2$ and $n \geq d + 4$ then $V_{d,n}$ is irreducible and $\dim V_{d,n} = d^2 + 2d + n - 3$.

Step 1

Find (bracket) equations that define $V_{d,n}$ set-theoretically.

Which Point Configurations are in $V_{d,n}$?

Theorem (CGMS[2])

A non-degenerate point configuration $\mathbf{p} \in (\mathbb{P}^d)^n$ is in $V_{d,n}$ if and only if it lies on a quasi-Veronese curve.

A *quasi-Veronese curve* $\mathcal{C}$ in $\mathbb{P}^d$ is a curve of degree d that is complete, connected, and non-degenerate.

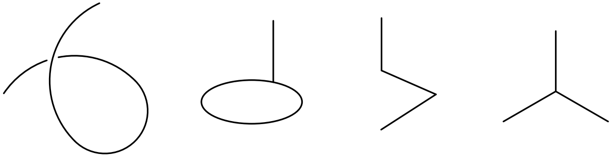


Figure: The degree three quasi-Veronese curves: twisted cubic, non-coplanar union of line and conic, chain of three lines, and non-coplanar union of three lines meeting at a point.

[2] A. Caminata, N. Giansiracusa, H. Moon, L. Schaffler, *Equations for point configurations to lie on a rational normal curve*, Adv. Math. 340, pp. 653–683, 2018.

Points on Plane Conics ($d = 2$)

Case $d = 2$, $n = 6$. Thanks to Coble's trick, $V_{2,6}$ is a hypersurface in $(\mathbb{P}^2)^6$ defined by the bracket equation:

$$\phi = [123][145][246][356] - [124][135][236][456] = 0$$

Points on Plane Conics ($d = 2$)

Case $d = 2$**,** $n = 6$**.** Thanks to Coble's trick, $V_{2,6}$ is a hypersurface in $(\mathbb{P}^2)^6$ defined by the bracket equation:

$$\phi = [123][145][246][356] - [124][135][236][456] = 0$$

Case $d = 2$**,** $n \geq 7$**.**

$$\pi_I : (\mathbb{P}^2)^n \to (\mathbb{P}^2)^6 \text{ forgetful map}$$
$$\mathbf{p} = (p_i) \mapsto (p_i)_{i \in I} \quad I \subseteq [n],\ |I| = 6.$$

Define $\phi_I := \pi_I^*(\phi)$.

Points on Plane Conics ($d = 2$)

Case $d = 2$, $n = 6$. Thanks to Coble's trick, $V_{2,6}$ is a hypersurface in $(\mathbb{P}^2)^6$ defined by the bracket equation:

$$\phi = [123][145][246][356] - [124][135][236][456] = 0$$

Case $d = 2$, $n \geq 7$.

$$\pi_I : (\mathbb{P}^2)^n \to (\mathbb{P}^2)^6 \quad \text{forgetful map}$$
$$\mathbf{p} = (p_i) \mapsto (p_i)_{i \in I} \quad I \subseteq [n],\ |I| = 6.$$

Define $\phi_I := \pi_I^*(\phi)$.

Theorem (CGMS)

1. $V_{2,n} = \bigcap_I \mathcal{Z}(\phi_I)$
2. $V_{2,n}$ is Cohen–Macaulay and normal
3. $V_{2,n}$ is Gorenstein $\iff n = 6$.

The Gale Transform

Fix $d \geq 3$, $n = d + 4$.
Consider $\mathbf{p} \in (\mathbb{P}^2)^n$, $\mathbf{q} \in (\mathbb{P}^d)^n$ not on a hyperplane.

$$\mathbf{p} = (p_1, \ldots, p_n) \in (\mathbb{P}^2)^n \xleftrightarrow{\text{Gale transform}} \mathbf{q} = (q_1, \ldots, q_n) \in (\mathbb{P}^d)^n$$

Construct two matrices with the coordinates of the points as columns:
$A = (p_1, \ldots, p_n)$ of size $3 \times n$ and
$B = (q_1, \ldots, q_n)$ of size $(d + 1) \times n$.

Definition

p and **q** are **Gale dual** if there exists a $n \times n$ diagonal matrix $D \neq 0$ such that $ADB^T = 0$.

The Gale Transform

Fix $d \geq 3$, $n = d + 4$.
Consider $\mathbf{p} \in (\mathbb{P}^2)^n$, $\mathbf{q} \in (\mathbb{P}^d)^n$ not on a hyperplane.

$$\mathbf{p} = (p_1, \ldots, p_n) \in (\mathbb{P}^2)^n \xleftrightarrow{\text{Gale transform}} \mathbf{q} = (q_1, \ldots, q_n) \in (\mathbb{P}^d)^n$$

Construct two matrices with the coordinates of the points as columns:
$A = (p_1, \ldots, p_n)$ of size $3 \times n$ and
$B = (q_1, \ldots, q_n)$ of size $(d+1) \times n$.

Definition

$\mathbf{p}$ and $\mathbf{q}$ are **Gale dual** if there exists a $n \times n$ diagonal matrix $D \neq 0$ such that $ADB^T = 0$.

Theorem (Goppa)

$\mathbf{p} \in \mathcal{C}_2$ r.n.c $\iff$ $\mathbf{q} \in \mathcal{C}_d$ r.n.c.

An Example ($d = 3$, $n = 7$)

Take 7 points $\mathbf{q}$ on the standard twisted cubic $\mathcal{C}_3 = \mathrm{Im} v_3 \subseteq \mathbb{P}^3$, $v_3([u : v]) = [u^3 : u^2 v : uv^2 : v^3]$.

$q_1 = [0 : 0 : 0 : 1]$, $q_2 = [1 : 0 : 0 : 0]$, $q_3 = [1 : 1 : 1 : 1]$, $q_4 = [1 : 2 : 4 : 8]$, $q_5 = [1 : 3 : 9 : 27]$, $q_6 = [1 : 4 : 16 : 64]$, $q_7 = [1 : 5 : 25 : 125]$

Write the corresponding matrix $B = (q_1, \ldots, q_7)$

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 4 & 9 & 16 & 25 \\ 1 & 0 & 1 & 8 & 27 & 64 & 125 \end{pmatrix}$$

Find the nullspace of B^T, that is a matrix A such that $AB^T = 0$.

$$A = \begin{pmatrix} 6 & 1 & -3 & 3 & -1 & 0 & 0 \\ 6 & 0 & 1 & -3 & 3 & -1 & 0 \\ -24 & 0 & -3 & 8 & -6 & 0 & 1 \end{pmatrix}$$

An Example ($d = 3$, $n = 7$)

$AB^T =$

$$\begin{pmatrix} 6 & 1 & -3 & 3 & -1 & 0 & 0 \\ 6 & 0 & 1 & -3 & 3 & -1 & 0 \\ -24 & 0 & -3 & 8 & -6 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 4 & 9 & 16 & 25 \\ 1 & 0 & 1 & 8 & 27 & 64 & 125 \end{pmatrix}^T = 0$$

The columns of $A = (p_1, \ldots, p_7)$ give 7 points $\mathbf{p}$ in $\mathbb{P}^2$.
$p_1 = [6 : 6 : -24]$, $p_2 = [1 : 0 : 0]$, $p_3 = [-3 : 1 : -3]$, $p_4 = [3 : -3 : 8]$, $p_5 = [-1 : 3 : -6]$, $p_6 = [0 : -1 : 0]$, $p_7 = [0 : 0 : 1]$.

By Goppa's Theorem $\mathbf{p}$ lie on a smooth conic, in fact they lie on

$$16x_0x_1 + 5x_0x_2 - x_1x_2 = 0$$

Equations for Rational Normal Curves ($d \geq 3$, $n = d + 4$)

ϕ_I defining equations for $V_{2,n}$ in bracket form, $I \subseteq [d+4]$, $|I| = 6$.

$$\phi_I \xmapsto{\text{Gale transform}} \psi_I$$

Example

$\mathbf{p} \in (\mathbb{P}^2)^7$, $\mathbf{q} \in (\mathbb{P}^3)^7$, choose $I = \{1, \ldots, 6\} \subseteq [7]$:

$$\phi_I(\mathbf{p}) = [123]_\mathbf{p}[145]_\mathbf{p}[246]_\mathbf{p}[356]_\mathbf{p} - [124]_\mathbf{p}[135]_\mathbf{p}[236]_\mathbf{p}[456]_\mathbf{p}$$

$$\psi_I(\mathbf{q}) = [4567]_\mathbf{q}[2367]_\mathbf{q}[1357]_\mathbf{q}[1247]_\mathbf{q} - [3567]_\mathbf{q}[2467]_\mathbf{q}[1457]_\mathbf{q}[1237]_\mathbf{q}$$

Equations for Rational Normal Curves ($d \geq 3$, $n = d+4$)

ϕ_I defining equations for $V_{2,n}$ in bracket form, $I \subseteq [d+4]$, $|I| = 6$.

$$\phi_I \xmapsto{\text{Gale transform}} \psi_I$$

Example

$\mathbf{p} \in (\mathbb{P}^2)^7$, $\mathbf{q} \in (\mathbb{P}^3)^7$, choose $I = \{1, \dots, 6\} \subseteq [7]$:

$$\phi_I(\mathbf{p}) = [123]_\mathbf{p}[145]_\mathbf{p}[246]_\mathbf{p}[356]_\mathbf{p} - [124]_\mathbf{p}[135]_\mathbf{p}[236]_\mathbf{p}[456]_\mathbf{p}$$

$$\psi_I(\mathbf{q}) = [4567]_\mathbf{q}[2367]_\mathbf{q}[1357]_\mathbf{q}[1247]_\mathbf{q} - [3567]_\mathbf{q}[2467]_\mathbf{q}[1457]_\mathbf{q}[1237]_\mathbf{q}$$

Theorem (CGMS)

$$\bigcap_I \mathcal{Z}(\psi_I) = V_{d,d+4} \cup Y_{d,d+4}$$

where $Y_{d,d+4} := \{\mathbf{p} \in (\mathbb{P}^d)^{d+4} : \ p_1, \dots, p_{d+4} \in \text{ hyperplane}\}$.

Part IV

Pascal's Theorem for Rational Normal Curves

The Cayley Factorization Problem

Grassmann–Cayley algebra	$(12 \wedge 45) \vee (23 \wedge 56) \vee (34 \wedge 61) = 0$
$\updownarrow$	$\updownarrow$
bracket algebra	$[123][145]\,[246][356] - [124][135][236][456] = 0$

Step 2

Find the GC algebra expression for the equations ψ_I.

The Cayley Factorization Problem

Grassmann–Cayley algebra	**??**
$\updownarrow$	$\updownarrow$
bracket algebra	$\{\psi_I : I \subseteq [d+4],\ \|I\| = 6\}$

Step 2

Find the GC algebra expression for the equations ψ_I.

The Cayley Factorization Problem

Grassmann–Cayley algebra	**??**
$\updownarrow$	$\updownarrow$
bracket algebra	$\{\psi_I : I \subseteq [d+4],\ \lvert I \rvert = 6\}$

Step 2

Find the GC algebra expression for the equations ψ_I.

Unfortunately this is not always possible!

The Cayley Factorization Problem

Grassmann–Cayley algebra	**??**
$\updownarrow$	$\updownarrow$
bracket algebra	$\{\psi_I : I \subseteq [d+4],\ \lvert I \rvert = 6\}$

Step 2

Find the GC algebra expression for the equations ψ_I.

Unfortunately this is not always possible!

Grassmann-Cayley algebra

$\downarrow$ **algorithm**

bracket algebra

The Cayley Factorization Problem

Grassmann–Cayley algebra	**??**
$\updownarrow$	$\updownarrow$
bracket algebra	$\{\psi_I : I \subseteq [d+4],\ \lvert I\rvert = 6\}$

Step 2

Find the GC algebra expression for the equations ψ_I.

Unfortunately this is not always possible!

Grassmann-Cayley algebra

$\uparrow$ **Cayley Factorization Problem**

bracket algebra

Example: $[123][456] + [124][356]$ does not factor in the GC algebra!

Turnbull–Young Problem

Open Question[3]: *Does there exist a synthetic condition à-la-Pascal for* 10 *points in* $\mathbb{P}^3$ *to lie on a quadric surface?*

Equivalent to find a GC algebra factorization of:

$$\det \begin{pmatrix} x_{0,1}^2 & x_{0,2}^2 & \cdots & x_{0,10}^2 \\ x_{1,1}^2 & & \cdots & x_{1,10}^2 \\ x_{2,1}^2 & \vdots & & \vdots \\ x_{3,1}^2 & & & x_{3,10}^2 \\ x_{0,1}x_{1,1} & & \cdots & x_{0,10}x_{1,10} \\ \vdots & & & \vdots \\ x_{2,1}x_{3,1} & & \cdots & x_{2,10}x_{3,10} \end{pmatrix}$$

Bracket form is known (White, 1990): $\simeq 138$ bracket monomials.

[3] H.W. Turnbull, A. Young, *The linear invariants of ten quaternary quadrics*, Trans. Cambridge Philos. Soc. 23, pp. 265–301, 1926.

Lifting Syzygies

Step 2: Find the GC algebra expression for the equations ψ_I.

It is possible! Thanks to the following lemma (and some extra work):

Lemma (C.–Schaffler)

There is a well-defined homomorphism of $\mathbb{k}$-algebras

$$\eta_{n+1} \colon \mathcal{B}_{n,d} \to \mathcal{B}_{n+1,d+1}$$

obtained by extending $[\lambda_{i_1} \cdots \lambda_{i_d}] \mapsto [\lambda_{i_1} \cdots \lambda_{i_d}\ n+1]$.

This allows us to lift syzygies to higher dimensions!

Example ($d = 2$, $n = 4$): The Plücker relation $[12][34] - [13][24] + [14][23] = 0$ lifts to a syzygy

$$[125][345] - [135][245] + [145][235] = 0$$

Main Theorem

Theorem (C.–Schaffler[4])

Let $d \geq 3$, let $1, \ldots, d+4$ be points in $\mathbb{P}^d$ not on a hyperplane. Then the following are equivalent:

1. $(1, \ldots, d+4) \in V_{d,d+4}$ *(equivalently, they lie on a quasi-Veronese curve of degree d);*
2. *For every $I \in \binom{[d+4]}{6}$, $I = \{i_1 < \cdots < i_6\}$, $I^c = \{j_1 < \cdots < j_{d-2}\}$ the following equality in the Grassmann–Cayley algebra holds:*

$$(i_1 i_2 \wedge i_4 i_5 j_1 \cdots j_{d-2}) \vee (i_2 i_3 \wedge i_5 i_6 j_1 \cdots j_{d-2}) \\ \vee (i_3 i_4 \wedge i_6 i_1 j_1 \cdots j_{d-2}) \vee (j_1 \cdots j_{d-2}) = 0.$$

[4] A. Caminata, L. Schaffler, *A Pascal's Theorem for rational normal curves*, arXiv:1903.00460.

Main Theorem

Theorem (C.–Schaffler[4])

Let $d \geq 3$, let $1, \dots, d+4$ be points in $\mathbb{P}^d$ in **general linear position**. *Then the following are equivalent:*

1. $(1, \dots, d+4) \in V_{d,d+4}$ *(equivalently, they lie on a* **rational normal curve** *of degree d);*
2. *For every $I \in \binom{[d+4]}{6}$, $I = \{i_1 < \dots < i_6\}$, $I^c = \{j_1 < \dots < j_{d-2}\}$ the following equality in the Grassmann–Cayley algebra holds:*

$$(i_1 i_2 \wedge i_4 i_5 j_1 \cdots j_{d-2}) \vee (i_2 i_3 \wedge i_5 i_6 j_1 \cdots j_{d-2}) \\ \vee (i_3 i_4 \wedge i_6 i_1 j_1 \cdots j_{d-2}) \vee (j_1 \cdots j_{d-2}) = 0.$$

[4] A. Caminata, L. Schaffler, *A Pascal's Theorem for rational normal curves*, arXiv:1903.00460.

The Case of Twisted Cubics

Let $1, \ldots, 7 \in \mathbb{P}^3$ be in general linear position. Then they lie on a twisted cubic if and only if for every $I \subseteq [7]$, $I = \{i_1 < \cdots < i_6\}$, $I^c = \{j\}$ the following 4 points lie on a plane:

- $\overline{i_1 i_2} \cap \overline{i_4 i_5 j}$
- $\overline{i_2 i_3} \cap \overline{i_5 i_6 j}$
- $\overline{i_3 i_4} \cap \overline{i_6 i_1 j}$
- j

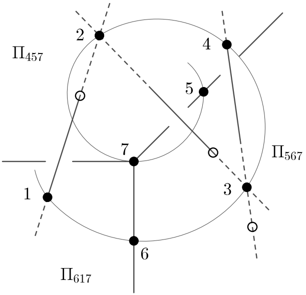


Figure: Condition for the choice $I = \{1, \ldots, 6\}$ and $I^c = \{7\}$

Thank you!