The Canny-Emiris conjecture for the sparse resultant

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Elementary Fact 1

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$$\begin{aligned} f_0 &= \alpha_0 t_0 + \alpha_1 t_1 \\ f_1 &= \beta_0 t_0^2 + \beta_1 t_0 t_1 + \beta_2 t_1^2 \end{aligned}$$

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$$\mathrm{Res}_t(f_0, f_1)$$

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$$\mathrm{Res}_t(f_0, f_1)$$

$$=$$

$$\det \begin{pmatrix} \alpha_0 & \alpha_1 & 0 \\ 0 & \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 & \beta_2 \end{pmatrix}$$

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$$\mathrm{Res}_t(f_0, f_1)$$

$$=$$

$$\det \begin{pmatrix} \alpha_0 & \alpha_1 & 0 \\ 0 & \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 & \beta_2 \end{pmatrix} = \beta_0\alpha_1^2 + \beta_1\alpha_0\alpha_1 + \beta_2\alpha_0^2$$

(Sylvester, 1845)

A smaller determinant?

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NO unless the entries of the matrix get more "complicated"

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$$\mathrm{Res}_t(f_0, f_1) = \det \begin{pmatrix} \alpha_0 & \alpha_1 \\ -\beta_0\alpha_1 & \beta_2\alpha_0 + \beta_1\alpha_1 \end{pmatrix}$$

A larger determinant?

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$$\det \begin{pmatrix} \alpha_0 & \alpha_1 & 0 & 0 & 0 \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}$$

A larger determinant?

$$\det \begin{pmatrix} \alpha_0 & \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_0 & \alpha_1 & 0 & 0 \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}$$

A larger determinant?

$$\det \begin{pmatrix} \alpha_0 & \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_0 & \alpha_1 & 0 \end{pmatrix}$$

A larger determinant?

$$\det \left(\begin{array}{ccccc} \alpha_0 & \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_0 & \alpha_1 \\ & & & & \end{array} \right.$$

A larger determinant?

$$\det\begin{pmatrix} \alpha_0 & \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_0 & \alpha_1 \\ 0 & 0 & \beta_0 & \beta_1 & \beta_2 \end{pmatrix}$$

A larger determinant?

$$\det \begin{pmatrix} \alpha_0 & \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_0 & \alpha_1 \\ 0 & 0 & \beta_0 & \beta_1 & \beta_2 \end{pmatrix} = \mathrm{Res}_t(f_0, f_1)$$

A larger determinant?

$$\det \begin{pmatrix} \alpha_0 & \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_0 & \alpha_1 \\ 0 & 0 & \beta_0 & \beta_1 & \beta_2 \end{pmatrix} = \mathrm{Res}_t(f_0, f_1) \cdot \alpha_0^2$$

A larger determinant?

$$\det \begin{pmatrix} \alpha_0 & \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_0 & \alpha_1 \\ 0 & 0 & \beta_0 & \beta_1 & \beta_2 \end{pmatrix} = \mathrm{Res}_t(f_0, f_1) \cdot \alpha_0^2$$

Obs:

A larger determinant?

$$\det\begin{pmatrix} \alpha_0 & \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_0 & \alpha_1 \\ 0 & 0 & \beta_0 & \beta_1 & \beta_2 \end{pmatrix} = \mathrm{Res}_t(f_0, f_1)\cdot \alpha_0^2$$

Obs: α_0^2 **is the determinant of several principal** 2×2 **minors above**

(Trivial) Macaulay style formula

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$$\begin{pmatrix} \alpha_0 & \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_0 & \alpha_1 \\ 0 & 0 & \beta_0 & \beta_1 & \beta_2 \end{pmatrix}$$

If $\deg(f_i) = d_i$

(Trivial) Macaulay style formula

$$\begin{pmatrix} \alpha_0 & \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_0 & \alpha_1 \\ 0 & 0 & \beta_0 & \beta_1 & \beta_2 \end{pmatrix}$$

If $\deg(f_i) = d_i$, for any $d \geq d_0 + d_1$ "Sylvester matrix",

(Trivial) Macaulay style formula

$$\begin{pmatrix} \alpha_0 & \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_0 & \alpha_1 \\ 0 & 0 & \beta_0 & \beta_1 & \beta_2 \end{pmatrix}$$

If $\deg(f_i) = d_i$, for any $d \geq d_0 + d_1$ "Sylvester matrix", $\mathrm{Res}_t(f_0, f_1)$ can be computed as the quotient of this determinant by the green $d - d_0 - d_1$ principal minor

(Less) Elementary Fact 2

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$$\begin{aligned} f_0 &= \alpha_0 t_0 + \alpha_1 t_1 + \alpha_2 t_2 \\ f_1 &= \beta_0 t_0^2 + \beta_1 t_0 t_1 + \beta_2 t_1^2 + \beta_3 t_0 t_2 + \beta_4 t_1 t_2 + \beta_5 t_2^2 \\ f_2 &= \gamma_0 t_0^2 + \gamma_1 t_0 t_1 + \gamma_2 t_1^2 + \gamma_3 t_0 t_2 + \gamma_4 t_1 t_2 + \gamma_5 t_2^2 \end{aligned}$$

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What is

$$\mathrm{Res}_t(f_0, f_1, f_2)$$

the "resultant" of f_0, f_1, f_2 ?

Answer: $\mathbf{Res}_t(f_0, f_1, f_2) =$

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$$
\begin{aligned}
&\beta_5^2\gamma_2^2\alpha_0^4 + \beta_2\beta_5\gamma_4^2\alpha_0^4 + \beta_2^2\gamma_5^2\alpha_0^4 - \beta_4\beta_5\gamma_2\gamma_4\alpha_0^4 + \beta_4^2\gamma_2\gamma_5\alpha_0^4 - 2\beta_2\beta_5\gamma_2\gamma_5\alpha_0^4 - \beta_2\beta_4\gamma_4\gamma_5\alpha_0^4 - 2\alpha_2\beta_3\beta_5\gamma_2^2\alpha_0^3\\
&-\alpha_2\beta_2\beta_3\gamma_4^2\alpha_0^3 - \alpha_1\beta_1\beta_5\gamma_4^2\alpha_0^3 - 2\alpha_1\beta_1\beta_2\gamma_5^2\alpha_0^3 - 2\alpha_1\beta_5^2\gamma_1\gamma_2\alpha_0^3 + \alpha_2\beta_4\beta_5\gamma_1\gamma_2\alpha_0^3 - \alpha_2\beta_4^2\gamma_2\gamma_3\alpha_0^3\\
&+2\alpha_2\beta_2\beta_5\gamma_2\gamma_3\alpha_0^3 + \alpha_1\beta_4\beta_5\gamma_2\gamma_3\alpha_0^3 - 2\alpha_2\beta_2\beta_5\gamma_1\gamma_4\alpha_0^3 + \alpha_1\beta_4\beta_5\gamma_1\gamma_4\alpha_0^3 + \alpha_2\beta_3\beta_4\gamma_2\gamma_4\alpha_0^3 + \alpha_2\beta_1\beta_5\gamma_2\gamma_4\alpha_0^3\\
&+\alpha_1\beta_3\beta_5\gamma_2\gamma_4\alpha_0^3 + \alpha_2\beta_2\beta_4\gamma_3\gamma_4\alpha_0^3 - 2\alpha_1\beta_2\beta_5\gamma_3\gamma_4\alpha_0^3 - \alpha_1\beta_4^2\gamma_1\gamma_5\alpha_0^3 + \alpha_2\beta_2\beta_4\gamma_1\gamma_5\alpha_0^3 + 2\alpha_1\beta_2\beta_5\gamma_1\gamma_5\alpha_0^3\\
&+2\alpha_2\beta_2\beta_3\gamma_2\gamma_5\alpha_0^3 - 2\alpha_2\beta_1\beta_4\gamma_2\gamma_5\alpha_0^3 - 2\alpha_1\beta_3\beta_4\gamma_2\gamma_5\alpha_0^3 + 2\alpha_1\beta_1\beta_5\gamma_2\gamma_5\alpha_0^3 - 2\alpha_2\beta_2^2\gamma_3\gamma_5\alpha_0^3 + \alpha_1\beta_2\beta_4\gamma_3\gamma_5\alpha_0^3\\
&+\alpha_2\beta_1\beta_2\gamma_4\gamma_5\alpha_0^3 + \alpha_1\beta_2\beta_3\gamma_4\gamma_5\alpha_0^3 + \alpha_1\beta_1\beta_4\gamma_4\gamma_5\alpha_0^3 + \alpha_1^2\beta_5^2\gamma_1^2\alpha_0^2 + \alpha_2^2\beta_2\beta_5\gamma_1^2\alpha_0^2 - \alpha_1\alpha_2\beta_4\beta_5\gamma_1^2\alpha_0^2\\
&+\alpha_2^2\beta_3^2\gamma_2^2\alpha_0^2 + 2\alpha_2^2\beta_0\beta_5\gamma_2^2\alpha_0^2 + \alpha_2^2\beta_2^2\gamma_3^2\alpha_0^2 - \alpha_1\alpha_2\beta_2\beta_4\gamma_3^2\alpha_0^2 + \alpha_1^2\beta_2\beta_5\gamma_3^2\alpha_0^2 + \alpha_2^2\beta_0\beta_2\gamma_4^2\alpha_0^2 + \alpha_1\alpha_2\beta_1\beta_3\gamma_4^2\alpha_0^2\\
&+\alpha_1^2\beta_0\beta_5\gamma_4^2\alpha_0^2 + \alpha_1^2\beta_1^2\gamma_5^2\alpha_0^2 + 2\alpha_1^2\beta_0\beta_2\gamma_5^2\alpha_0^2 + \alpha_2^2\beta_4^2\gamma_0\gamma_2\alpha_0^2 + 2\alpha_1^2\beta_5^2\gamma_0\gamma_2\alpha_0^2 - 2\alpha_2^2\beta_2\beta_5\gamma_0\gamma_2\alpha_0^2\\
&-2\alpha_1\alpha_2\beta_4\beta_5\gamma_0\gamma_2\alpha_0^2 - \alpha_2^2\beta_3\beta_4\gamma_1\gamma_2\alpha_0^2 - \alpha_2^2\beta_1\beta_5\gamma_1\gamma_2\alpha_0^2 + 3\alpha_1\alpha_2\beta_3\beta_5\gamma_1\gamma_2\alpha_0^2 + \alpha_1\alpha_2\beta_4^2\gamma_1\gamma_3\alpha_0^2\\
&-\alpha_2^2\beta_2\beta_4\gamma_1\gamma_3\alpha_0^2 - \alpha_1^2\beta_4\beta_5\gamma_1\gamma_3\alpha_0^2 - 2\alpha_2^2\beta_2\beta_3\gamma_2\gamma_3\alpha_0^2 + 2\alpha_2^2\beta_1\beta_4\gamma_2\gamma_3\alpha_0^2 + \alpha_1\alpha_2\beta_3\beta_4\gamma_2\gamma_3\alpha_0^2\\
&-3\alpha_1\alpha_2\beta_1\beta_5\gamma_2\gamma_3\alpha_0^2 - \alpha_1^2\beta_3\beta_5\gamma_2\gamma_3\alpha_0^2 - \alpha_2^2\beta_2\beta_4\gamma_0\gamma_4\alpha_0^2 + 4\alpha_1\alpha_2\beta_2\beta_5\gamma_0\gamma_4\alpha_0^2 - \alpha_1^2\beta_4\beta_5\gamma_0\gamma_4\alpha_0^2\\
&+2\alpha_2^2\beta_2\beta_3\gamma_1\gamma_4\alpha_0^2 - \alpha_1\alpha_2\beta_3\beta_4\gamma_1\gamma_4\alpha_0^2 + \alpha_1\alpha_2\beta_1\beta_5\gamma_1\gamma_4\alpha_0^2 - \alpha_1^2\beta_3\beta_5\gamma_1\gamma_4\alpha_0^2 - \alpha_1\alpha_2\beta_3^2\gamma_2\gamma_4\alpha_0^2\\
&-\alpha_2^2\beta_1\beta_3\gamma_2\gamma_4\alpha_0^2 - \alpha_2^2\beta_0\beta_4\gamma_2\gamma_4\alpha_0^2 - 2\alpha_1\alpha_2\beta_0\beta_5\gamma_2\gamma_4\alpha_0^2 - \alpha_2^2\beta_1\beta_2\gamma_3\gamma_4\alpha_0^2 + \alpha_1\alpha_2\beta_2\beta_3\gamma_3\gamma_4\alpha_0^2\\
&-\alpha_1\alpha_2\beta_1\beta_4\gamma_3\gamma_4\alpha_0^2 + 2\alpha_1^2\beta_1\beta_5\gamma_3\gamma_4\alpha_0^2 + 2\alpha_2^2\beta_2^2\gamma_0\gamma_5\alpha_0^2 + \alpha_1^2\beta_4^2\gamma_0\gamma_5\alpha_0^2 - 2\alpha_1\alpha_2\beta_2\beta_4\gamma_0\gamma_5\alpha_0^2\\
&-2\alpha_1^2\beta_2\beta_5\gamma_0\gamma_5\alpha_0^2 - \alpha_2^2\beta_1\beta_2\gamma_1\gamma_5\alpha_0^2 - 3\alpha_1\alpha_2\beta_2\beta_3\gamma_1\gamma_5\alpha_0^2 + \alpha_1\alpha_2\beta_1\beta_4\gamma_1\gamma_5\alpha_0^2 + 2\alpha_1^2\beta_3\beta_4\gamma_1\gamma_5\alpha_0^2\\
&-2\alpha_1^2\beta_1\beta_5\gamma_1\gamma_5\alpha_0^2 + \alpha_2^2\beta_1^2\gamma_2\gamma_5\alpha_0^2 + \alpha_1^2\beta_3^2\gamma_2\gamma_5\alpha_0^2 - 2\alpha_2^2\beta_0\beta_2\gamma_2\gamma_5\alpha_0^2 + 4\alpha_1\alpha_2\beta_0\beta_4\gamma_2\gamma_5\alpha_0^2\\
&-2\alpha_1^2\beta_0\beta_5\gamma_2\gamma_5\alpha_0^2 + 3\alpha_1\alpha_2\beta_1\beta_2\gamma_3\gamma_5\alpha_0^2 - \alpha_1^2\beta_2\beta_3\gamma_3\gamma_5\alpha_0^2 - \alpha_1^2\beta_1\beta_4\gamma_3\gamma_5\alpha_0^2 - \alpha_1\alpha_2\beta_1^2\gamma_4\gamma_5\alpha_0^2\\
&-2\alpha_1\alpha_2\beta_0\beta_2\gamma_4\gamma_5\alpha_0^2 - \alpha_1^2\beta_1\beta_3\gamma_4\gamma_5\alpha_0^2 - \alpha_1^2\beta_0\beta_4\gamma_4\gamma_5\alpha_0^2 - \alpha_2^3\beta_2\beta_3\gamma_1^2\alpha_0 + \alpha_1\alpha_2^2\beta_3\beta_4\gamma_1^2\alpha_0\\
&-\alpha_1^2\alpha_2\beta_3\beta_5\gamma_1^2\alpha_0 - 2\alpha_2^3\beta_0\beta_3\gamma_2^2\alpha_0 - \alpha_1\alpha_2^2\beta_1\beta_2\gamma_3^2\alpha_0 + \alpha_1^2\alpha_2\beta_1\beta_4\gamma_3^2\alpha_0 - \alpha_1^3\beta_1\beta_5\gamma_3^2\alpha_0 - \alpha_1\alpha_2^2\beta_0\beta_1\gamma_4^2\alpha_0\\
&-\alpha_1^2\alpha_2\beta_0\beta_3\gamma_4^2\alpha_0 - 2\alpha_1^3\beta_0\beta_1\gamma_5^2\alpha_0 - \alpha_1\alpha_2^2\beta_4^2\gamma_0\gamma_1\alpha_0 - 2\alpha_1^3\beta_5^2\gamma_0\gamma_1\alpha_0 + \alpha_2^3\beta_2\beta_4\gamma_0\gamma_1\alpha_0 - 2\alpha_1\alpha_2^2\beta_2\beta_5\gamma_0\gamma_1\alpha_0\\
&+3\alpha_1^2\alpha_2\beta_4\beta_5\gamma_0\gamma_1\alpha_0 + 2\alpha_2^3\beta_2\beta_3\gamma_0\gamma_2\alpha_0 - 2\alpha_2^3\beta_1\beta_4\gamma_0\gamma_2\alpha_0 + 4\alpha_1\alpha_2^2\beta_1\beta_5\gamma_0\gamma_2\alpha_0 - 2\alpha_1^2\alpha_2\beta_3\beta_5\gamma_0\gamma_2\alpha_0\\
&-\alpha_1\alpha_2^2\beta_3^2\gamma_1\gamma_2\alpha_0 + \alpha_2^3\beta_1\beta_3\gamma_1\gamma_2\alpha_0 + \alpha_2^3\beta_0\beta_4\gamma_1\gamma_2\alpha_0 - 2\alpha_1\alpha_2^2\beta_0\beta_5\gamma_1\gamma_2\alpha_0 - 2\alpha_2^3\beta_2^2\gamma_0\gamma_3\alpha_0
\end{aligned}
$$

Answer: $\mathbf{Res}_t(f_0, f_1, f_2) =$

$$\beta_5^2\gamma_2^2\alpha_0^4 + \beta_2\beta_5\gamma_4^2\alpha_0^4 + \beta_2^2\gamma_5^2\alpha_0^4 - \beta_4\beta_5\gamma_2\gamma_4\alpha_0^4 + \beta_4^2\gamma_2\gamma_5\alpha_0^4 - 2\beta_2\beta_5\gamma_2\gamma_5\alpha_0^4 - \beta_2\beta_4\gamma_4\gamma_5\alpha_0^4 - 2\alpha_2\beta_3\beta_5\gamma_2^2\alpha_0^3$$
$$-\alpha_2\beta_2\beta_3\gamma_4^2\alpha_0^3 - \alpha_1\beta_1\beta_5\gamma_4^2\alpha_0^3 - 2\alpha_1\beta_1\beta_2\gamma_5^2\alpha_0^3 - 2\alpha_1\beta_5^2\gamma_1\gamma_2\alpha_0^3 + \alpha_2\beta_4\beta_5\gamma_1\gamma_2\alpha_0^3 - \alpha_2\beta_4^2\gamma_2\gamma_3\alpha_0^3$$
$$+2\alpha_2\beta_2\beta_5\gamma_2\gamma_3\alpha_0^3 + \alpha_1\beta_4\beta_5\gamma_2\gamma_3\alpha_0^3 - 2\alpha_2\beta_2\beta_5\gamma_1\gamma_4\alpha_0^3 + \alpha_1\beta_4\beta_5\gamma_1\gamma_4\alpha_0^3 + \alpha_2\beta_3\beta_4\gamma_2\gamma_4\alpha_0^3 + \alpha_2\beta_1\beta_5\gamma_2\gamma_4\alpha_0^3$$
$$+\alpha_1\beta_3\beta_5\gamma_2\gamma_4\alpha_0^3 + \alpha_2\beta_2\beta_4\gamma_3\gamma_4\alpha_0^3 - 2\alpha_1\beta_2\beta_5\gamma_3\gamma_4\alpha_0^3 - \alpha_1\beta_4^2\gamma_1\gamma_5\alpha_0^3 + \alpha_2\beta_2\beta_4\gamma_1\gamma_5\alpha_0^3 + 2\alpha_1\beta_2\beta_5\gamma_1\gamma_5\alpha_0^3$$
$$+2\alpha_2\beta_2\beta_3\gamma_2\gamma_5\alpha_0^3 - 2\alpha_2\beta_1\beta_4\gamma_2\gamma_5\alpha_0^3 - 2\alpha_1\beta_3\beta_4\gamma_2\gamma_5\alpha_0^3 + 2\alpha_1\beta_1\beta_5\gamma_2\gamma_5\alpha_0^3 - 2\alpha_2\beta_2^2\gamma_3\gamma_5\alpha_0^3 + \alpha_1\beta_2\beta_4\gamma_3\gamma_5\alpha_0^3$$
$$+\alpha_2\beta_1\beta_2\gamma_4\gamma_5\alpha_0^3 + \alpha_1\beta_2\beta_3\gamma_4\gamma_5\alpha_0^3 + \alpha_1\beta_1\beta_4\gamma_4\gamma_5\alpha_0^3 + \alpha_1^2\beta_5^2\gamma_1^2\alpha_0^2 + \alpha_2^2\beta_2\beta_5\gamma_1^2\alpha_0^2 - \alpha_1\alpha_2\beta_4\beta_5\gamma_1^2\alpha_0^2$$
$$+\alpha_2^2\beta_3^2\gamma_2^2\alpha_0^2 + 2\alpha_2^2\beta_0\beta_5\gamma_2^2\alpha_0^2 + \alpha_2^2\beta_2^2\gamma_3^2\alpha_0^2 - \alpha_1\alpha_2\beta_2\beta_4\gamma_3^2\alpha_0^2 + \alpha_1^2\beta_2\beta_5\gamma_3^2\alpha_0^2 + \alpha_2^2\beta_0\beta_2\gamma_4^2\alpha_0^2 + \alpha_1\alpha_2\beta_1\beta_3\gamma_4^2\alpha_0^2$$
$$+\alpha_1^2\beta_0\beta_5\gamma_4^2\alpha_0^2 + \alpha_1^2\beta_1^2\gamma_5^2\alpha_0^2 + 2\alpha_1^2\beta_0\beta_2\gamma_5^2\alpha_0^2 + \alpha_2^2\beta_4^2\gamma_0\gamma_2\alpha_0^2 + 2\alpha_1^2\beta_5^2\gamma_0\gamma_2\alpha_0^2 - 2\alpha_2^2\beta_2\beta_5\gamma_0\gamma_2\alpha_0^2$$
$$-2\alpha_1\alpha_2\beta_4\beta_5\gamma_0\gamma_2\alpha_0^2 - \alpha_2^2\beta_3\beta_4\gamma_1\gamma_2\alpha_0^2 - \alpha_2^2\beta_1\beta_5\gamma_1\gamma_2\alpha_0^2 + 3\alpha_1\alpha_2\beta_3\beta_5\gamma_1\gamma_2\alpha_0^2 + \alpha_1\alpha_2\beta_4^2\gamma_1\gamma_3\alpha_0^2$$
$$-\alpha_2^2\beta_2\beta_4\gamma_1\gamma_3\alpha_0^2 - \alpha_1^2\beta_4\beta_5\gamma_1\gamma_3\alpha_0^2 - 2\alpha_2^2\beta_2\beta_3\gamma_2\gamma_3\alpha_0^2 + 2\alpha_2^2\beta_1\beta_4\gamma_2\gamma_3\alpha_0^2 + \alpha_1\alpha_2\beta_3\beta_4\gamma_2\gamma_3\alpha_0^2$$
$$-3\alpha_1\alpha_2\beta_1\beta_5\gamma_2\gamma_3\alpha_0^2 - \alpha_1^2\beta_3\beta_5\gamma_2\gamma_3\alpha_0^2 - \alpha_2^2\beta_2\beta_4\gamma_0\gamma_4\alpha_0^2 + 4\alpha_1\alpha_2\beta_2\beta_5\gamma_0\gamma_4\alpha_0^2 - \alpha_1^2\beta_4\beta_5\gamma_0\gamma_4\alpha_0^2$$
$$+2\alpha_2^2\beta_2\beta_3\gamma_1\gamma_4\alpha_0^2 - \alpha_1\alpha_2\beta_3\beta_4\gamma_1\gamma_4\alpha_0^2 + \alpha_1\alpha_2\beta_1\beta_5\gamma_1\gamma_4\alpha_0^2 - \alpha_1^2\beta_3\beta_5\gamma_1\gamma_4\alpha_0^2 - \alpha_1\alpha_2\beta_3^2\gamma_2\gamma_4\alpha_0^2$$
$$-\alpha_2^2\beta_1\beta_3\gamma_2\gamma_4\alpha_0^2 - \alpha_2^2\beta_0\beta_4\gamma_2\gamma_4\alpha_0^2 - 2\alpha_1\alpha_2\beta_0\beta_5\gamma_2\gamma_4\alpha_0^2 - \alpha_2^2\beta_1\beta_2\gamma_3\gamma_4\alpha_0^2 + \alpha_1\alpha_2\beta_2\beta_3\gamma_3\gamma_4\alpha_0^2$$
$$-\alpha_1\alpha_2\beta_1\beta_4\gamma_3\gamma_4\alpha_0^2 + 2\alpha_1^2\beta_1\beta_5\gamma_3\gamma_4\alpha_0^2 + 2\alpha_2^2\beta_2^2\gamma_0\gamma_5\alpha_0^2 + \alpha_1^2\beta_4^2\gamma_0\gamma_5\alpha_0^2 - 2\alpha_1\alpha_2\beta_2\beta_4\gamma_0\gamma_5\alpha_0^2$$
$$-2\alpha_1^2\beta_2\beta_5\gamma_0\gamma_5\alpha_0^2 - \alpha_2^2\beta_1\beta_2\gamma_1\gamma_5\alpha_0^2 - 3\alpha_1\alpha_2\beta_2\beta_3\gamma_1\gamma_5\alpha_0^2 + \alpha_1\alpha_2\beta_1\beta_4\gamma_1\gamma_5\alpha_0^2 + 2\alpha_1^2\beta_3\beta_4\gamma_1\gamma_5\alpha_0^2$$
$$-2\alpha_1^2\beta_1\beta_5\gamma_1\gamma_5\alpha_0^2 + \alpha_2^2\beta_1^2\gamma_2\gamma_5\alpha_0^2 + \alpha_1^2\beta_3^2\gamma_2\gamma_5\alpha_0^2 - 2\alpha_2^2\beta_0\beta_2\gamma_2\gamma_5\alpha_0^2 + 4\alpha_1\alpha_2\beta_0\beta_4\gamma_2\gamma_5\alpha_0^2$$
$$-2\alpha_1^2\beta_0\beta_5\gamma_2\gamma_5\alpha_0^2 + 3\alpha_1\alpha_2\beta_1\beta_2\gamma_3\gamma_5\alpha_0^2 - \alpha_1^2\beta_2\beta_3\gamma_3\gamma_5\alpha_0^2 - \alpha_1^2\beta_1\beta_4\gamma_3\gamma_5\alpha_0^2 - \alpha_1\alpha_2\beta_1^2\gamma_4\gamma_5\alpha_0^2$$
$$-2\alpha_1\alpha_2\beta_0\beta_2\gamma_4\gamma_5\alpha_0^2 - \alpha_1^2\beta_1\beta_3\gamma_4\gamma_5\alpha_0^2 - \alpha_1^2\beta_0\beta_4\gamma_4\gamma_5\alpha_0^2 - \alpha_2^3\beta_2\beta_3\gamma_1^2\alpha_0 + \alpha_1\alpha_2^2\beta_3\beta_4\gamma_1^2\alpha_0$$
$$-\alpha_1^2\alpha_2\beta_3\beta_5\gamma_1^2\alpha_0 - 2\alpha_2^3\beta_0\beta_3\gamma_2^2\alpha_0 - \alpha_1\alpha_2^2\beta_1\beta_2\gamma_3^2\alpha_0 + \alpha_1^2\alpha_2\beta_1\beta_4\gamma_3^2\alpha_0 - \alpha_1^3\beta_1\beta_5\gamma_3^2\alpha_0 - \alpha_1\alpha_2^2\beta_0\beta_1\gamma_4^2\alpha_0$$
$$-\alpha_1^2\alpha_2\beta_0\beta_3\gamma_4^2\alpha_0 - 2\alpha_1^3\beta_0\beta_1\gamma_5^2\alpha_0 - \alpha_1\alpha_2^2\beta_4^2\gamma_0\gamma_1\alpha_0 - 2\alpha_1^3\beta_5^2\gamma_0\gamma_1\alpha_0 + \alpha_2^3\beta_2\beta_4\gamma_0\gamma_1\alpha_0 - 2\alpha_1\alpha_2^2\beta_2\beta_5\gamma_0\gamma_1\alpha_0$$
$$+3\alpha_1^2\alpha_2\beta_4\beta_5\gamma_0\gamma_1\alpha_0 + 2\alpha_2^3\beta_2\beta_3\gamma_0\gamma_2\alpha_0 - 2\alpha_2^3\beta_1\beta_4\gamma_0\gamma_2\alpha_0 + 4\alpha_1\alpha_2^2\beta_1\beta_5\gamma_0\gamma_2\alpha_0 - 2\alpha_1^2\alpha_2\beta_3\beta_5\gamma_0\gamma_2\alpha_0$$
$$-\alpha_1\alpha_2^2\beta_3^2\gamma_1\gamma_2\alpha_0 + \alpha_2^3\beta_1\beta_3\gamma_1\gamma_2\alpha_0 + \alpha_2^3\beta_0\beta_4\gamma_1\gamma_2\alpha_0 - 2\alpha_1\alpha_2^2\beta_0\beta_5\gamma_1\gamma_2\alpha_0 - 2\alpha_2^3\beta_2^2\gamma_0\gamma_3\alpha_0 \ldots$$

$$
\begin{aligned}
&-\alpha_1^2\alpha_2\beta_4^2\gamma_0\gamma_3\alpha_0 + 3\alpha_1\alpha_2^2\beta_2\beta_4\gamma_0\gamma_3\alpha_0 - 2\alpha_1^2\alpha_2\beta_2\beta_5\gamma_0\gamma_3\alpha_0 + \alpha_1^3\beta_4\beta_5\gamma_0\gamma_3\alpha_0 + \alpha_2^3\beta_1\beta_2\gamma_1\gamma_3\alpha_0 \\
&+\alpha_1\alpha_2^2\beta_2\beta_3\gamma_1\gamma_3\alpha_0 - \alpha_1\alpha_2^2\beta_1\beta_4\gamma_1\gamma_3\alpha_0 - \alpha_1^2\alpha_2\beta_3\beta_4\gamma_1\gamma_3\alpha_0 + \alpha_1^2\alpha_2\beta_1\beta_5\gamma_1\gamma_3\alpha_0 + \alpha_1^3\beta_3\beta_5\gamma_1\gamma_3\alpha_0 \\
&-\alpha_2^3\beta_1^2\gamma_2\gamma_3\alpha_0 + 2\alpha_2^3\beta_0\beta_2\gamma_2\gamma_3\alpha_0 + \alpha_1\alpha_2^2\beta_1\beta_3\gamma_2\gamma_3\alpha_0 - 3\alpha_1\alpha_2^2\beta_0\beta_4\gamma_2\gamma_3\alpha_0 + 4\alpha_1^2\alpha_2\beta_0\beta_5\gamma_2\gamma_3\alpha_0 \\
&+\alpha_2^3\beta_1\beta_2\gamma_0\gamma_4\alpha_0 - 3\alpha_1\alpha_2^2\beta_2\beta_3\gamma_0\gamma_4\alpha_0 + \alpha_1\alpha_2^2\beta_1\beta_4\gamma_0\gamma_4\alpha_0 + \alpha_1^2\alpha_2\beta_3\beta_4\gamma_0\gamma_4\alpha_0 - 3\alpha_1^2\alpha_2\beta_1\beta_5\gamma_0\gamma_4\alpha_0 \\
&+\alpha_1^3\beta_3\beta_5\gamma_0\gamma_4\alpha_0 + \alpha_1^2\alpha_2\beta_3^2\gamma_1\gamma_4\alpha_0 - 2\alpha_2^3\beta_0\beta_2\gamma_1\gamma_4\alpha_0 - \alpha_1\alpha_2^2\beta_1\beta_3\gamma_1\gamma_4\alpha_0 + \alpha_1\alpha_2^2\beta_0\beta_4\gamma_1\gamma_4\alpha_0 \\
&+\alpha_2^3\beta_0\beta_1\gamma_2\gamma_4\alpha_0 + 3\alpha_1\alpha_2^2\beta_0\beta_3\gamma_2\gamma_4\alpha_0 + \alpha_1\alpha_2^2\beta_1^2\gamma_3\gamma_4\alpha_0 - \alpha_1^2\alpha_2\beta_1\beta_3\gamma_3\gamma_4\alpha_0 + \alpha_1^2\alpha_2\beta_0\beta_4\gamma_3\gamma_4\alpha_0 \\
&-2\alpha_1^3\beta_0\beta_5\gamma_3\gamma_4\alpha_0 - 2\alpha_1\alpha_2^2\beta_1\beta_2\gamma_0\gamma_5\alpha_0 + 4\alpha_1^2\alpha_2\beta_2\beta_3\gamma_0\gamma_5\alpha_0 - 2\alpha_1^3\beta_3\beta_4\gamma_0\gamma_5\alpha_0 + 2\alpha_1^3\beta_1\beta_5\gamma_0\gamma_5\alpha_0 \\
&-\alpha_1^3\beta_3^2\gamma_1\gamma_5\alpha_0 + 4\alpha_1\alpha_2^2\beta_0\beta_2\gamma_1\gamma_5\alpha_0 + \alpha_1^2\alpha_2\beta_1\beta_3\gamma_1\gamma_5\alpha_0 - 3\alpha_1^2\alpha_2\beta_0\beta_4\gamma_1\gamma_5\alpha_0 + 2\alpha_1^3\beta_0\beta_5\gamma_1\gamma_5\alpha_0 \\
&-2\alpha_1\alpha_2^2\beta_0\beta_1\gamma_2\gamma_5\alpha_0 - 2\alpha_1^2\alpha_2\beta_0\beta_3\gamma_2\gamma_5\alpha_0 - \alpha_1^2\alpha_2\beta_1^2\gamma_3\gamma_5\alpha_0 - 2\alpha_1^2\alpha_2\beta_0\beta_2\gamma_3\gamma_5\alpha_0 + \alpha_1^3\beta_1\beta_3\gamma_3\gamma_5\alpha_0 \\
&+\alpha_1^3\beta_0\beta_4\gamma_3\gamma_5\alpha_0 + 3\alpha_1^2\alpha_2\beta_0\beta_1\gamma_4\gamma_5\alpha_0 + \alpha_1^3\beta_0\beta_3\gamma_4\gamma_5\alpha_0 + \alpha_2^4\beta_2^2\gamma_0^2 + \alpha_1^2\alpha_2^2\beta_4^2\gamma_0^2 + \alpha_1^4\beta_5^2\gamma_0^2 \\
&-2\alpha_1\alpha_2^3\beta_2\beta_4\gamma_0^2 + 2\alpha_1^2\alpha_2^2\beta_2\beta_5\gamma_0^2 - 2\alpha_1^3\alpha_2\beta_4\beta_5\gamma_0^2 + \alpha_2^4\beta_0\beta_2\gamma_1^2 - \alpha_1\alpha_2^3\beta_0\beta_4\gamma_1^2 + \alpha_1^2\alpha_2^2\beta_0\beta_5\gamma_1^2 + \alpha_2^4\beta_0^2\gamma_2^2 \\
&+\alpha_1^2\alpha_2^2\beta_0\beta_2\gamma_3^2 - \alpha_1^3\alpha_2\beta_0\beta_4\gamma_3^2 + \alpha_1^4\beta_0\beta_5\gamma_3^2 + \alpha_1^2\alpha_2^2\beta_0^2\gamma_4^2 + \alpha_1^4\beta_0^2\gamma_5^2 - \alpha_2^4\beta_1\beta_2\gamma_0\gamma_1 + \alpha_1\alpha_2^3\beta_2\beta_3\gamma_0\gamma_1 \\
&+\alpha_1\alpha_2^3\beta_1\beta_4\gamma_0\gamma_1 - \alpha_1^2\alpha_2^2\beta_3\beta_4\gamma_0\gamma_1 - \alpha_1^2\alpha_2^2\beta_1\beta_5\gamma_0\gamma_1 + \alpha_1^3\alpha_2\beta_3\beta_5\gamma_0\gamma_1 + \alpha_2^4\beta_1^2\gamma_0\gamma_2 + \alpha_1^2\alpha_2^2\beta_3^2\gamma_0\gamma_2 \\
&-2\alpha_2^4\beta_0\beta_2\gamma_0\gamma_2 - 2\alpha_1\alpha_2^3\beta_1\beta_3\gamma_0\gamma_2 + 2\alpha_1\alpha_2^3\beta_0\beta_4\gamma_0\gamma_2 - 2\alpha_1^2\alpha_2^2\beta_0\beta_5\gamma_0\gamma_2 - \alpha_2^4\beta_0\beta_1\gamma_1\gamma_2 + \alpha_1\alpha_2^3\beta_0\beta_3\gamma_1\gamma_2 \\
&+\alpha_1\alpha_2^3\beta_1\beta_2\gamma_0\gamma_3 - \alpha_1^2\alpha_2^2\beta_2\beta_3\gamma_0\gamma_3 - \alpha_1^2\alpha_2^2\beta_1\beta_4\gamma_0\gamma_3 + \alpha_1^3\alpha_2\beta_3\beta_4\gamma_0\gamma_3 + \alpha_1^3\alpha_2\beta_1\beta_5\gamma_0\gamma_3 - \alpha_1^4\beta_3\beta_5\gamma_0\gamma_3 \\
&-2\alpha_1\alpha_2^3\beta_0\beta_2\gamma_1\gamma_3 + 2\alpha_1^2\alpha_2^2\beta_0\beta_4\gamma_1\gamma_3 - 2\alpha_1^3\alpha_2\beta_0\beta_5\gamma_1\gamma_3 + \alpha_1\alpha_2^3\beta_0\beta_1\gamma_2\gamma_3 - \alpha_1^2\alpha_2^2\beta_0\beta_3\gamma_2\gamma_3 - \alpha_1\alpha_2^3\beta_1^2\gamma_0\gamma_4 \\
&-\alpha_1^3\alpha_2\beta_3^2\gamma_0\gamma_4 + 2\alpha_1\alpha_2^3\beta_0\beta_2\gamma_0\gamma_4 + 2\alpha_1^2\alpha_2^2\beta_1\beta_3\gamma_0\gamma_4 - 2\alpha_1^2\alpha_2^2\beta_0\beta_4\gamma_0\gamma_4 + 2\alpha_1^3\alpha_2\beta_0\beta_5\gamma_0\gamma_4 + \alpha_1\alpha_2^3\beta_0\beta_1\gamma_1\gamma_4 \\
&-\alpha_1^2\alpha_2^2\beta_0\beta_3\gamma_1\gamma_4 - 2\alpha_1\alpha_2^3\beta_0^2\gamma_2\gamma_4 - \alpha_1^2\alpha_2^2\beta_0\beta_1\gamma_3\gamma_4 + \alpha_1^3\alpha_2\beta_0\beta_3\gamma_3\gamma_4 + \alpha_1^2\alpha_2^2\beta_1^2\gamma_0\gamma_5 + \alpha_1^4\beta_3^2\gamma_0\gamma_5 \\
&-2\alpha_1^2\alpha_2^2\beta_0\beta_2\gamma_0\gamma_5 - 2\alpha_1^3\alpha_2\beta_1\beta_3\gamma_0\gamma_5 + 2\alpha_1^3\alpha_2\beta_0\beta_4\gamma_0\gamma_5 - 2\alpha_1^4\beta_0\beta_5\gamma_0\gamma_5 - \alpha_1^2\alpha_2^2\beta_0\beta_1\gamma_1\gamma_5 + \alpha_1^3\alpha_2\beta_0\beta_3\gamma_1\gamma_5 \\
&+2\alpha_1^2\alpha_2^2\beta_0^2\gamma_2\gamma_5 + \alpha_1^3\alpha_2\beta_0\beta_1\gamma_3\gamma_5 - \alpha_1^4\beta_0\beta_3\gamma_3\gamma_5 - 2\alpha_1^3\alpha_2\beta_0^2\gamma_4\gamma_5.
\end{aligned}
$$

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\begin{aligned}
&-\alpha_1^2\alpha_2\beta_4^2\gamma_0\gamma_3\alpha_0 + 3\alpha_1\alpha_2^2\beta_2\beta_4\gamma_0\gamma_3\alpha_0 - 2\alpha_1^2\alpha_2\beta_2\beta_5\gamma_0\gamma_3\alpha_0 + \alpha_1^3\beta_4\beta_5\gamma_0\gamma_3\alpha_0 + \alpha_2^3\beta_1\beta_2\gamma_1\gamma_3\alpha_0\\
&+\alpha_1\alpha_2^2\beta_2\beta_3\gamma_1\gamma_3\alpha_0 - \alpha_1\alpha_2^2\beta_1\beta_4\gamma_1\gamma_3\alpha_0 - \alpha_1^2\alpha_2\beta_3\beta_4\gamma_1\gamma_3\alpha_0 + \alpha_1^2\alpha_2\beta_1\beta_5\gamma_1\gamma_3\alpha_0 + \alpha_1^3\beta_3\beta_5\gamma_1\gamma_3\alpha_0\\
&-\alpha_2^3\beta_1^2\gamma_2\gamma_3\alpha_0 + 2\alpha_2^3\beta_0\beta_2\gamma_2\gamma_3\alpha_0 + \alpha_1\alpha_2^2\beta_1\beta_3\gamma_2\gamma_3\alpha_0 - 3\alpha_1\alpha_2^2\beta_0\beta_4\gamma_2\gamma_3\alpha_0 + 4\alpha_1^2\alpha_2\beta_0\beta_5\gamma_2\gamma_3\alpha_0\\
&+\alpha_2^3\beta_1\beta_2\gamma_0\gamma_4\alpha_0 - 3\alpha_1\alpha_2^2\beta_2\beta_3\gamma_0\gamma_4\alpha_0 + \alpha_1\alpha_2^2\beta_1\beta_4\gamma_0\gamma_4\alpha_0 + \alpha_1^2\alpha_2\beta_3\beta_4\gamma_0\gamma_4\alpha_0 - 3\alpha_1^2\alpha_2\beta_1\beta_5\gamma_0\gamma_4\alpha_0\\
&+\alpha_1^3\beta_3\beta_5\gamma_0\gamma_4\alpha_0 + \alpha_1^2\alpha_2\beta_3^2\gamma_1\gamma_4\alpha_0 - 2\alpha_2^3\beta_0\beta_2\gamma_1\gamma_4\alpha_0 - \alpha_1\alpha_2^2\beta_1\beta_3\gamma_1\gamma_4\alpha_0 + \alpha_1\alpha_2^2\beta_0\beta_4\gamma_1\gamma_4\alpha_0\\
&+\alpha_2^3\beta_0\beta_1\gamma_2\gamma_4\alpha_0 + 3\alpha_1\alpha_2^2\beta_0\beta_3\gamma_2\gamma_4\alpha_0 + \alpha_1\alpha_2^2\beta_1^2\gamma_3\gamma_4\alpha_0 - \alpha_1^2\alpha_2\beta_1\beta_3\gamma_3\gamma_4\alpha_0 + \alpha_1^2\alpha_2\beta_0\beta_4\gamma_3\gamma_4\alpha_0\\
&-2\alpha_1^3\beta_0\beta_5\gamma_3\gamma_4\alpha_0 - 2\alpha_1\alpha_2^2\beta_1\beta_2\gamma_0\gamma_5\alpha_0 + 4\alpha_1^2\alpha_2\beta_2\beta_3\gamma_0\gamma_5\alpha_0 - 2\alpha_1^3\beta_3\beta_4\gamma_0\gamma_5\alpha_0 + 2\alpha_1^3\beta_1\beta_5\gamma_0\gamma_5\alpha_0\\
&-\alpha_1^3\beta_3^2\gamma_1\gamma_5\alpha_0 + 4\alpha_1\alpha_2^2\beta_0\beta_2\gamma_1\gamma_5\alpha_0 + \alpha_1^2\alpha_2\beta_1\beta_3\gamma_1\gamma_5\alpha_0 - 3\alpha_1^2\alpha_2\beta_0\beta_4\gamma_1\gamma_5\alpha_0 + 2\alpha_1^3\beta_0\beta_5\gamma_1\gamma_5\alpha_0\\
&-2\alpha_1\alpha_2^2\beta_0\beta_1\gamma_2\gamma_5\alpha_0 - 2\alpha_1^2\alpha_2\beta_0\beta_3\gamma_2\gamma_5\alpha_0 - \alpha_1^2\alpha_2\beta_1^2\gamma_3\gamma_5\alpha_0 - 2\alpha_1^2\alpha_2\beta_0\beta_2\gamma_3\gamma_5\alpha_0 + \alpha_1^3\beta_1\beta_3\gamma_3\gamma_5\alpha_0\\
&+\alpha_1^3\beta_0\beta_4\gamma_3\gamma_5\alpha_0 + 3\alpha_1^2\alpha_2\beta_0\beta_1\gamma_4\gamma_5\alpha_0 + \alpha_1^3\beta_0\beta_3\gamma_4\gamma_5\alpha_0 + \alpha_2^4\beta_2^2\gamma_0^2 + \alpha_1^2\alpha_2^2\beta_4^2\gamma_0^2 + \alpha_1^4\beta_5^2\gamma_0^2\\
&-2\alpha_1\alpha_2^3\beta_2\beta_4\gamma_0^2 + 2\alpha_1^2\alpha_2^2\beta_2\beta_5\gamma_0^2 - 2\alpha_1^3\alpha_2\beta_4\beta_5\gamma_0^2 + \alpha_2^4\beta_0\beta_2\gamma_1^2 - \alpha_1\alpha_2^3\beta_0\beta_4\gamma_1^2 + \alpha_1^2\alpha_2^2\beta_0\beta_5\gamma_1^2 + \alpha_2^4\beta_0^2\gamma_2^2\\
&+\alpha_1^2\alpha_2^2\beta_0\beta_2\gamma_3^2 - \alpha_1^3\alpha_2\beta_0\beta_4\gamma_3^2 + \alpha_1^4\beta_0\beta_5\gamma_3^2 + \alpha_1^2\alpha_2^2\beta_0^2\gamma_4^2 + \alpha_1^4\beta_0^2\gamma_5^2 - \alpha_2^4\beta_1\beta_2\gamma_0\gamma_1 + \alpha_1\alpha_2^3\beta_2\beta_3\gamma_0\gamma_1\\
&+\alpha_1\alpha_2^3\beta_1\beta_4\gamma_0\gamma_1 - \alpha_1^2\alpha_2^2\beta_3\beta_4\gamma_0\gamma_1 - \alpha_1^2\alpha_2^2\beta_1\beta_5\gamma_0\gamma_1 + \alpha_1^3\alpha_2\beta_3\beta_5\gamma_0\gamma_1 + \alpha_2^4\beta_1^2\gamma_0\gamma_2 + \alpha_1^2\alpha_2^2\beta_3^2\gamma_0\gamma_2\\
&-2\alpha_2^4\beta_0\beta_2\gamma_0\gamma_2 - 2\alpha_1\alpha_2^3\beta_1\beta_3\gamma_0\gamma_2 + 2\alpha_1\alpha_2^3\beta_0\beta_4\gamma_0\gamma_2 - 2\alpha_1^2\alpha_2^2\beta_0\beta_5\gamma_0\gamma_2 - \alpha_2^4\beta_0\beta_1\gamma_1\gamma_2 + \alpha_1\alpha_2^3\beta_0\beta_3\gamma_1\gamma_2\\
&+\alpha_1\alpha_2^3\beta_1\beta_2\gamma_0\gamma_3 - \alpha_1^2\alpha_2^2\beta_2\beta_3\gamma_0\gamma_3 - \alpha_1^2\alpha_2^2\beta_1\beta_4\gamma_0\gamma_3 + \alpha_1^3\alpha_2\beta_3\beta_4\gamma_0\gamma_3 + \alpha_1^3\alpha_2\beta_1\beta_5\gamma_0\gamma_3 - \alpha_1^4\beta_3\beta_5\gamma_0\gamma_3\\
&-2\alpha_1\alpha_2^3\beta_0\beta_2\gamma_1\gamma_3 + 2\alpha_1^2\alpha_2^2\beta_0\beta_4\gamma_1\gamma_3 - 2\alpha_1^3\alpha_2\beta_0\beta_5\gamma_1\gamma_3 + \alpha_1\alpha_2^3\beta_0\beta_1\gamma_2\gamma_3 - \alpha_1^2\alpha_2^2\beta_0\beta_3\gamma_2\gamma_3 - \alpha_1\alpha_2^3\beta_1^2\gamma_0\gamma_4\\
&-\alpha_1^3\alpha_2\beta_3^2\gamma_0\gamma_4 + 2\alpha_1\alpha_2^3\beta_0\beta_2\gamma_0\gamma_4 + 2\alpha_1^2\alpha_2^2\beta_1\beta_3\gamma_0\gamma_4 - 2\alpha_1^2\alpha_2^2\beta_0\beta_4\gamma_0\gamma_4 + 2\alpha_1^3\alpha_2\beta_0\beta_5\gamma_0\gamma_4 + \alpha_1\alpha_2^3\beta_0\beta_1\gamma_1\gamma_4\\
&-\alpha_1^2\alpha_2^2\beta_0\beta_3\gamma_1\gamma_4 - 2\alpha_1\alpha_2^3\beta_0^2\gamma_2\gamma_4 - \alpha_1^2\alpha_2^2\beta_0\beta_1\gamma_3\gamma_4 + \alpha_1^3\alpha_2\beta_0\beta_3\gamma_3\gamma_4 + \alpha_1^2\alpha_2^2\beta_1^2\gamma_0\gamma_5 + \alpha_1^4\beta_3^2\gamma_0\gamma_5\\
&-2\alpha_1^2\alpha_2^2\beta_0\beta_2\gamma_0\gamma_5 - 2\alpha_1^3\alpha_2\beta_1\beta_3\gamma_0\gamma_5 + 2\alpha_1^3\alpha_2\beta_0\beta_4\gamma_0\gamma_5 - 2\alpha_1^4\beta_0\beta_5\gamma_0\gamma_5 - \alpha_1^2\alpha_2^2\beta_0\beta_1\gamma_1\gamma_5 + \alpha_1^3\alpha_2\beta_0\beta_3\gamma_1\gamma_5\\
&+2\alpha_1^2\alpha_2^2\beta_0^2\gamma_2\gamma_5 + \alpha_1^3\alpha_2\beta_0\beta_1\gamma_3\gamma_5 - \alpha_1^4\beta_0\beta_3\gamma_3\gamma_5 - 2\alpha_1^3\alpha_2\beta_0^2\gamma_4\gamma_5.
\end{aligned}
$$

Definition of $\mathbf{Res}_t(f_0, f_1, f_2)$?

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Definition of **Res**$_t(f_0, f_1, f_2)$?

Is it a determinant?

If not

Definition of $\mathbf{Res}_t(f_0, f_1, f_2)$?

Is it a determinant?

If not, how do you define it?

Back to the classics...

$$\begin{cases} f_0 = \alpha_0 t_0^{d_0} + \alpha_1 t_0^{d_0-1} t_1 + \ldots + \alpha_{d_0} t_1^{d_0} \\ f_1 = \beta_0 t_0^{d_1} + \beta_1 t_0^{d_1-1} t_1 + \ldots + \beta_{d_1} t_1^{d_1} \end{cases}$$

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What is $\mathrm{Res}_t(f_0, f_1)$

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What is $\mathrm{Res}_t(f_0, f_1)$ (or Res_{d_0,d_1}) the resultant of f_0 and f_1?

Algebraic Definitions

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Algebraic Definitions

$\text{Res}_t(f_0, f_1)$ is:

- the "condition" to test when $\gcd(f_0, f_1) = 1$ (Sylvester)
- One of the (two) generators of the elimination ideal

$$\langle f_0(1, t_1), \; f_1(1, t_1)\rangle \cap \mathbb{Z}[\alpha_0, \ldots, \alpha_{d_0}, \; \beta_0, \ldots, \beta_{d_1}]$$

Geometry

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$$V = \{(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}; \xi_0 : \xi_1) : f_0(\tilde{\boldsymbol{\alpha}}, \xi_0, \xi_1) = f_1(\tilde{\boldsymbol{\beta}}, \xi_0, \xi_1) = 0\}$$

Geometry

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$$V=\{(\tilde{\alpha},\tilde{\beta};\xi_0:\xi_1):f_0(\tilde{\alpha},\xi_0,\xi_1)=f_1(\tilde{\beta},\xi_0,\xi_1)=0\}$$

$$\begin{array}{ccc} V & \subset & \mathbb{K}^{d_0+d_1+2}\times\mathbb{P}^1_{\mathbb{K}} \\ \pi_1|_V\downarrow & & \downarrow\pi_1 \\ \pi_1(V) & \subset & \mathbb{K}^{d_1+d_2+2} \end{array}$$

Geometry

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$$\{\mathrm{Res}_{d_0,d_1} = 0\} = \pi_1(V)$$

Computation

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must fail to be an isomorphism

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$$\mathrm{Res}_{d_0,d_1} = \det(\phi)$$

d-Sylvester matrices

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must fail to be **onto**.

d-Sylvester matrices

For $\tilde{f}_0, \tilde{f}_1 \in \mathbb{K}[t_0, t_1]$ to have a nontrivial common factor and any $d \geq d_0 + d_1 - 1$, the map

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must fail to be **onto**.

The d-Sylvester matrix is a maximal submatrix of ϕ_d

Back to multivariate Resultants

$$\begin{cases} f_0 = \sum_{|\mathbf{a}|=d_0} \alpha_{1,\mathbf{a}} t_0^{a_0} \dots t_n^{a_n} \\ f_1 = \sum_{|\mathbf{a}|=d_1} \alpha_{2,\mathbf{a}} t_0^{a_0} \dots t_n^{a_n} \\ \vdots \\ f_n = \sum_{|\mathbf{a}|=d_n} \alpha_{n,\mathbf{a}} t_0^{a_0} \dots t_n^{a_n} \end{cases}$$

Back to multivariate Resultants

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What is $\mathrm{Res}_t(f_0, \dots, f_n) / \mathrm{Res}_{d_0,\dots,d_n}$?

Answer

- A generator of

$$\langle f_0(1,\mathbf{t}),\ldots,f_n(1,\mathbf{t})\rangle \cap \mathbb{Z}[\alpha_{i,\boldsymbol{a}},\ i=0,\ldots,n]$$

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$$\begin{array}{ccc} V=\{(u_{i,\boldsymbol{a}},\xi):\ f_i(u_{i,\boldsymbol{a}},\xi)=0\} & \subset & \mathbb{K}^N\times\mathbb{P}^n \\ \downarrow & & \downarrow \pi_1 \\ \pi_1(V) & \subset & \mathbb{K}^N \end{array}$$

Answer

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$$\langle f_0(1,\mathbf{t}),\ldots,f_n(1,\mathbf{t})\rangle \cap \mathbb{Z}[\alpha_{i,\boldsymbol{a}},\ i=0,\ldots,n]$$

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- A condition for the f_i's to have a **common zero** in $\mathbb{P}^{n-1}_{\mathbb{K}}$

$\mathbf{Res}_{d_0,\ldots,d_n} = \det(\phi_d)$?

$\mathbf{Res}_{d_0,\ldots,d_n} = \det(\phi_d)$?

- Sylvester,

$\mathbf{Res}_{d_0,\ldots,d_n} = \det(\phi_d)$?

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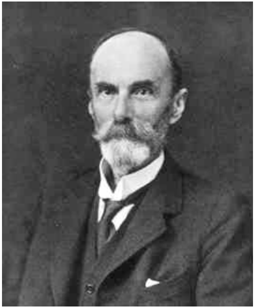
- Zariski, Jouanolou, Kapranov, Sturmfels, Zelevinski

$\mathbf{Res}_{d_0,\ldots,d_n} = \det(\phi_d)$?

- Sylvester, Bézout, Cayley, Dixon, Macaulay... (1800)
- Zariski, Jouanolou, Kapranov, Sturmfels, Zelevinski, ... (1900)

So far Macaulay's formulation (1902) is the only one that works for **any** data $(d_0, \ldots, d_n)$



Macaulay's formula (1902)

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$$
\begin{aligned}
f_0 &= \alpha_0 t_0 + \alpha_1 t_1 + \alpha_2 t_2 \\
f_1 &= \beta_0 t_0^2 + \beta_1 t_0 t_1 + \beta_2 t_1^2 + \beta_3 t_0 t_2 + \beta_4 t_1 t_2 + \beta_5 t_2^2 \\
f_2 &= \gamma_0 t_0^2 + \gamma_1 t_0 t_1 + \gamma_2 t_1^2 + \gamma_3 t_0 t_2 + \gamma_4 t_1 t_2 + \gamma_5 t_2^2
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Any monomial $t_0^a t_1^b t_2^c$ of degree 3 or more is either

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Macaulay Matrices

Macaulay Matrices

$$t_0^3 \quad t_0^2\,t_1 \quad t_0^2\,t_2 \quad t_0\,t_1\,t_2 \quad t_0\,t_1^2 \quad t_0\,t_2^2 \quad t_1^3 \quad t_1^2\,t_2 \quad t_1\,t_2^2 \quad t_2^3$$

$$\left[\phantom{\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}}\right.$$

Macaulay Matrices

	t_0^3	$t_0^2\, t_1$	$t_0^2\, t_2$	$t_0\, t_1\, t_2$	$t_0\, t_1^2$	$t_0\, t_2^2$	t_1^3	$t_1^2\, t_2$	$t_1\, t_2^2$	t_2^3
$t_0^2 f_0$	α_0	α_1	α_2	0	0	0	0	0	0	0

Macaulay Matrices

	t_0^3	$t_0^2\,t_1$	$t_0^2\,t_2$	$t_0\,t_1\,t_2$	$t_0\,t_1^2$	$t_0\,t_2^2$	t_1^3	$t_1^2\,t_2$	$t_1\,t_2^2$	t_2^3
$t_0^2 f_0$	α_0	α_1	α_2	0	0	0	0	0	0	0
$t_0\,t_1 f_0$	0	α_0	0	α_2	α_1	0	0	0	0	0

Macaulay Matrices

	t_0^3	$t_0^2\ t_1$	$t_0^2\ t_2$	$t_0\ t_1\ t_2$	$t_0\ t_1^2$	$t_0\ t_2^2$	t_1^3	$t_1^2\ t_2$	$t_1\ t_2^2$	t_2^3
$t_0^2 f_0$	α_0	α_1	α_2	0	0	0	0	0	0	0
$t_0\ t_1 f_0$	0	α_0	0	α_2	α_1	0	0	0	0	0
$t_0\ t_2 f_0$	0	0	α_0	α_1	0	α_2	0	0	0	0

Macaulay Matrices

	t_0^3	$t_0^2 t_1$	$t_0^2 t_2$	$t_0 t_1 t_2$	$t_0 t_1^2$	$t_0 t_2^2$	t_1^3	$t_1^2 t_2$	$t_1 t_2^2$	t_2^3
$t_0^2 f_0$	α_0	α_1	α_2	0	0	0	0	0	0	0
$t_0 t_1 f_0$	0	α_0	0	α_2	α_1	0	0	0	0	0
$t_0 t_2 f_0$	0	0	α_0	α_1	0	α_2	0	0	0	0
$t_1 t_2 f_0$	0	0	0	α_0	0	0	α_1	0	α_2	0

Macaulay Matrices

	t_0^3	$t_0^2\, t_1$	$t_0^2\, t_2$	$t_0\, t_1\, t_2$	$t_0\, t_1^2$	$t_0\, t_2^2$	t_1^3	$t_1^2\, t_2$	$t_1\, t_2^2$	t_2^3
$t_0^2 f_0$	α_0	α_1	α_2	0	0	0	0	0	0	0
$t_0\, t_1 f_0$	0	α_0	0	α_2	α_1	0	0	0	0	0
$t_0\, t_2 f_0$	0	0	α_0	α_1	0	α_2	0	0	0	0
$t_1\, t_2 f_0$	0	0	0	α_0	0	0	α_1	0	α_2	0
$t_1^2 f_0$	0	0	0	0	α_0	0	α_2	α_1	0	0

Macaulay Matrices

	t_0^3	$t_0^2\ t_1$	$t_0^2\ t_2$	$t_0\ t_1\ t_2$	$t_0\ t_1^2$	$t_0\ t_2^2$	t_1^3	$t_1^2\ t_2$	$t_1\ t_2^2$	t_2^3
$t_0^2 f_0$	α_0	α_1	α_2	0	0	0	0	0	0	0
$t_0\ t_1 f_0$	0	α_0	0	α_2	α_1	0	0	0	0	0
$t_0\ t_2 f_0$	0	0	α_0	α_1	0	α_2	0	0	0	0
$t_1\ t_2 f_0$	0	0	0	α_0	0	0	α_1	0	α_2	0
$t_1^2 f_0$	0	0	0	0	α_0	0	α_2	α_1	0	0
$t_2^2 f_0$	0	0	0	0	0	α_0	0	0	α_1	α_2

Macaulay Matrices

	t_0^3	$t_0^2\,t_1$	$t_0^2\,t_2$	$t_0\,t_1\,t_2$	$t_0\,t_1^2$	$t_0\,t_2^2$	t_1^3	$t_1^2\,t_2$	$t_1\,t_2^2$	t_2^3
$t_0^2 f_0$	α_0	α_1	α_2	0	0	0	0	0	0	0
$t_0\,t_1 f_0$	0	α_0	0	α_2	α_1	0	0	0	0	0
$t_0\,t_2 f_0$	0	0	α_0	α_1	0	α_2	0	0	0	0
$t_1\,t_2 f_0$	0	0	0	α_0	0	0	α_1	0	α_2	0
$t_1^2 f_0$	0	0	0	0	α_0	0	α_2	α_1	0	0
$t_2^2 f_0$	0	0	0	0	0	α_0	0	0	α_1	α_2
$t_1 f_1$	0	β_0	0	β_3	β_1	0	β_2	β_4	β_5	0
$t_2 f_1$	0	0	β_0	β_1	0	β_3	0	β_2	β_4	β_5
$t_1 f_2$	0	γ_0	0	γ_3	γ_1	0	γ_2	γ_4	γ_5	0
$t_2 f_2$	0	0	γ_0	γ_1	0	γ_3	0	γ_2	γ_4	γ_5

Macaulay Matrices

	t_0^3	$t_0^2\,t_1$	$t_0^2\,t_2$	$t_0\,t_1\,t_2$	$t_0\,t_1^2$	$t_0\,t_2^2$	t_1^3	$t_1^2\,t_2$	$t_1\,t_2^2$	t_2^3
$t_0^2 f_0$	α_0	α_1	α_2	0	0	0	0	0	0	0
$t_0\,t_1 f_0$	0	α_0	0	α_2	α_1	0	0	0	0	0
$t_0\,t_2 f_0$	0	0	α_0	α_1	0	α_2	0	0	0	0
$t_1\,t_2 f_0$	0	0	0	α_0	0	0	α_1	0	α_2	0
$t_1^2 f_0$	0	0	0	0	α_0	0	α_2	α_1	0	0
$t_2^2 f_0$	0	0	0	0	0	α_0	0	0	α_1	α_2
$t_1 f_1$	0	β_0	0	β_3	β_1	0	β_2	β_4	β_5	0
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$t_1 f_2$	0	γ_0	0	γ_3	γ_1	0	γ_2	γ_4	γ_5	0
$t_2 f_2$	0	0	γ_0	γ_1	0	γ_3	0	γ_2	γ_4	γ_5

2 matrices:

Macaulay Matrices

	t_0^3	$t_0^2\, t_1$	$t_0^2\, t_2$	$t_0\, t_1\, t_2$	$t_0\, t_1^2$	$t_0\, t_2^2$	t_1^3	$t_1^2\, t_2$	$t_1\, t_2^2$	t_2^3
$t_0^2 f_0$	α_0	α_1	α_2	0	0	0	0	0	0	0
$t_0\, t_1 f_0$	0	α_0	0	α_2	α_1	0	0	0	0	0
$t_0\, t_2 f_0$	0	0	α_0	α_1	0	α_2	0	0	0	0
$t_1\, t_2 f_0$	0	0	0	α_0	0	0	α_1	0	α_2	0
$t_1^2 f_0$	0	0	0	0	α_0	0	α_2	α_1	0	0
$t_2^2 f_0$	0	0	0	0	0	α_0	0	0	α_1	α_2
$t_1 f_1$	0	β_0	0	β_3	β_1	0	β_2	β_4	β_5	0
$t_2 f_1$	0	0	β_0	β_1	0	β_3	0	β_2	β_4	β_5
$t_1 f_2$	0	γ_0	0	γ_3	γ_1	0	γ_2	γ_4	γ_5	0
$t_2 f_2$	0	0	γ_0	γ_1	0	γ_3	0	γ_2	γ_4	γ_5

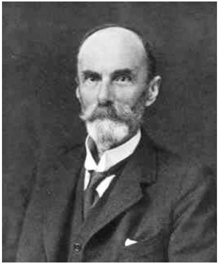
2 matrices: $\mathbb{M}^3_{1,2,2}$ of size 10×10,

Macaulay Matrices

$$
\begin{array}{c|cccccccccc}
 & t_0^3 & t_0^2\,t_1 & t_0^2\,t_2 & t_0\,t_1\,t_2 & t_0\,t_1^2 & t_0\,t_2^2 & t_1^3 & t_1^2\,t_2 & t_1\,t_2^2 & t_2^3 \\
t_0^2 f_0 & \alpha_0 & \alpha_1 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t_0\,t_1 f_0 & 0 & \alpha_0 & 0 & \alpha_2 & \alpha_1 & 0 & 0 & 0 & 0 & 0 \\
t_0\,t_2 f_0 & 0 & 0 & \alpha_0 & \alpha_1 & 0 & \alpha_2 & 0 & 0 & 0 & 0 \\
t_1\,t_2 f_0 & 0 & 0 & 0 & \alpha_0 & 0 & 0 & \alpha_1 & 0 & \alpha_2 & 0 \\
t_1^2 f_0 & 0 & 0 & 0 & 0 & \alpha_0 & 0 & \alpha_2 & \alpha_1 & 0 & 0 \\
t_2^2 f_0 & 0 & 0 & 0 & 0 & 0 & \alpha_0 & 0 & 0 & \alpha_1 & \alpha_2 \\
t_1 f_1 & 0 & \beta_0 & 0 & \beta_3 & \beta_1 & 0 & \beta_2 & \beta_4 & \beta_5 & 0 \\
t_2 f_1 & 0 & 0 & \beta_0 & \beta_1 & 0 & \beta_3 & 0 & \beta_2 & \beta_4 & \beta_5 \\
t_1 f_2 & 0 & \gamma_0 & 0 & \gamma_3 & \gamma_1 & 0 & \gamma_2 & \gamma_4 & \gamma_5 & 0 \\
t_2 f_2 & 0 & 0 & \gamma_0 & \gamma_1 & 0 & \gamma_3 & 0 & \gamma_2 & \gamma_4 & \gamma_5
\end{array}
$$

2 matrices: $\mathbb{M}^3_{1,2,2}$ of size 10×10, and $\mathbb{F}^3_{1,1,2}$ of size 2×2

Macaulay formula (1902)



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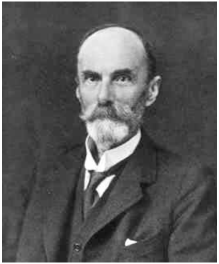

$$\mathrm{Res}_{d_0,\ldots,d_n} = \frac{\det(\mathbb{M}^d_{d_0,\ldots,d_n})}{\det(\mathbb{F}^d_{d_0,\ldots,d_n})}$$

Back to $n = 1$

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$$\begin{cases} f_0 = \alpha_0 t_0 + \alpha_1 t_1 \\ f_1 = \beta_0 t_0^2 + \beta_1 t_0 t_1 + \beta_2 t_1^2 \end{cases}$$

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$$\mathbb{M}_{1,2}^4 = \begin{array}{c} \\ t_0^3 f_0 \\ t_0^2 t_1 f_0 \\ t_0 t_1^2 f_0 \\ t_1^3 f_0 \\ t_1^2 f_1 \end{array} \begin{array}{c} \begin{array}{ccccc} t_0^4 & t_0^3 t_1 & t_0^2 t_1^2 & t_0 t_1^3 & t_1^4 \end{array} \\ \begin{bmatrix} \alpha_0 & \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_0 & \alpha_1 \\ 0 & 0 & \beta_0 & \beta_1 & \beta_2 \end{bmatrix} \end{array}$$

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$$\mathbb{F}_{1,2}^4 = \begin{array}{c} \\ t_0 t_1^2 f_0 \\ t_1^3 f_0 \end{array} \begin{array}{c} \begin{matrix} t_0^2 t_1^2 & t_0 t_1^3 \end{matrix} \\ \begin{bmatrix} \alpha_0 & \alpha_1 \\ 0 & \alpha_0 \end{bmatrix} \end{array}$$

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Given $d \geq d_0 + \ldots + d_n - n$, for the polynomials $\tilde{f}_0, \ldots, \tilde{f}_n \in \mathbb{K}[\mathbf{t}]$ to have a common zero in $\mathbb{P}^n_{\mathbb{K}}$

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- $\mathbb{M}^d_{d_0,\ldots,d_n}$ is a maximal submatrix of the matrix of ϕ_d
- $\implies \mathrm{Res}_{d_0,\ldots,d_n} \mid \det(\mathbb{M}^d_{d_0,\ldots,d_n})$

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you need a formulation **for any**

$$d \geq d_0 + \ldots + d_n - n$$

Sparse Resultants, 100 years later...

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$$\begin{cases} f_0 = \alpha_0 + \alpha_1 t_1^2\, t_2^2 + \alpha_2 t_1\, t_2^3 \\ f_1 = \beta_0 + \beta_1 t_1^2 + \beta_2 t_1\, t_2^2 \\ f_2 = \gamma_0 t_1^3 + \gamma_1 t_1\, t_2 \end{cases}$$

Sparse Resultants

- $\mathcal{A}_0, \ldots, \mathcal{A}_n \subset \mathbb{Z}^n$
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What is

$\mathrm{Res}(f_0, \ldots, f_n) = \mathrm{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n}$?

Geometric definition

Geometric definition

(**D**-Sombra), PLMS 2015

$$\begin{array}{ccc} V = \{(\alpha_{i,\mathbf{a}}, \xi) : \, f_i(\boldsymbol{\alpha}, \boldsymbol{\xi}) = 0 \, \forall i\} & \subset & \mathbb{K}^N \times (\mathbb{K}^\times)^n \\ \downarrow & & \downarrow \, \pi_1 \\ \pi_1(V) & \subset & \mathbb{K}^N \end{array}$$

$\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}$ is the defining equation of the direct image $\pi_{1*}V$

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  $\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2} = \det(\alpha_i, \beta_j, \gamma_k)$

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- Classically (algebraic definition):
  $\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2} = \det(\alpha_i, \beta_j, \gamma_k)$
- With the new definition,
  $\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2} = \det(\alpha_i, \beta_j, \gamma_k)^4$

Properties

Properties

(**D**-Sombra), PLMS 2015

- $\text{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}$ may not be irreducible

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  v finite integer vectors, $h_{\mathcal{A}_0}$ is the "lattice distance" to $\mathcal{A}_0$

Algebra meets Geometry

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(if $\pi_1(V)$ has codimension one)

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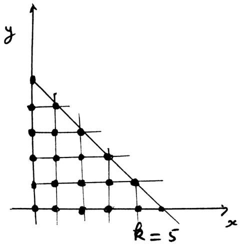
$X_{\mathcal{A}}$ is the **toric variety** being the image of

$$\begin{array}{ccc} (\mathbb{C}^\times)^n & \to & \mathbb{P}^{\#\mathcal{A}_0-1}\times\ldots\times\mathbb{P}^{\#\mathcal{A}_n-1} \\ \boldsymbol{\xi} & \mapsto & \left(\boldsymbol{\xi}^{\boldsymbol{a}_i}\right)_{i=0,\ldots,n,\boldsymbol{a}_i\in\mathcal{A}_i} \end{array}$$

Sparse generalizes classical



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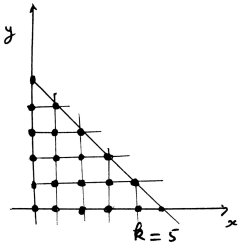

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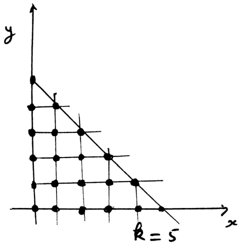

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then we have $\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n} = \mathrm{Res}_{d_0,\ldots,d_n}$

How do you compute it?

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- As a determinant?

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Canny-Emiris' construction

An Efficient Algorithm For The Sparse Resultant

Article · February 1995

Source: CiteSeer

2 authors:

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Ioannis Z Emiris
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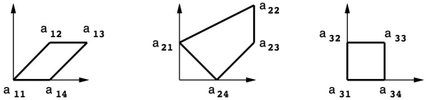
Generalize Macaulay matrices for computing $\mathrm{Res}_{d_0,\ldots,d_n}$

The method (pics. from that paper)



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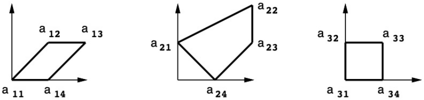

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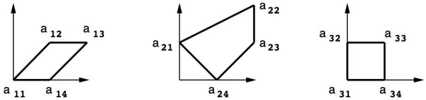

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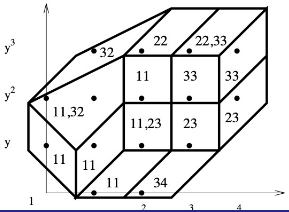
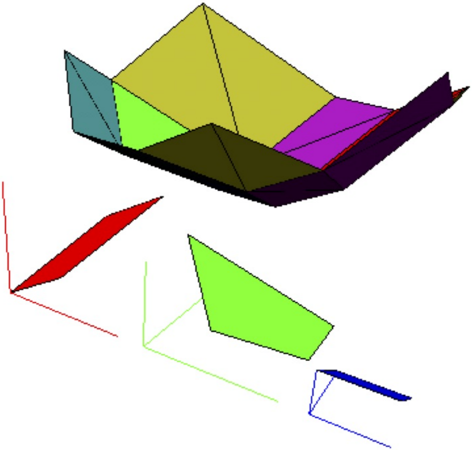


Subdivision via projection



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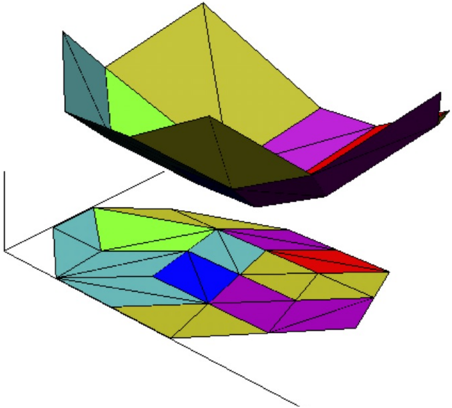
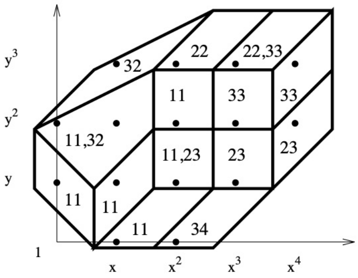


Rigid vs flexible monomials



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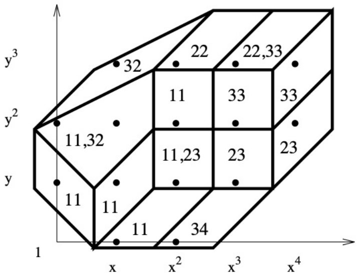

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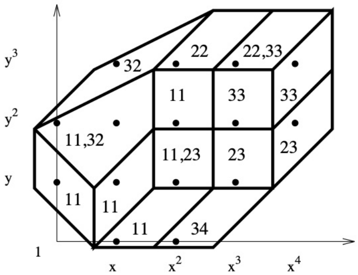

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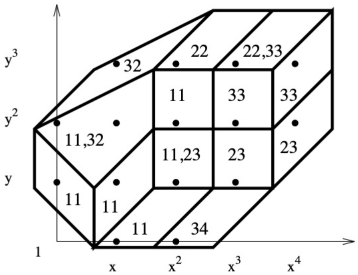

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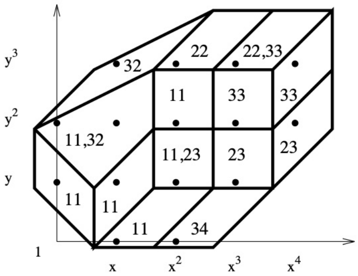

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$$\det(\mathbb{M}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}) = \mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n} \cdot \mathbb{E},$$

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Conjecture

$$\mathbb{E} = \det(\mathbb{F}_{\mathcal{A}_0,\ldots,\mathcal{A}_n})$$

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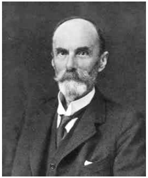
- Even in the classical case it would fail (with a bad lifting).

How would Macaulay prove this?



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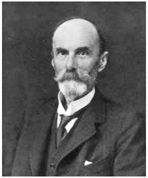

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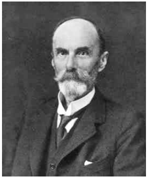

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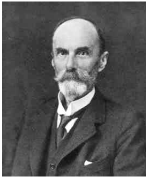

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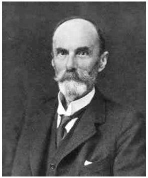

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What is the sparse analogue of $f_n \mapsto t_n^{d_n}$?

Are there "larger" matrices to do the recursion?

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Some connections to the original C-E's conjecture in the "handable" cases worked out by Emiris and Konaxis 2011

End of the story

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arXiv.org > math > arXiv:2004.14622

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Mathematics > Commutative Algebra

[Submitted on 30 Apr 2020]

The Canny–Emiris conjecture for the sparse resultant

Carlos D'Andrea, Gabriela Jeronimo, Martin Sombra

We present a product formula for the initial parts of the sparse resultant associated to an arbitrary family of supports, generalizing a previous result by Sturmfels. This allows to compute the homogeneities and degrees of the sparse resultant, and its evaluation at systems of Laurent polynomials with smaller supports. We obtain a similar product formula for some of the initial parts of the principal minors of the Sylvester–type square matrix associated to a mixed subdivision of a polytope. Applying these results, we prove that the sparse resultant can be computed as the quotient of the determinant of such a square matrix by a certain principal minor, under suitable hypothesis. This generalizes the classical Macaulay formula for the homogeneous resultant, and confirms a conjecture of Canny and~Emiris.

Comments: 48 pages, comments are welcome
Subjects: **Commutative Algebra (math.AC)**; Computational Geometry (cs.CG); Algebraic Geometry (math.AG)
MSC classes: 13P15, 52B20
Cite as: arXiv:2004.14622 [math.AC]
(or arXiv:2004.14622v1 [math.AC] for this version)

Main Theorem

(**D**-Jeronimo-Sombra 2020)

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For $i = 0, \ldots, n$, let $\omega_i : \mathcal{A}_i \to \mathbb{R}$ be the lifting function.

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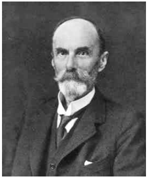
Main Theorem

(**D**-Jeronimo-Sombra 2020)
For $i = 0, \dots, n$, let $\omega_i : \mathcal{A}_i \to \mathbb{R}$ be the lifting function. If the subdivision is "admissible" (for instance if $\omega_0 \gg \omega_1 \gg \dots$) then the Canny-Emiris Conjecture holds

Proof “a la Macaulay”



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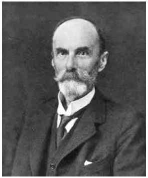

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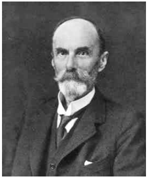

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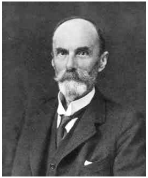

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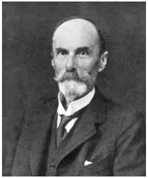

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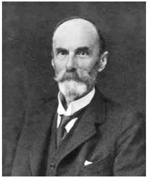

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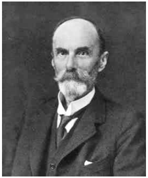

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