# Rees Algebras Associated to Rational Sextics

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# Outline

**Rational Curves via Commutative Algebra: Three Case Studies**

- June 4: Rees Algebras Associated to Rational Sextics
  - I. Two Moving Lines
  - II. Three Perspectives
  - III. Singularities
  - IV. Degree 6
- June 5: Singularities of Multiplicity $c$ on Rational Plane Curves of Degree $2c$.
- June 6: Singularities of Rational Plane Quartics

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# I. Two Moving Lines

Imagine two moving lines in $\mathbb{P}^2$:

$$p = p_1x + p_2y + p_3z = 0$$
$$q = q_1x + q_2y + q_3z = 0$$

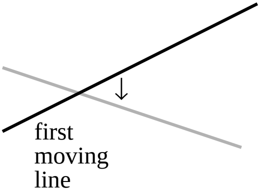
where $p_i, q_i \in R = k[s,t]$ are homogeneous and

$$\deg(p_i) = d_1,\ \deg(q_i) = d_2.$$

We assume the lines are always distinct.

# Picture

The point of intersection of the two moving lines traces out a rational curve $C \subseteq \mathbb{P}^2$:



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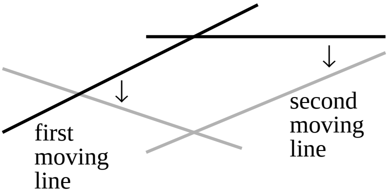

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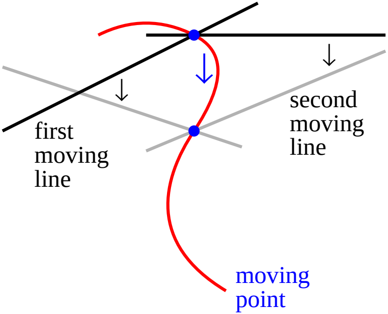


Assume the parametrization $\mathbb{P}^1 \to C$ is birational.

# Properties of $C \subset \mathbb{P}^2$

- $\deg(C) = d_1 + d_2 = n$.
- The map $\mathbb{P}^1 \to \mathbb{P}^2$ is given by
$$B = (a, b, c) = 2 \times 2 \text{ minors of } A = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \\ p_3 & q_3 \end{pmatrix}.$$
- $\gcd(a, b, c) = 1$.
- $I = \langle a, b, c \rangle \subseteq R = k[s, t]$ has free resolution
$$0 \to R(-n - d_1) \oplus R(-n - d_2) \xrightarrow{A} R(-n)^3 \xrightarrow{B} I \to 0.$$
- $p, q$ generate $\mathrm{Syz}(a, b, c) \simeq R(-n-d_1) \oplus R(-n-d_2)$.
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We will think about our setup from three points of view:

- Geometric Modeling
- Projective Elimination Theory
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# Geometric Modeling

- A moving curve in $\mathbb{P}^2$ is $\sum_{i+j+k=m} A_{ijk}\, x^i y^j z^k$ where $A_{ijk} \in R = k[s,t]$ are homogeneous of the same degree.
- It follows the parametrization $(a,b,c)$ if $\sum_{i+j+k=m} A_{ijk}\, a^i b^j c^k = 0$.
- The two given moving lines $p$ (deg $d_1 = \mu$) and $q$ (deg $d_2 = n - \mu$) follow the parametrization, and

$$\mathrm{Syz}(a,b,c) \simeq R(-n-\mu) \oplus R(-n-(n-\mu))$$

implies they generate all moving lines that follow the parametrization ($\mu$-basis).

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# Moving Curve Ideal

The collection of all moving curves that follow the parametrization is the moving curve ideal

$$MC \subseteq k[s,t,x,y,z] \ \leftarrow \ \text{bigraded}$$

**Example**

$n = 4, \mu = 2$. The moving lines $p, q$ determine the implicit equation $F = 0$ via

$$F = \mathrm{Res}_{s,t}(p,q) \ \leftarrow \ 4 \times 4 \text{ determinant.}$$

There are two moving conics $C_1, C_2$ that follow the parametrization of degree 1 is $s, t$ with

$$F = \mathrm{Res}_{s,t}(C_1,C_2) \ \leftarrow \ 2 \times 2 \text{ determinant.}$$

# Example, Continued

- A moving conic is

$$A\,x^2 + B\,xy + C\,xz + D\,y^2 + E\,yz + F\,z^2.$$

- The kernel of

$$\underbrace{R_1^6}_{\dim 12} \xrightarrow{a^2,\dots,c^2} \underbrace{R_9}_{\dim 10} \quad (R = k[s,t])$$

  has dim 2, giving moving conics $C_1, C_2$ that follow the parametrization of degree 1 is $s, t$.

- For $n = 4, \mu = 2$, the minimal generators of $MC$ are

$$p,\ q,\ C_1,\ C_2,\ F.$$

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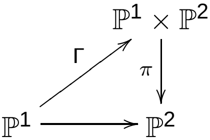
$k[s,t,x,y,z]$ is the coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^2$, and $p = q = 0$ defines the graph $\Gamma$ of the parametrization




Since $C = \pi(\Gamma)$, eliminating $s, t$ from $p = q = 0$ gives the implicit equation

$$F = 0.$$

# Elimination Theory, Continued

- (Affine) Given polynomials

$$f_1, \ldots, f_s \in k[x_1, \ldots, x_n, y_1, \ldots, y_m],$$

eliminating $x_1, \ldots, x_n$ means computing

$$\langle f_1, \ldots, f_s \rangle \cap k[y_1, \ldots, y_m].$$

- (Projective) $\langle p, q \rangle \cap k[x, y, z] = \{0\}$ for degree reasons. Instead, we need to compute

$$(\langle p, q \rangle : \langle s, t \rangle^\infty) \cap k[x, y, z].$$

- Hence we have the ideals $\langle p, q \rangle \subseteq \langle p, q \rangle : \langle s, t \rangle^\infty$.

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# Commutative Algebra

An ideal $I$ in a commutative ring $R$ gives two important $R$-algebras:

- The symmetric algebra: $\mathrm{Sym}(I) = \bigoplus_d \mathrm{Sym}^d(I)$
- The Rees algebra: $\mathcal{R}(I) = \bigoplus_d I^d e^d \subseteq R[e]$

They are related by a natural surjection:

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When $I = \langle f_1, \dots, f_s \rangle$, we have surjections:

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# The Viewpoints are the Same!

- $MC = \langle p,q\rangle : \langle s,t\rangle^\infty$
- We have a commutative diagram:

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k[s,t,x,y,z] & = & k[s,t,x,y,z] \\
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- $\ker(\mathrm{Sym}(I) \to \mathcal{R}(I)) =$ $R$-torsion in $\mathrm{Sym}(I)$.
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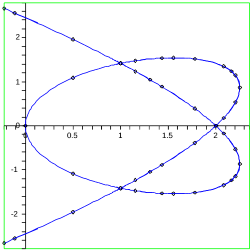
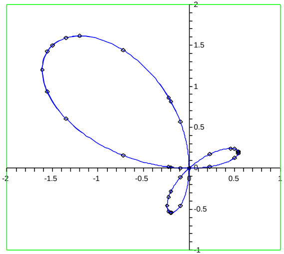
# III. Singularities

Two examples with $n = 4$:



$\mu = 2$



$\mu = 1$

Bi-degrees of minimal generators of $MC$:

- $\mu = 2$: (2,1), (2,1), (1,2), (1,2), (0,4)
- $\mu = 1$: (1,1), (3,1), (2,2), (1,3), (0,4)

# Main Question

When $n = 4$, we have $\mu = 1$ or 2. Furthermore:

- $\mu = 1 \iff$ there is a point of mult 3.
- $\mu = 2 \iff$ all singularities have mult 2.
- The bi-degrees of the minimal generators of *MC* are determined by the singularities of the curve.

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# Two Theorems

**Theorem** (Ascenzi, Song/Chen/Goldman, Chen/Wang/Liu)

*Assume $C \subseteq \mathbb{P}^2$ is a rational plane curve of degree $n$ with $\mu \le n - \mu$. Then the singular points of $C$ have:*

- $(\mu = n - \mu)$ *Multiplicity* $\le \mu$.
- $(\mu < n - \mu)$ *Multiplicity* $\le \mu$ *or* $= n - \mu$, *and there is at most one of multiplicity* $= n - \mu$.

**Theorem** (C/Hoffman/Wang, Hong/Simis/Vasconcelos, Cortadellas/D'Andrea)

*Assume $n \ge 3$. Then:*

- $\mu = 1 \iff \exists!$ *singular point of multiplicity* $n - 1$.
- *If $\mu = 1$, then MC has minimal generators of bi-degrees:*

$$(1,1),(n-1,1),(n-2,2),\ldots,(1,n-1),(0,n).$$

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*Assume $C \subseteq \mathbb{P}^2$ is a rational plane curve of degree $n$ with $\mu \leq n - \mu$. Then the singular points of $C$ have:*

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**Theorem** (C/Hoffman/Wang, Hong/Simis/Vasconcelos, Cortadellas/D'Andrea)

*Assume $n \geq 3$. Then:*

- $\mu = 1 \iff \exists!$ *singular point of multiplicity* $n - 1$.
- *If $\mu = 1$, then MC has minimal generators of bi-degrees:*

$$(1,1), (n-1,1), (n-2,2), \ldots, (1,n-1), (0,n).$$

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- Joint work with Andy Kustin, Claudia Polini and Bernd Ulrich.
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# $n = 6, \ \mu = 2$

Previous results:

- Singular points have multiplicity $= 2$ or $4$.
- It one of mult 4 occurs, it is unique.
- (Song/Chen/Goldman) Multiplicity 4 occurs if and only if the moving line of degree 2 spins about fixed point.
- (Busé) Bi-degrees of generators of $MC$ are known when a certain depth condition is satisfied.

**Theorem (CKPU)**

*If $n = 6, \ \mu = 2$, then there are two possibilities for the bi-degrees of the minimal generators of MC, corresponding to whether the moving line of degree 2 spins about fixed point.*

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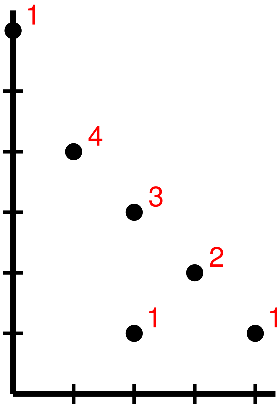
The bi-degrees of the minimal generators of $MC$:

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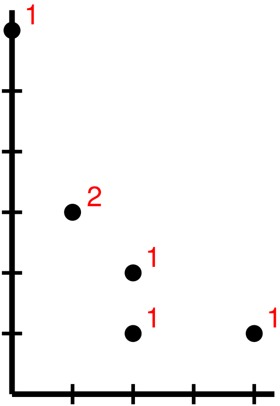


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# $n = 6, \mu = 3$: Algebra

To analyze this case, write

$$\begin{aligned} p &= s^3C_{01} + s^2tC_{11} + st^2C_{21} + t^3C_{31} \\ q &= s^3C_{02} + s^2tC_{12} + st^2C_{22} + t^3C_{32} \end{aligned}$$

where $C_{ij}$ is linear in $x, y, z$. Let $I_2(C)$ be the ideal generated by the $2 \times 2$ minors of the $2 \times 4$ matrix $C = (C_{ij})$.

**Theorem (KPU)**

*When $n = 6$, $\mu = 3$:*

- *There are four possibilities* for the bi-degrees of the minimal generators of MC.*
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# $n=6, \mu=3$: Geometry

Here, the singular points of the curve have multiplicity $= 2$ or $3$.

**Theorem (CKPU)**

*When $n = 6$, $\mu = 3$, the number of triple points (including infinitely near) is $0, 1, 2$ or $3$ and is equal to*

$$6 - \#\,\text{minimal generators of } I_2(C).$$

**Corollary**

*When $n = 6$, $\mu = 3$, the number of triple points (including infinitely near) determines the bi-degrees of the minimal generators of MC.*

My next lecture will include a generalization of this theorem.

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# An Unresolved Question

**For $n = 6$, we can have:**

- $\mu = 2$ with 10 ordinary double points.
- $\mu = 3$ with 10 ordinary double points.
- The ideals of these points in $\mathbb{P}^2$ have the same betti diagrams.
- Yet the associated Rees algebras look quite different.

The next slide shows the bi-degrees of the minimal generators of $MC$ in these two cases.

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# Bi-Degrees of Minimal Generators of $MC$

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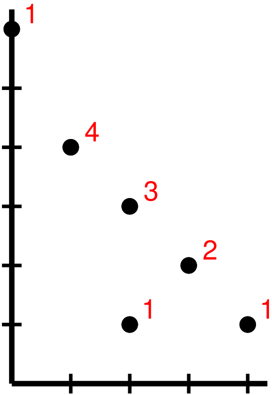


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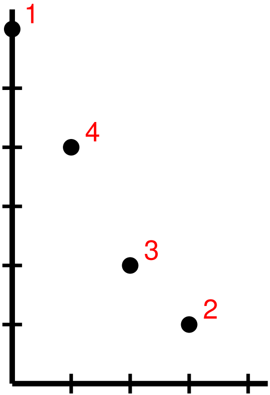


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# Can Geometry Distinguish These Cases?

Consider a sextic curve in $\mathbb{P}^2$ with 10 ordinary double points. Fix 9 of the points.

## Suggestion of Izzet Coskun

- Consider the linear system of sextic curves spanned by the implicit equation $F$ of the sextic and the square $G^2$ of the unique cubic going through the 9 points.
- Does this linear system have a reduced member that is not irreducible?

## Suggestion of Damiano Testa

- Is the unique cubic through these 9 points singular?

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