Singularities of Multiplicity c on Rational Plane Curves of degree 2c

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Explain the Title

- Let $\mathcal{C} \subseteq \mathbb{P}^2$ be irreducible and rational of degree d. Then
$$0 = (d-1)(d-2)/2 - \sum_P m_P(m_P - 1)/2,$$
where the sum is over all singular and ∞-near singular points of $\mathcal{C}$.
- Now assume $d = 2c$ and $\mu = c$. The latter implies that all singular points of $\mathcal{C}$ have multiplicity $\leq c$. We also assume $c \geq 3$.
- Let $s = \#$ singular points of multiplicity c (including ∞-near). Then
$$\frac{(2c-1)(2c-2)}{2} = \frac{1}{2}\sum_P m_P(m_P - 1) \geq s \cdot \frac{1}{2}c(c-1).$$
This easily implies that
$$s \leq 3.$$

Goal

Study these singularities using the methods of commutative algebra.

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Outline

Joint work with Andy Kustin, Claudia Polini and Bernd Ulrich

- Computer: Describe the main results
- Blackboard: Give some of the details

Full details appear in Sections 1–7 of *A Study of Singularities on Rational Curves via Syzygies*.

This paper will appear in the Memoirs of the American Mathematical Society.

Setup

Let $\mathcal{C} \subseteq \mathbb{P}^2$ be irreducible and rational of degree $d = 2c$. Parametrize $\mathcal{C}$:

$$\mathbb{P}^1 \longrightarrow \mathcal{C} \subseteq \mathbb{P}^2$$

where $(s,t) \mapsto (a(s,t), b(s,t), c(s,t))$ have degree d, $\gcd(a,b,c) = 1$.

Our assumptions $d = 2c$ and $\mu = c$ imply $I = \langle a,b,c\rangle \subseteq R = k[s,t]$ has a free resolution

$$0 \to R(-3c) \oplus R(-3c) \xrightarrow{\varphi} R(-2c)^3 \xrightarrow{(a,b,c)} I.$$

We call φ the Hilbert-Burch Matrix and write it as

$$\varphi = \begin{pmatrix} Q_1 & Q_4 \\ Q_2 & Q_5 \\ Q_3 & Q_6 \end{pmatrix}.$$

The Hilbert-Burch Matrix

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has the following properties:

- The columns Q_1, Q_2, Q_3 and Q_4, Q_5, Q_6 have degree c.
- They generate the syzygy module $\mathrm{Syz}(a, b, c)$.
- We also have $\bigwedge^2 \varphi = (a, b, c)$.

Our Approach

We will use the Hilbert-Burch matrix φ to study singularities of multiplicity c of the rational curve $\mathcal{C} \subseteq \mathbb{P}^2$.

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The Partition For $d = 2c$, $\mu = c$

Define $\mathcal{U} \subseteq R^3_{2c}$ to be

$$\mathcal{U} = \{(a,b,c) \in R^3_{2c} \mid (a,b,c) \text{ gives a parametrization with } \mu = c\}$$

The set $\mathcal{U}$ is open and dense in R^3_{2c}. Then partition $\mathcal{U}$ as follows:

$$\begin{aligned}
S_\emptyset &= \{(a,b,c) \in \mathcal{U} \mid \text{no points of mult } c\}\\
S_c &= \{(a,b,c) \in \mathcal{U} \mid \text{one point of mult } c\}\\
S_{c,c} &= \{(a,b,c) \in \mathcal{U} \mid \text{two distinct points of mult } c\}\\
S_{c,c,c} &= \{(a,b,c) \in \mathcal{U} \mid \text{three distinct points of mult } c\}\\
S_{c:c} &= \{(a,b,c) \in \mathcal{U} \mid \text{one point \& one } \infty\text{-near point of mult } c\}\\
S_{c:c,c} &= \{(a,b,c) \in \mathcal{U} \mid \text{one point \& one } \infty\text{-near point of mult } c,\\
&\qquad\qquad \text{and one additional point of mult } c\}\\
S_{c:c:c} &= \{(a,b,c) \in \mathcal{U} \mid \text{one point \& two } \infty\text{-near points of mult } c\}
\end{aligned}$$

We call the S_i the strata of $\mathcal{U}$.

The Main Result

Theorem (CKPU)

The strata S_i are open in their closures, irreducible, and fit into the diagram:

$$
\begin{array}{ccccccccc}
 & & & S^{4c+6}_{c:c} & & & & & \\
 & & \nearrow & & \searrow & & & & \\
S^{3c+7}_{c:c:c} & \longrightarrow & S^{3c+8}_{c:c,c} & & & S^{4c+7}_{c,c} & \longrightarrow & S^{5c+5}_{c} & \longrightarrow & S^{6c+3}_{\emptyset} \\
 & & \searrow & & \nearrow & & & & \\
 & & & S^{3c+9}_{c,c,c} & & & & &
\end{array}
$$

Furthermore:

- *Arrows mean "is contained in the closure of".*
- *Superscripts indicate the dimension of each strata.*
- *Note* $\dim \mathcal{U} = \dim R^3_{2c} = 3(2c+1) = 6c+3$.

A Hint of the Proof

To compute the dimension of $S_{c:c,c}$, the first step is to show that for suitable coordinates in $\mathbb{P}^2$ and a suitable basis of the syzygy module, we have the normal form

$$\varphi = \begin{pmatrix} Q_1 & 0 \\ Q_2 & Q_3 \\ 0 & Q_2 \end{pmatrix}$$

where the point of multiplicity $c{:}c$ is $(0,0,1)$ and the other point of multiplicity c is $(1,0,0)$.

Let $N_{c:c,c} = \left\{ \begin{pmatrix} Q_1 & 0 \\ Q_2 & Q_3 \\ 0 & Q_2 \end{pmatrix} \right\}$ be the set of normal forms.

A Hint of the Proof, Continued

We have the normal form $\begin{pmatrix} Q_1 & 0 \\ Q_2 & Q_3 \\ 0 & Q_2 \end{pmatrix}$ and the maps

$$\mathrm{GL}(3) \times \mathrm{GL}(2) \times N_{c:c,c} \xrightarrow{\Phi} \{\text{HB matrices for } c:c,c\} \xrightarrow{\Psi} S_{c:c,c},$$

where $\Psi(\varphi) = \bigwedge^2 \varphi$.

Furthermore:

- Φ and Ψ are surjective.
- The generic fiber of Φ has dimension 5 (takes proof).
- the generic fiber of Ψ has dimension 3 (easy to see).

Hence $S_{c:c,c}$ is irreducible of dimension

$$\dim S_{c:c,c} = 9 + 4 + 3(c+1) - 5 - 3 = 3c + 8.$$