Singularities of Rational Plane Quartics

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Outline

Joint work with Andy Kustin, Claudia Polini and Bernd Ulrich

Goal

Study the singularities of rational plane quartics using the methods of commutative algebra

- Computer: Describe the singularities and give the strategy
- Blackboard: Give some of the details

Full details appear in Sections 8 and 9 of *A Study of Singularities on Rational Curves via Syzygies*.

Singular Points of Rational Quartics

For an irreducible rational quartic in $\mathbb{P}^2$, there are nine possible singular points:

Name	Name	Mult:∞-near;Branches
node	A_1	2;2
cusp	A_2	2;1
tacnode	A_3	2:2;2
ramphoid cusp	A_4	2:2;1
oscnode	A_5	2:2:2;2
A_6-cusp	A_6	2:2:2;1
ord triple pt	D_4	3;3
tacnode-cusp	D_5	3;2
mult-3 cusp	E_6	3;1

Configurations of Singular Points

For an irreducible rational quartic in $\mathbb{P}^2$, there are 13 possible configurations of singular points:

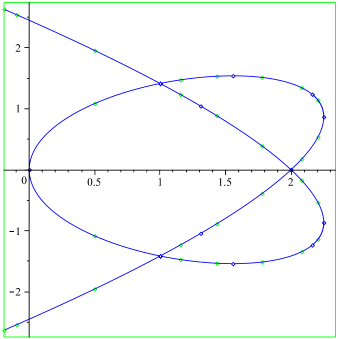
3 with $\mu = 1$:

- 1 ordinary triple point
- 1 tacnode-cusp
- 1 multiplicity-3 cusp

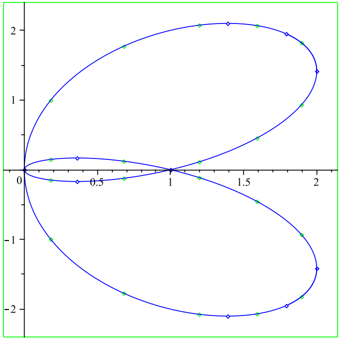
10 with $\mu = 2$:

- 4 with i nodes and j cusps, $i + j = 3$
- 2 with one tacnode plus one node or cusp
- 2 with a ramphoid cusp plus one node or cusp
- 1 oscnode
- 1 A_6-cusp

Two Pictures

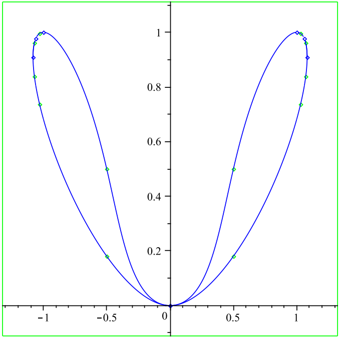


Three nodes

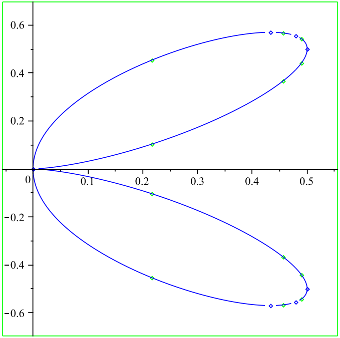


Tacnode and node

Two More Pictures



Oscnode



Tacnode-cusp

Organize the Configurations

We focus on the 10 configurations with $\mu = 2$.

A Coarse Stratification

We organize these configurations as follows:

- $S_{2,2,2}$: $\begin{cases} \text{3 nodes} & \text{(6 branches)} \\ \text{2 nodes and 1 cusp} & \text{(5 branches)} \\ \text{1 node and 2 cusps} & \text{(4 branches)} \\ \text{3 cusps} & \text{(3 branches)} \end{cases}$
- $S_{2:2,2}$: $\begin{cases} \text{1 tacnode and 1 node} & \text{(4 branches)} \\ \text{1 tacnode and 1 cusp} & \text{(3 branches)} \\ \text{1 ramphoid cusp and 1 node} & \text{(3 branches)} \\ \text{1 ramphoid cusp and 1 cusp} & \text{(2 branches)} \end{cases}$
- $S_{2:2:2}$: $\begin{cases} \text{1 oscnode} & \text{(2 branches)} \\ 1\ A_6\text{-cusp} & \text{(1 branch)} \end{cases}$

The notation $S_{2,2,2}$, $S_{2:2,2}$, $S_{2:2:2}$ is from my second lecture.

Count Branches

To count branches, we use the following:

Theorem (CKPU)

Let (a, b, c) give a parametrization $\mathbb{P}^1 \to \mathcal{C} \subseteq \mathbb{P}^2$ and set

$$N = \begin{pmatrix} \frac{\partial a}{\partial s} & \frac{\partial b}{\partial s} & \frac{\partial c}{\partial s} \\ \frac{\partial a}{\partial t} & \frac{\partial b}{\partial t} & \frac{\partial c}{\partial t} \end{pmatrix}.$$

If s_P is the number of branches at P, then

$$\deg \gcd I_2(N) = \sum_P m_P - s_P.$$

If we know the multiplicities, this gives the number of branches.

Strategy

Three Steps

- Step 1: Stratify using the number of visible singular points ($S_{2,2,2}$, $S_{2:2,2}$, $S_{2:2:2}$).
- Step 2: Refine the stratification using the number of branches.
- Step 3: Curves with 1 tacnode and 1 cusp and with 1 ramphoid cusp and 1 node both have two visible singular points and three total branches. Separate these using the conductor.

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The Conductor

Let (a, b, c) of degree d parametrize $\mathcal{C} \subseteq \mathbb{P}^2$, and let

$$B^{(d)} = \bigoplus_{\ell=0}^{\infty} B_{\ell d} \subseteq B = k[s,t]$$

be the dth Veronese subring. Then

$$\mathfrak{c} = \{r \in k[a,b,c] \mid r \cdot B^{(d)} \subseteq k[a,b,c]\}$$

is the conductor of

$$k[a,b,c] \subseteq B^{(d)}.$$

Theorem 1 (General d)

$\mathfrak{c}B = F(s,t)\langle s,t\rangle^{d-2}$, where $\deg F = (d-1)(d-2)$.

Theorem 2 ($d = 4$)

- 1 tacnode and 1 cusp $\Rightarrow F = \ell_1^2 \ell_2^2 \ell_3^2$.
- 1 ramphoid cusp and 1 node $\Rightarrow F = \ell_1^4 \ell_2 \ell_3$.

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