Strong homology and set theory

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Abstract

Various connections between strong homology theory and set theory are discussed. The Continuum Hypothesis, the Proper Forcing Axiom, and other set-theoretic axioms imply different values of higher derived limits, and strong homology groups.

The topic we are going to talk about, has quite a long history. In [MP88] it was proved that strong homology neither is additive nor has compact supports, provided the Continuum Hypothesis (CH) is assumed. In that paper, an abelian pro-group $A$ was constructed such that the groups $\lim^sA$, $s \geq 1$, serve as the obstructions both to additivity and having compact supports. Assuming CH, it was proved that $\lim^1A \neq 0$. Later, in [Pra05], under a weaker assumption $\vartheta = \aleph_1$, it was proved that the cardinality of $\lim^1A$ is quite large:

$$|\lim^1A| = \aleph_1^{\aleph_1}.$$  

In [DSV89], assuming the Proper Forcing Axiom (PFA), it was proved that $\lim^1A = 0$.

However, recently in [Ber15], it was shown that, assuming PFA,

$$\lim^2A \neq 0,$$

$$\lim^sA = 0, s \neq 0, 2.$$  

An interesting result in [Tod98, Theorem 1] should be also mentioned: $\lim^1A \neq 0$ implies that there is a subset $X$ of $\mathbb{R} - \mathbb{Q}$ which is not analytic but its intersection with every compact subset of $\mathbb{R} - \mathbb{Q}$ is $F_\sigma$. Another interesting result [Far11, Theorem 1.1] uses the methods of [MP88]: the Continuum Hypothesis implies there is an outer automorphism of the Calkin algebra. Moreover, the restriction of this automorphism to any separable subalgebra is inner.

What if only ZFC (the Zermelo-Fraenkel axioms plus the Axiom of Choice) is assumed? Günther [Gün92, Example before Theorem 8] proved the following.

Let $X = \omega_1$, and let $A$ be the subspace of limit ordinals. $X$ is a normal Hausdorff space. It was proved that

$$\mathcal{H}_n(X, A; \mathbb{Z}_2) \neq \mathcal{H}_n(X/A, *; \mathbb{Z}_2)$$

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where $\overline{H}_n$ is strong homology, while $\overline{H}_n^c$ is strong homology with compact supports. It follows then that one (or both) of the following statements is valid:

1. $\overline{H}_n^c (X, \mathbb{Z}_2) \neq \overline{H}_n (X, \mathbb{Z}_2)$

2. $\overline{H}_n^c (A, \mathbb{Z}_2) \neq \overline{H}_n (A, \mathbb{Z}_2)$

Therefore, strong homology does not have compact supports.

Mardešić (see [Mar96] and [Mar00, Ch. 21.5]) constructed paracompact spaces $X$ such that $\overline{H}_n^c (X) \neq \overline{H}_n (X)$.

Lisica [Lis05] constructed a separable metric space $X$ with $\overline{H}_n^c (X) \neq \overline{H}_n (X)$.

The problem of additivity was finally solved in the negative in [Pra05]: it was proved that, assuming only ZFC, strong homology is not additive.

It could seem that strong homology is a source of mostly negative results. However, there are a lot of clearly positive results, like in [MP98, Theorem 1 and Theorem 10]:

Let $X$ be a compact Hausdorff space, represented as a limit of finite polyhedra:

$$X = \lim_{i \in I} X_i.$$ 

Given an abelian group $G$ and integer $m \in \mathbb{Z}$, the strong homology group $\overline{H}_m (X; G)$ has a natural (on $X$ and $G$) filtration

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 = \overline{H}_m (X, G)$$

together with the following natural isomorphisms:

1. $F_1 \simeq Ext (\overline{H}_m^{m+1} (X), G) \simeq \lim_{i \in I} H_{m+1} (X_i, G) \simeq \lim_{i \in I} Hom (H_{m+1} (X_i), G)$;

2. $F_2 \simeq Ext (\overline{H}_m^{m+1} (X), G)$;

3. $F_2/F_1 \simeq \lim_{i \in I} Ext (H_{m+1} (X_i), G)$;

4. $F_3/F_1 \simeq \overline{H}_m (X, G)$;

5. $F_3/F_2 \simeq Hom (\overline{H}_m (X), G) \simeq \lim_{i \in I} (H_{m} (X_i), G)$,

where $\overline{H}_*$ and $\overline{H}^*$ are respectively Čech homology and cohomology. Moreover, $\overline{H}_m (X, G) = 0$ if $m < 0$. 


1 Pro-category

Given a category \( C \), consider a sequence of full embeddings
\[
C \xrightarrow{j_i} \text{Pro}(C) \xrightarrow{j} \left(\text{Set}^C\right)^{\text{op}}
\]
where \( j_i \) is the second Yoneda embedding
\[
j_i = h_i^* : C \rightarrow \left(\text{Set}^C\right)^{\text{op}},
\]
and \( \text{Pro}(C) \) is the full subcategory of \( \left(\text{Set}^C\right)^{\text{op}} \) consisting of pro-representable functors:
\[
\mathcal{X} \in \text{Pro}(C) \iff \mathcal{X} = \lim_{i \in I} (h_{X_i})
\]
where \( I \) is the small cofiltrant category.

Remark 1.1 Cofiltrant means dual to filtrant (filtered, filtering):

1. For any \( i, j \in I \) there exist a \( k \in I \), and morphisms
   \[
   k \rightarrow i, k \rightarrow j.
   \]

2. For any two morphisms
   \[
   i \xrightarrow{\alpha} j, \quad i \xrightarrow{\beta} j
   \]
   there exists a morphism
   \[
   \gamma : k \rightarrow i
   \]
   with \( \alpha \gamma = \beta \gamma \).

Remark 1.2 A (pre)poset \( I \) is called cofiltrant or codirected if the corresponding category:
\[
\text{Hom}_I(x, y) = \begin{cases} 
(x, y) & \text{if } x \leq y \\
\varnothing & \text{otherwise}
\end{cases}
\]
is cofiltrant.

Remark 1.3 Objects in \( \text{Pro}(C) \) can be described by diagrams
\[
\mathcal{X} : I \rightarrow C
\]
where \( I \) is a small cofiltrant category. Morphisms can be described like this:
\[
\text{Hom}_{\text{Pro}(C)}(\mathcal{X} : I \rightarrow C, \mathcal{Y} : J \rightarrow C) = \lim_{j \in J, i \in I} \text{Hom}_C(\mathcal{X}(i), \mathcal{Y}(j)).
\]

Using Mardešić trick [MS82, Theorem I.1.4], one can assume that \( I \) and \( J \) are cofiltrant (pre)posets.
Example 1.4 Let \( \text{Fin} \) be the category of finite sets. Then \( \text{Pro}(\text{Fin}) \) is equivalent to the category of \textit{profinite sets}, i.e. 0-dimensional compact Hausdorff spaces.

Example 1.5 Let \( \text{FinGr} \) be the category of finite groups. Then \( \text{Pro}(\text{FinGr}) \) is equivalent to the category of \textit{profinite groups}, i.e. 0-dimensional compact Hausdorff groups. The equivalence is given by the following:

\[
G \mapsto (G/U)_{U \in \text{Norm}(G)} \in \text{Pro}(\text{FinGr}),
\]

\[
\text{Pro}(\text{FinGr}) \ni (G_i)_{i \in I} \mapsto \lim_{i} G_i,
\]

where \( \text{Norm}(G) \) is the set of open normal subgroups of \( G \), and \( \lim_{i} \) is taken in the category of topological groups.

Remark 1.6 The ind-category \( \text{Ind}(C) \) is defined dually:

\[
C \overset{j'}{\longrightarrow} \text{Ind}(C) = (\text{Pro}(C^{\text{op}}))^{\text{op}} \overset{j'}{\longrightarrow} \text{Set}^{C^{\text{op}}}
\]

where \( j' \) equals the first Yoneda embedding:

\[ j' : C \longrightarrow \text{Set}^{C^{\text{op}}}. \]

2 Shape theory

The main reference is [MS82].

Let \( P \) be the class of topological spaces having the homotopy type of a polyhedron. By the same letter we will denote the corresponding full subcategory of \( \text{Top} \).

Remark 2.1 \( P \) can be described equivalently as the class of topological spaces having the homotopy type of an absolute neighborhood retract (ANR).

Definition 2.2 Let \( X \) be a space. Consider the set \( \text{Cov}(X) \) of normal (numerable) coverings on \( X \). The set is pre-ordered by the following:

\[ U \leq V \iff U \text{ refines } V. \]

The family of homotopy types of \( Č \text{ech} \) nerves \( (NU)_{U \in \text{Cov}(X)} \) represents an object of \( \text{Pro}(H(P)) \) which is called the \textit{shape} of \( X \).

The \textit{pointed shape} is defined similarly.

Remark 2.3 If \( X \) is (Hausdorff) paracompact then the set of normal coverings can be replaced by the set of all coverings.

Definition 2.4 Given a space \( X \) and an abelian group \( G \), let

\[ \text{Pro-H}_{n}(X, G) := (H_{n}(NU, G))_{U \in \text{Cov}(X)} \in \text{Pro}(\text{Ab}) \]

be its \textit{pro-homology}.
Definition 2.5 Given a pointed space $X$, let
\[ \text{Pro}-\pi_n(X) := (\pi_n(NU))_{U \in \text{Cov}(X)} \in \text{Pro}(C) \]
where
\[ C = \begin{cases} 
\text{Set} & \text{if } n = 0 \\
\text{Gr} & \text{if } n = 1 \\
\text{Ab} & \text{if } n \geq 2 
\end{cases} \]
be its pro-homotopy.

Example 2.6 The Warsaw circle.

Example 2.7 Given a family $(X_\alpha)_{\alpha \in A}$ of pointed spaces, let $Y$ be their bouquet, i.e. the coproduct of $X_\alpha$ in the category of pointed sets. $Y$ is naturally a subset of the product:
\[ Y \subseteq \prod_{\alpha \in A} X_\alpha. \]

Let the cluster (or wedge, or compact bouquet) of the family $(X_\alpha)_{\alpha \in A}$ be $Y$ with the subspace topology. Denote it by
\[ Y = \bigvee_{\alpha \in A} X_\alpha. \]

Remark 2.8 The cluster is not isomorphic to the coproduct
\[ \bigvee_{\alpha \in A} X_\alpha \]
in the category $\text{Top}_*$ of pointed topological spaces.

Example 2.9 The Hawaiian ear-ring $X^{(k)}$ is the cluster of countably many copies of the $k$-sphere:
\[ X^{(k)} = \bigvee_{n \in \mathbb{N}} S^k. \]

3 Strong shape theory

The main reference is [Mar00].

Definition 3.1 Let $X$ be a space. Consider the family of Čech nerves $(NU)_{U \in \text{Cov}(X)}$ which represents an object of $\text{Pro}(H(P))$. The diagram
\[ U \mapsto NU : \text{Cov}(X) \to P \]
is commutative only up to homotopy, and does therefore not represent an object of $\text{Pro}(P)$. The homotopies are homotopic up to a second order homotopy. The second order homotopies are homotopic up to a third order homotopy etc. This family of higher order homotopies defines a coherent homotopy commutative diagram which represents the strong shape of $X$. The morphisms between such diagrams are defined similarly, up to a coherent homotopy.

The pointed strong shape is defined similarly.
Remark 3.2 In fact, the objects of the strong shape category $\text{SSh}$ can be represented as objects of $\text{Pro}(\mathcal{P})$, i.e. strictly commutative diagrams, while the morphisms between such diagrams are described as coherent families of mappings. However, it is possible [Pra01] to define $\text{SSh}$ as a full subcategory of the localization $\text{Pro}(\mathcal{P})[\Sigma^{-1}]$ for an appropriate family $\Sigma$ of morphisms.

4 Strong homology

Let $\mathcal{X} \in \text{Pro}(\mathcal{P})$ be given by a diagram

$$\mathcal{X} : I \longrightarrow \mathcal{P}.$$

Consider a cochain bicomplex $C^{\bullet\bullet}(\mathcal{X}, G)$:

$$C^{\text{st}}(\mathcal{X}, G) = \prod_{(i_0 \to i_1 \to \cdots \to i_s) \in I} C^{\text{sing}}_{-t}(X(i_s), G)$$

where $C^{\text{sing}}_{-t}(X(i_s), G)$ is the singular chain complex.

Definition 4.1 Strong homology of $X$ is defined as the cohomology of the total cochain complex

$$\overline{H}_n(X, G) = H^{-n}(\text{Tot}(C^{\bullet\bullet}(\mathcal{X}), G))$$

where $\mathcal{X} \in \text{Pro}(\mathcal{P})$ represents the strong shape of $X$.

The relative strong homology groups $\overline{H}_n(X, A; G)$ are defined similarly.

Remark 4.2 Since the bicomplex $C^{\bullet\bullet}(\mathcal{X}, G)$ occupies the IV quadrant, it is possible that nontrivial strong homology groups $\overline{H}_n(X, G)$ exist for $n < 0$. However, it is impossible for compact Hausdorff spaces.

In [Pra01], strong homology $\overline{H}_n(X, E)$ with the coefficients in a spectrum $E$ is defined. When $E = K(G)$ is the Eilenberg - Mac Lane spectrum, there exists a natural isomorphism with strong homology from the above definition:

$$\overline{H}_n(X, E) \simeq \overline{H}_n(X, G).$$

Strong homology satisfy the seven Eilenberg-Steenrod Axioms and Axioms 8 and 9 from [Mil95]. On the category of compact metric spaces, strong homology is isomorphic to the Steenrod homology. Moreover, there exists a conditionally convergent spectral sequence [Pra89], [Pra13, Theorem 4.5]:

$$E_2^{st} = \lim^s H_{-t}(\mathcal{X}, G) \Rightarrow \overline{H}_{-s-t}(X, G).$$
5 Additivity

In [Mil62], Milnor formulated the Additivity axiom: given a homology theory $h_*$, and a family $(X_\alpha)_{\alpha \in A}$ of topological spaces, there is a natural isomorphism:

$$h_* \left( \prod_{\alpha \in A} X_\alpha \right) \simeq \bigoplus_{\alpha \in A} h_* (X_\alpha).$$

**Remark 5.1** The axiom is evidently valid for a finite family $(X_\alpha)$.

Let $X^{(k)}$ be the $k$-dimensional Hawaiian ear-ring, i.e. the cluster of countably many copies of the $k$-sphere:

$$X^{(k)} = \bigvee_{n \in \mathbb{N}} S^k,$$

and let

$$Y^{(k)} = \prod_{n \in \mathbb{N}} X^{(k)}$$

$Y^{(k)}$ is a locally compact metric space, which can be embedded into $\mathbb{R}^{k+1}$.

**Definition 5.2** Let $\leq$ be the pointwise order on $\mathbb{N}^\mathbb{N}$, and let

$$I = (\mathbb{N}^\mathbb{N}, \leq).$$

We define the following pro-group $A$:

$$A : I \longrightarrow \text{Ab},$$

$$A(f) = \bigoplus_{n \in \mathbb{N}} \bigoplus_{m \leq f(n)} \mathbb{Z}.$$  

**Definition 5.3** Let $\leq^*$ be another pre-order on $\mathbb{N}^\mathbb{N}$:

$$f \leq^* g \iff |n : f(n) > g(n)| < \infty.$$  

The dominating number $d$ equals the cofinality of $I^* = (\mathbb{N}^\mathbb{N}, \leq^*)$. The bounding number $b$ equals the least cardinality of an unbounded subset of $I^*$.

**Theorem 5.4** [MP88], [Pra05], [DSV89], [Ber15]

Let

$$S_{m,k} := \text{coker} \left( \bigoplus_{n \in \mathbb{N}} \overline{H}_m \left( X^{(k)} \right) \rightarrow \overline{H}_m \left( Y^{(k)} \right) \right).$$

Then:

1. 

$$S_{m,k} = \left\{ \begin{array}{ll} 0 & \text{if } m \geq k \\ \lim_{k \to m} A & \text{if } m < k \end{array} \right.$$
2. If $\mathfrak{d} = \mathfrak{N}_1$ is assumed, then $\lim^1 \mathcal{A} \neq 0$. Moreover,

$$\left| \lim^1 \mathcal{A} \right| = \mathfrak{N}_1^\mathfrak{N}_1.$$

3. If PFA is assumed, then:

$$\lim^2 \mathcal{A} \neq 0,$$

$$\lim^s \mathcal{A} = 0, s \neq 0, 2.$$

**Remark 5.5** The last statement can be proved under a weaker assumption

$$(b = d = \mathfrak{c} = \mathfrak{N}_2) \& \diamondsuit \left( S^2_1 \right).$$

### References


