SIMBa
Junior Week of Mathematics 2018
Organizing Committee
October, 2018
Abstracts

Algebra

On the structure of Binomial Ideals

L. Colarte

Abstract

Determining the radical of an ideal or whether it is prime are notoriously difficult problems in Algebra. In general, given an ideal, (even a finitely generated one), it is extremely hard to deduce anything about its structure. For instance, we cannot infer many information about the shape of its primary decomposition from a set of generators. There are a few but notable exceptions: monomial and binomial ideals.

Let us consider $k$ to be an algebraically closed field of characteristic zero and $k[x_0,\ldots,x_n]$ the polynomial ring. By a binomial in $k[x_0,\ldots,x_n]$ we mean a polynomial $ax^\alpha + bx^\beta$, where $a,b \in k$, $\alpha = (\alpha_0,\ldots,\alpha_n)$, $\beta = (\beta_0,\ldots,\beta_n) \in \mathbb{Z}_+^n$ and $x^\alpha$ (respectively $x^\beta$) represents the monomial $x_0^{\alpha_0}\cdots x_n^{\alpha_n}$. A binomial ideal in $k[x_0,\ldots,x_n]$ is just defined to be an ideal generated by binomials.

Through this course, we intend to study the structure of binomial ideals, and see that the radical and the primary decomposition of binomial ideals is again binomial. In addition, we wish to provide a criterion to determine whether a binomial ideal is prime.

Course Outline

We will start studying a primality criterion for binomial ideals, introducing first the Laurent polynomial ring $k[x^\pm]$ and Laurent binomial ideals. We will see that Laurent binomial ideals are uniquely determined by partial characters, that in turn defines binomial ideals that inherit good properties from Laurent binomial ideals.
Afterwards, we will analyze the radical and the associated primes of a binomial ideal. Our goal is to prove that both the radical and the primary decomposition of ideals are again binomial.

The complexity of the topics does not allow us to divide the course in different sessions, but must attend to several features of this construction during the length of the course.
Complex Analysis

An introduction to the $\overline{\partial}$-problem

C. Cruz

Abstract

The Cauchy-Riemann equation (also known as the $\partial$-equation) is provided in most basic courses of complex analysis. In 1965, L. Hörmander proved the existence of solution to the $\overline{\partial}$-equation with $L^2$-estimates. Although Hörmander regarded his estimates as a particular case of a certain technique (applicable to more general differential operators), his result is important in complex analysis and complex differential geometry.

Session Outline

The course will be planned in three sessions that are outlined as follows:

1. **The $\overline{\partial}$-equation on $\mathbb{C}$:** We present a brief summary of basic concepts to study this problem. The key historical moment was the Hörmander’s $L^2$-estimates, so we will study this result in $\mathbb{C}$. However, the Hörmander’s result does not give an explicit expression of the solution, so we can analyze the Jones’ explicit solution and one application of these famous $L^2$-estimates.

2. **The $\overline{\partial}$-equation with compact support:** In this session we will focus on this problem in higher dimensions. Hence we can study this problem in the particular case with compact support.

3. **The tangential $\overline{\partial}$-equation:** Finally, we can study when a given function in the boundary $\partial \Omega$ of a domain $\Omega$ is a restriction of a holomorphic function. This allows to analyze briefly the Dirichlet problem for $\overline{\partial}$ in 1 dimension and obtain conditions in terms of the tangential $\overline{\partial}$ in dimension $n > 1$. 

Abstract

Though *Computation* spawned from mathematics when creating approaches to classical problems (e.g. Hilbert’s *Entscheidungsproblem*), and was originally regarded as a sub-branch of logic, it has grown as a field of its own and diversified into a myriad of diverse disciplines: Computer Sciences, Computational Sciences, Software Engineering, etc.

However omnipresent -and omnipotent as some believe- in matters of real-world applications, Computation is still a young discipline. As time passes by, it grows apart even further from mathematics, losing its primordial rigor (inherited by logic), and substituting it with concepts and practices from several disciplines, that obscure fundamental issues in matters of philosophy and epistemology of Computation, and that revert in seemingly inescapable issues of computerized solutions (e.g. *Software*), in matters of performance, complexity, and sustainability.

Newer paradigms like *Deep Learning* or *particle based simulation methods* make the use of mathematical models irrelevant in benefit of brute force solutions, while at the same discussions arise on how to approach theorem proof using computers.

What is nowadays the nature of the relation between Computation and Mathematics? Will we be overrun by neural networks? How can we join forces with the Computation communities of the 21st Century? These and other questions will be elaborated throughout this brief course, while several elements of computation (in its different inceptions) are explored, to provide mathematicians with tools to be more assertive when working alongside with Software Engineers or Computer Scientists.

Session Outline

The course will be planned in three sessions that are outlined as follows:

1. **Paradigms of Computation:** In this session we will review the history of computation and its paradigms, to understand the different roles mathematicians can play in computation, computer sciences, and *Software* Engineering.

2. **Modeling vs Specification:** During this session we will review two work-flows within Computation that showcase the tension between mathematics and computation. We will focus on the tension between classical modeling as proposed by Brito & Vallina.
and its nemesis (the simulation paradigm) proposed by S. Wolfram, K. Sims, and others.

3. Programming vs Software-making: During this session we will review several elements that embed the activity of programming and the discipline of software making that sets apart pure -academic- research, from R+D+i (Research, Development and innovation) to understand where do mathematicians (theoretical and applied) fit into the production chain of technology.
Abstract

The harmonic analysis deals with the decomposition of elements, in certain topological vector spaces of functions, using some linear basis which are typically distinguished by some nice representation theoretically-wise. This decomposition can be a sum, but more general it is a decomposition involving integrals. Historically, these basis elements were thought of as basic waves (fundamental harmonics).

In these sessions we will talk about weight theory, which is related with some classical operators in harmonic analysis. We say that $w$ is a weight if it is a positive and locally integrable function. In particular, we will deal with the class of Muckenhoupt weights $A_p$, $p \geq 1$. In this line, we will study the Hardy-Littlewood’s maximal operator, which is extremely related with the Muckenhoupt weights. Indeed, we will construct the $A_p$ weights from the Hardy-Littlewood’s maximal operator and we will see that the boundedness of the Hardy-Littlewood’s maximal operator over the $p$-Lebesgue weighted spaces depends on the weight being a Muckenhoupt weight. The motivation of studying such operator is due to it takes an important role in extrapolation theory, allowing to bound “easily” classical operators such as the Hilbert’s transform, the generalized Calderón-Zygmund operators, etc.

Session Outline

The sessions will be planned in three talks that are outlined as follows:

1. The Hardy-Littlewood Maximal Operator: In the first session we will introduce some concepts of measure theory, such as weak and strong type inequalities, and the Marcinkiewicz interpolation theorem. Moreover, we will define the Hardy-Littlewood maximal operator and we will show that satisfies a weak-type inequality.

2. The $A_p$ Condition: During this session we will define the Muckenhoupt weights using the Hardy-Littlewood maximal operator. What’s more, we will see that the boundedness of the Hardy-Littlewood maximal operator in a weighted Lebesgue space will be related with the fact of the weight being a Muckenhoupt weight.

3. An Extrapolation Theorem: Finally, in the last session we will see an interesting result of extrapolation by Rubio de Francia and we will study how it is related with
the Muckenhoupt weights. This will allow us to bound some classic linear operators in harmonic analysis.
Geometry

Introduction to vector bundles

A. Rojas, M. Salat

Abstract

A vector bundle on a topological space $X$ can be intuitively thought of, as a continuous way of attaching a vector space to every point $p \in X$. Geometric properties of $X$ can be translated to the language of vector bundles, so that problems can be more easily explored. For instance the problem of classification, specially when $X$ has an extra structure (e.g. $X$ is a smooth manifold or a scheme).

This course aims at introducing the theory of Vector Bundles over complex manifolds, and showcase particular interactions of this theory with algebraic geometry and complex analysis.

Session Outline

• **Introduction**: Topological motivation, basic definitions about complex manifolds and holomorphic functions on them. Motivating example: Tangent spaces. Other examples from an algebraic geometry viewpoint.

• **Vector Bundles**: Definition of a vector bundle over a complex manifold. Motivating example continued: the tangent bundle. Sections of vector bundles. Examples: tangent vector fields as sections of the tangent bundle.

• **Vector bundles and sheaves**: Basic definitions about sheaves of modules. The locally free sheaf of sections of a vector bundle. Relation between vector bundles and locally free sheaves.
Representation Theory

An introductory course to Galois representations

D. Gil

Abstract

Representation theory intends to improve the process of obtaining more information about the structure of a group, by embedding it into a group of automorphisms of a vector space.

In order to start working in representation theory, we must study first, the theory of representation in the context of finite groups and introduce the theory of characters attached to them.

Particularly, we wish to draw interest in the representations of profinite groups by endowing them with a natural topology (i.e. the Krull topology) attached to the Galois group of an extension of fields.

This gives rise to the concept of Galois representation, which is a representation of the Galois group of a (non-necessary finite) Galois extension, that will be presented as examples during the course.

Session Outline

1. **Representations of finite groups.** We begin by defining the notion of representation of a finite group and introduce some basic notions related to them, such as subrepresentations and irreducible representations. We shall define the notion of character attached to a representation and study the theory of characters.

2. **Representations of profinite groups.** The notion of representation of a non-finite group requires the introduction of a topology. We shall review the notion of projective limit in order to define profinite groups and study the topology attached to them. We use this to define representations of profinite groups and we connect it to Galois theory by the construction of the Krull topology of the Galois group of a (non-necessary finite) Galois extension.

3. **Galois representations.** A Galois representation is nothing but a representation of the Galois group of some extension. In this session we will study some examples related to algebraic number theory. We shall review the \( l \)-adic Tate module attached to an elliptic curve \( E \) and construct the \( l \)-adic representation attached to \( E \). We shall introduce the Artin and Swan characters and study briefly the corresponding representations.