Four definitions for the fractional Laplacian

N. Accomazzo (UPV/EHU), S. Baena (UB), A. Becerra Tomé (US), J. Martínez (BCAM), A. Rodríguez (UCM), I. Soler (UM)

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[1] International Women's Day

A pointwise definition of the fractional Laplacian

Laplace fractional operator: several points of view

- Functional analysis: M. Riesz, S. Bochner, W. Feller, E. Hille, R. S. Phillips, A. V. Balakrishnan, T. Kato, Martínez–Carracedo y Sanz–Alix, K. Yosida
- potencial theory for fractional laplacian: N. S. Landkof
- Lévy's processes: K. Bogdan e.a.
- Partial Derivative Ecuations: L. Caffarelli y L. Silvestre
- Scattering theory: C. R. Graham y M. Zworski, S-Y. A. Chang y M.d.M. González
- Kato's square root (solved by P. Auscher e.a.)

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Basic example of fractional operator : **fractional laplacean**

The working space

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$\mathscr{S}(\mathbb{R}^n)$ is the space $C^\infty(\mathbb{R}^n)$ of functions that

$$\|f\|_p = \sup_{|\alpha|\leq p} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^{p/2} |\partial^\alpha f(x)| < \infty \quad p \in \mathbb{N}\cup\{0\}$$

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This space endowed with the metric topology

$$d(f,g) = \sum_{p=0}^{\infty} 2^{-p} \frac{\|f-g\|_p}{1+\|f-g\|_p}$$

First definition motivation

Let $f \in C^2(a, b)$, then for every $x \in (a, b)$ one has

$$-f''(x) = \lim_{y \to 0} \frac{2f(x) - f(x+y) - f(x-y)}{y^2}$$

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If we introduce the spherical and solid averaging operators

$$\mathscr{M}_y f(x) = \frac{f(x+y) + f(x-y)}{2} \qquad \mathscr{A}_y f(x) = \frac{1}{2y} \int_{x-y}^{x+y} f(t)\, dt$$

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then we can reformulate $-f''(x)$ like this

$$-f''(x) = 2 \lim_{y \to 0} \frac{f(x) - \mathscr{M}_y f(x)}{y^2} = 6 \lim_{y \to 0} \frac{f(x) - \mathscr{A}_y f(x)}{y^2}$$

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Thus, if we bear in mind that $-\Delta f = -\sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2}$ and making an extension of the spherical and solid averaging operators

$$\mathscr{M}_y f(x) = \frac{1}{\sigma_{(n-1)} r^{(n-1)}} \int_{S(x,r)} f(y)\, d\sigma(y) \quad \mathscr{A}_y f(x) = \frac{1}{\omega_n r^n} \int_{B(x,r)} f(y)\, dy$$

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then we have

$$-\Delta f(x) = 2n \lim_{y \to 0} \frac{f(x) - \mathscr{M}_y f(x)}{y^2} = 2(n+2) \lim_{y \to 0} \frac{f(x) - \mathscr{A}_y f(x)}{y^2}$$

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Observation 3: since as $s \to 1^-$ the fractional Laplacean tends (at least, formally right now) to $(-\Delta)$, one might surmise that in the regime $1/2 < s < 1$ the operator $(-\Delta)^s$ should display properties closer to those of the classical Laplacian, whereas since $(-\Delta)^s \to I \;\; as \;\; s \to 0^+$, the stronger discrepancies might present themselves in the range $0 < s < 1/2$

$(-\Delta u(x))$ is well-defined

It is important to observe that the integral in the right-hand side is convergent. In order to see this, it suffices to write:

$$\int_{\mathbb{R}^n} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\,dy = \int_{|y|\le 1} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\,dy +$$

$$+ \int_{|y|\ge 1} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\,dy = (a)+(b)$$

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$$|(a)| \le \int_{|y|\le 1} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{n+2s}}\, dy$$

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Using Taylor $2u(x)-u(x+y)-u(x-y) = -\langle \nabla^2 u(x)y, y\rangle + o(|x|^3)$

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$$\leq \int_{|y|\leq 1} \frac{|\nabla^2 u(x) y||y| + |o(|x|^3)|}{|y|^{n+2s}}\, dy \leq \int_{|y|\leq 1} \frac{|\nabla^2 u(x)||y|^2 + |o(|x|^3)|}{|y|^{n+2s}}\, dy$$

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Now, for (b):

$$\left| \int_{|y|\geq 1} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\, dy \right| \leq 4\|u\|_{L_\infty(\mathbb{R}^n)} \int_{|y|\geq 1} \frac{1}{|y|^{n+2s}}\, dy < \infty$$

Translations and dilations

Let $h \in \mathbb{R}^n$ and $\lambda > 0$, the translation and dilation operators are defined, respectively, by

$$\tau_h f(x) = f(x+h); \quad \delta_\lambda f(x) = f(\lambda x)$$

for every $f\colon \mathbb{R}^n \to \mathbb{R}$ and every $x \in \mathbb{R}^n$

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Proposition 1

Let $u \in \mathcal{S}(\mathbb{R}^n)$, then for every $h \in \mathbb{R}^n$ and $\lambda > 0$ we have

$$(-\Delta)^s(\tau_h u) = \tau_h((-\Delta)^s u)$$

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In particular, $(-\Delta)^s$ is a homogeneous operator of order $2s$.

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Recall that the orthogonal group is

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We say that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ has spherical symmetry if $f(x) = f^*(|x|)$ for some $f^*\colon \mathbb{R}^n \to \mathbb{R}$ or, equivalently, if $f(Tx) = f(x)$ for every $T \in \mathbb{O}(n)$ and every $x \in \mathbb{R}$

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Proposition 2

Let $u \in \mathcal{S}(\mathbb{R}^n)$ (actually, it is enough that $u \in \mathcal{C}^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$) be a function with spherical symmetry. Then, $(-\Delta)^s$ has spherical symmetry.

Orthogonal group (Cont.)

PROOF.-

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Let's see that $(-\Delta)^s u(Tx) = (-\Delta)^s u(x)$ for each $T \in \mathbb{O}(n)$ and each $x \in \mathbb{R}^n$

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Alternative expression for the fractional Laplacian

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Theorem

Let $u \in \mathcal{S}(\mathbb{R}^n)$, then

$$(-\Delta)^s u(x) = \gamma(n,s) P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy$$

where P.V. means the Cauchy's principal value, i.e.

$$P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy$$

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Another two definitions of the fractional Laplacian

Fourier transform

We recall the definition of the Fourier transform, $\mathcal{F}$, of a function $f \in \mathcal{S}(\mathbb{R}^n)$:

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx, \qquad \xi \in \mathbb{R}^n,$$

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whose inverse function is given by

$$\mathcal{F}^{-1}(f)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\xi) e^{ix\cdot\xi} d\xi, \qquad x \in \mathbb{R}^n,$$

Fourier transform

We recall the definition of the Fourier transform, $\mathcal{F}$, of a function $f \in \mathcal{S}(\mathbb{R}^n)$:

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx, \qquad \xi \in \mathbb{R}^n,$$

whose inverse function is given by

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so that

$$f(x) = \mathcal{F}^{-1} \circ \mathcal{F}(f)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i x \cdot \xi} d\xi, \quad x \in \mathbb{R}^n.$$

Definition of $(-\Delta)^s$ via the heat semigroup $e^{t\Delta}$

We will define $(-\Delta)^s f$ in terms of the heat semigroup $e^{t\Delta}$, which is nothing but an operator such that maps every function $f \in \mathcal{S}(\mathbb{R}^n)$ to the solution of the heat equation with initial data given by f:

$$\begin{cases} v_t = \Delta v, & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ v(x,0) = f(x), & x \in \mathbb{R}^n. \end{cases}$$

Using Fourier transform and its inverse and with a bit of magic, we can write

$$e^{t\Delta} f(x) := v(x,t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-t|\xi|^2} \hat{f}(\xi) e^{ix\cdot\xi} d\xi = \int_{\mathbb{R}^n} W_t(x-z) f(z) dz,$$

where

$$W_t(x) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n,$$

is the Gauss-Weierstrass kernel.

Definition of $(-\Delta)^s$ via the heat semigroup $e^{t\Delta}$

Inspired by the following numerical identity: for $\lambda > 0$,

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\lambda} - 1) \frac{dt}{t^{1+s}}, \qquad 0 < s < 1,$$

where

$$\Gamma(-s) = \int_0^\infty (e^{-r} - 1) \frac{dr}{r^{1+s}} < 0;$$

we can think of $(-\Delta)^s$ as the following operator

$$(-\Delta)^s f(x) \sim \frac{1}{\Gamma(-s)} \int_0^\infty (e^{t\Delta} f(x) - f(x)) \frac{dt}{t^{1+s}}, \qquad 0 < s < 1.$$

Definition of $(-\Delta)^s$ via the Fourier Transform

By the well-known properties of $\mathcal{F}$ with respect to derivatives, we have that, for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\mathcal{F}[-\Delta f](\xi) = |\xi|^2 \mathcal{F}(f)(\xi), \qquad \xi \in \mathbb{R}^n,$$

so it is reasonable to write something like

$$(-\Delta)^s f(x) \sim \mathcal{F}^{-1}[|\cdot|^{2s} \mathcal{F}(f)](x), \qquad x \in \mathbb{R}^n,\ 0 < s < 1.$$

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By the well-known properties of $\mathcal{F}$ with respect to derivatives, we have that, for $f \in \mathcal{S}(\mathbb{R}^n)$,

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Theorem (Lemma 2.1. P. Stinga's PhD thesis)

Given $f \in \mathcal{S}(\mathbb{R}^n)$ and $0 < s < 1$,

$$\mathcal{F}^{-1}[|\cdot|^{2s}\mathcal{F}(f)](x) = \frac{1}{\Gamma(-s)}\int_0^\infty (e^{t\Delta}f(x) - f(x))\frac{dt}{t^{1+s}}, \quad x \in \mathbb{R}^n$$

and this two functions coincide in a pointwise way with $(-\Delta)^s f(x)$ when the constant $\gamma(n,s)$ in its definition is given by

$$\gamma(n,s) = \frac{4^s\Gamma(n/2+s)}{-\pi^{n/2}\Gamma(-s)} > 0.$$

Everybody wants to be the fractional Laplacian

Let $x \in \mathbb{R}^n$. By Fubini's theorem and inverse Fourier formula,

$$\begin{aligned}\frac{1}{\Gamma(-s)}\int_0^\infty (e^{t\Delta}f(x)-f(x))\frac{dt}{t^{1+s}} &= \frac{1}{\Gamma(-s)}\int_{\mathbb{R}^n}\int_0^\infty (e^{-t|\xi|^2}-1)\frac{dt}{t^{1+s}}\hat{f}(\xi)e^{ix\cdot\xi}d\xi \\ &= \frac{1}{\Gamma(-s)}\int_{\mathbb{R}^n}\int_0^\infty (e^{-r}-1)\frac{dr}{r^{1+s}}|\xi|^{2s}\hat{f}(y)e^{ix\cdot\xi}dy \\ &= \int_{\mathbb{R}^n}|\xi|^{2s}\hat{f}(\xi)e^{ix\cdot\xi}d\xi = \mathcal{F}^{-1}[|\cdot|^{2s}\mathcal{F}(f)](x).\end{aligned}$$

Since $f \in \mathcal{S}(\mathbb{R}^n)$, we have that

$$\int_0^\infty |e^{t\Delta}f(x)-f(x)|\frac{dt}{t^{1+s}} < \infty,$$

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Next, we will see that

$$\frac{1}{\Gamma(-s)}\int_0^\infty (e^{t\Delta}f(x)-f(x))\frac{dt}{t^{1+s}} = \frac{4^s\Gamma(n/2+s)}{-\pi^{n/2}\Gamma(-s)}\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{f(x)-f(z)}{|x-z|^{n+2s}}dz, \quad x\in\mathbb{R}^n.$$

Let $\varepsilon > 0$. Using that $\|W_t(x-\cdot)\|_{L^1(\mathbb{R}^n)} = 1$ for any $x \in \mathbb{R}^n$,

$$\begin{aligned}\int_0^\infty (e^{t\Delta}f(x)-f(x))\frac{dt}{t^{1+s}} &= \int_0^\infty\int_{\mathbb{R}^n} W_t(x-z)(f(z)-f(x))dz\frac{dt}{t^{1+s}}\\ &= I_\varepsilon + II_\varepsilon.\end{aligned}$$

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Using Fubini's theorem,

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I_\varepsilon &= \int_{|x-z|>\varepsilon} \int_0^\infty (4\pi t)^{-n/2} e^{-\frac{|x-z|^2}{4t}} (f(z) - f(x)) \frac{dt}{t^{1+s}} dz \\
&= \int_{|x-z|>\varepsilon} (f(z) - f(x)) \int_0^\infty (4\pi t)^{-n/2} e^{-\frac{|x-z|^2}{4t}} \frac{dt}{t^{1+s}} dz \\
&= \int_{|x-z|>\varepsilon} (f(x) - f(z)) \frac{4^s \Gamma(n/2+s)}{-\pi^{n/2}} \frac{1}{|x-z|^{n+2s}} dz
\end{aligned}
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where we used the change of variables $r = \frac{|x-z|^2}{4t}$.
Observe that I_ε converges absolutely for any $\varepsilon > 0$ since f is bounded, so the use of Fubini's theorem is licit.

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Using polar coordinates,

$$II_\varepsilon = \int_0^\infty \int_{|x-z|<\varepsilon} W_t(x-z)(f(z)-f(x))dz \frac{dt}{t^{1+s}}$$
$$= \int_0^\infty (4\pi t)^{-n/2} \int_0^\varepsilon e^{-\frac{r^2}{4t}} r^{n-1} \int_{|z'|=1} (f(x+rz')-f(z))dS(z')dr \frac{dt}{t^{1+s}}.$$

By Taylor's theorem, using the symmetry of the sphere,

$$\int_{|z'|=1} (f(x+rz')-f(z))dS(z') = C_n r^2 \Delta f(x) + O(r^3),$$

thus

$$|II_\varepsilon| \le C_{n,\Delta f(x)} \int_0^\varepsilon r^{n+1} \int_0^\infty \frac{e^{-\frac{r^2}{4t}}}{t^{n/2+s}} \frac{dt}{t}$$
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Everybody IS the fractional Laplacian

This proves that $II_\varepsilon \to 0$ as $\varepsilon \to 0$, so

$$\int_0^\infty \int_{\mathbb{R}^n} W_t(x-z)(f(z)-f(x))dz\frac{dt}{t^{1+s}} = \lim_{\varepsilon\to 0} I_\varepsilon + II_\varepsilon$$
$$= \frac{4^s\Gamma(n/2+s)}{-\pi^{n/2}}\text{P.V.}\int_{\mathbb{R}^n}\frac{f(x)-f(z)}{|x-z|^{n+2s}}dz$$

This kind of computations (bearing in mind the exact expression of the constant $\gamma(n,s)$) also prove the following pointwise convergence

$$(-\Delta)^s f(x) \to -\Delta f(x), \qquad x \in \mathbb{R}^n \text{ as } s \to 0^+,$$

when $f \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ (observe that, in $\mathcal{S}(\mathbb{R}^n)$ this is obvious by the definition via Fourier transform).

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This kind of computations (bearing in mind the exact expression of the constant $\gamma(n,s)$) also prove the following pointwise convergence

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And last but not least

Extension Problem

Let $s \in (0,1)$ and consider $a = 1-2s$. We want to solve the extension problem

$$\begin{cases} L_a U(x,y) = \operatorname{div}_{x,y}(y^a \nabla_{x,y} U) = 0, & x \in \mathbb{R}^n_+, y > 0, \\ U(x,0) = u(x), \\ U(x,y) \to 0 \text{ as } y \to \infty. \end{cases}$$

The previous system can be written as

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is the Poisson Kernel for the extension problem in the half-space $\mathbb{R}^{n+1}_+$. For U as in (2) one has

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If we take a partial Fourier transform of (1)

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$$y^2 Y'' + (1-2\alpha)yY'(y) + [\beta^2\gamma^2 y^{2\gamma} + (\alpha - \nu^2\gamma^2)]Y(y) = 0 \tag{5}$$

$$\alpha = s, \gamma = 1, \nu = s, \beta = 2\pi|\xi|.$$

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The general solutions of (5) are given by

$$\hat{U}(\xi, y) = Ay^s I_s(2\pi|\xi|y) + By^s K_s(2\pi|\xi|y)$$

where I_s and K_s are the Bessel functions of second and third kind, both independent solutions of the modified Bessel equation of order s

$$z^2\phi'' + z\phi' - (z^2 + s^2)\phi = 0 \tag{6}$$

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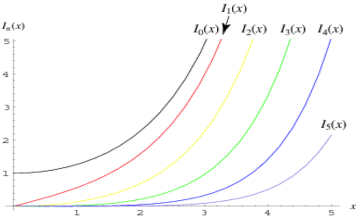

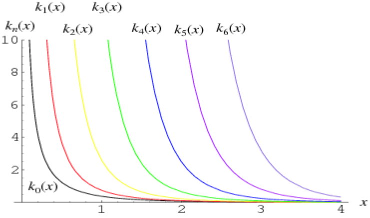

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The condition $\hat{U}(\xi, y) \to 0$ as $y \to \infty$ forces $A = 0$. Using I_s asymptotic behavior,

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Theorem 2 (Fourier-Bessel Representation)

Let $u(x) = f(|x|)$, and suppose that

$$t \mapsto t^{n/2} f(t) J_{n/2-1}(2\pi|\xi|t) \in L^1(\mathbb{R}^n).$$

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We finally reach the conclusion

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Differentiating both sides respect to y we obtain

$$y^{1-2s}\partial_y U(x,y) = 2s\frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)} \int_{\mathbb{R}^n} \frac{u(z)-u(x)}{(y^2+|z-x|^2)^{(n+2s)/2}}\, dz + O(y^2).$$

Let $u \in \mathcal{S}(\mathbb{R}^n)$ and consider the solution $U(x,y) = (P_s(\cdot,y) \star u)(x)$ to the extension problem (1). We can write

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Now, letting $y \to 0^+$ and using the Lebesgue dominated convergence theorem, we thus find

$$\begin{aligned}\lim_{y\to 0^+} y^{1-2s}\partial_y U(x,y) &= 2s\frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)}\text{P.V.}\int_{\mathbb{R}^n} \frac{u(z)-u(x)}{(|z-x|^2)^{(n+2s)/2}}\,dz \\ &= -2s\frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)}\gamma(n,s)^{-1}(-\Delta)^s u(x).\end{aligned}$$

Finally, recall that

$$\gamma(n,s) = \frac{s2^{2s}\Gamma(n/2+s)}{\pi^{n/2}\Gamma(1-s)}.$$

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□

La exposición está basada en:

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Thanks for your attention
Eskerrik asko zuen arretarengatik