Stable homology of spaces of embedded surfaces:
Closed background manifolds

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The space of embedded surfaces in a manifold

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$$\mathcal{E}_g(M) \longrightarrow \mathcal{E}_g(M)$$

which is also a weak homotopy equivalence.
Theorem A (C. – Randal-Williams)

If $M$ is simply connected and of dimension at least 5, and $\partial M \neq \emptyset$, then the scanning map

$$\mathcal{I}_g : \mathcal{E}_g^\nu(M) \to \Gamma_c(S(TM) \to M)_g$$

induces an isomorphism in integral homology in degrees $k \leq \frac{2}{3}(g - 1)$.

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From an inner product vector space $V$, we can construct the following:

- The Grassmannian of oriented linear 2-planes in $V$,

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  Forgetting the vector $v$ we obtain a vector bundle of rank $\dim V - 2$:
  \[
  \gamma^+_2(V) \longrightarrow \text{Gr}^+_2(V)
  \]

- The Thom space of this vector bundle,
  \[
  S(V) := \text{Th}(\gamma^+_2(V) \rightarrow \text{Gr}^+_2(V))
  \]
Consider now a vector bundle $E \to M$ endowed with a metric.

**Definition**

The fibre bundle $S(E) \to M$ is the result of applying the construction $S$ fibrewise to the fibre bundle $E \to M$.

If $E_p$ is the fibre of $E$ over $p \in M$, then we obtain a fibre bundle

$$S(E_p) \longrightarrow S(E) \longrightarrow M.$$ 

In particular, for the tangent bundle of a *Riemannian* manifold $M$, we obtain a fibre bundle

$$S(T_pM) \longrightarrow S(TM) \longrightarrow M.$$
The scanning map $\mathcal{I}_g : \mathcal{E}_g(M) \rightarrow \Gamma_c(S(TM) \rightarrow M)_g$

The scanning map approximates each oriented surface $W \subset M$ with its tangent bundle.

First, if $p \in W$, we have the Gauss map. Second, if $\pi : U \rightarrow W \subset U$ is a tubular neighbourhood of $W$, we can identify $T_p M$ as a translation of $T_{\pi(p)} M$, and $T_{\pi(p)} W$ as an affine subspace of $T_p M$. Third, we may send any other point to the point at infinity (interpreted as the empty subspace).
The scanning map

We have obtained the *scanning map*:

\[ \mathcal{I}_g : \mathcal{E}^\nu_g(M) \to \Gamma_c(S(TM) \to M) \]
\[ (W, u) \mapsto \mathcal{I}_g(W, u). \]
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\textbf{Lemma}

If $M$ is simply connected and of dimension at least 5, then

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*The space of compactly supported genus* \(g\) *sections* \(\Gamma_c(S(TM) \rightarrow M)_g\) *is the union of those components labeled by* \(H_2(M; \mathbb{Z}) \times \{2 - 2g\}\).
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**Lemma**

*The image of* \( \mathcal{I}_g \) *is contained in* \( \Gamma_c(S(TM) \to M)_g \).
Theorem A (C. – Randal-Williams)

If $M$ is simply connected and of dimension at least 5, and $\partial M \neq \emptyset$, then the scanning map

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**Relation to previous works**

<table>
<thead>
<tr>
<th>$B\Sigma_n$</th>
<th>$C_n(M) := \text{Emb}([n], M)/\Sigma_n$</th>
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<tbody>
<tr>
<td>Thm B</td>
<td>Nakaoka '60</td>
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<td>Thm A</td>
<td>Barratt–Priddy '72</td>
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<td>$BDiff^+(\Sigma_g)$</td>
<td>$E_g(M) := \text{Emb}(\Sigma_g, M)/\text{Diff}^+(\Sigma_g)$</td>
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**Thm B** Martin Palmer: Stability for embedded disconnected submanifolds.
Definition

A semi-simplicial space $X_\bullet$ is a simplicial space without degeneracies, that is, a functor $X_\bullet : \Delta_{\text{inj}} \to \text{Spaces}$ from the full subcategory $\Delta_{\text{inj}} \subset \Delta$ whose morphisms are the inclusions. A maps of semi-simplicial spaces is a natural transformation.

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An augmented semi-simplicial space is a triple consisting of
- a space $X$,
- a semi-simplicial space $X_\bullet$ and
- a map $\epsilon : X_0 \rightarrow X$ (called augmentation) that equalizes the face maps $\partial_0 : X_1 \rightarrow X_0$ and $\partial_1 : X_1 \rightarrow X_0$.

We denote by $\epsilon_i : X_i \rightarrow X$ the unique composition of face maps and $\epsilon$. A map between augmented semi-simplicial spaces is a pair $(X \rightarrow Y, X_\bullet \rightarrow Y_\bullet)$ that commutes with the augmentation maps.
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An augmented semi-simplicial space $(X, X \bullet, \epsilon)$ is the same as a map from $X \bullet$ to the constant semi-simplicial space $X$ whose face maps are identities.
Example (Hatcher, *Algebraic Topology*)

A semi-simplicial space with values in discrete spaces (aka sets) is called a $\Delta$-set.
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There is a functor (the *realization*)

$$\| - \| : \text{Semi-simplicial spaces} \rightarrow \text{Spaces},$$

that sends the constant semi-simplicial space $X$ to $X$, hence an augmentation map $X_0 \rightarrow X$ induces a map $\|X_{\bullet}\| \rightarrow X$, which we call *realized augmentation*. 
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**Definition**

*We say that a semi-simplicial space $X_\bullet$ is a resolution of a space $X$ if $X_\bullet$ is augmented over $X$ and the realized augmentation is a weak homotopy equivalence. A resolution of a map $f : X \to Y$ is a pair $X_\bullet, Y_\bullet$ of resolutions of $X, Y$ and a map $f_\bullet : X_\bullet \to Y_\bullet$ that extends the map $f$.*
Let \((X, X_\bullet, \epsilon)\) be an augmented semi-simplicial space.

**Lemma**

*If \(x \in X\), then there is a homotopy fibre sequence*

\[
\|\text{hofib}_x(\epsilon_\bullet)\| \to \|X_\bullet\| \to X.
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We say that \((X, X_\bullet, \epsilon)\) is an augmented *topological flag complex* if in addition

- the product map \(X_i \to X_0 \times_X \cdots \times_X X_0\) is an open embedding;
- a tuple \((x_0, \ldots, x_i)\) is in \(X_i \iff (x_j, x_k) \in X_1\) for all \(0 \leq j < k \leq i\).
Techniques I: How to prove that something is a resolution

Let \((X, X_\bullet, \epsilon)\) be an augmented semi-simplicial space.

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**Lemma (Galatius–Randal-Williams ’12)**

Suppose in addition that

1. \(\epsilon: X_0 \to X\) has local sections;
2. given any finite collection \(\{x_1, \ldots, x_n\} \subset X_0\) in a single fibre of \(\epsilon\) over some \(x \in X\), there is a \(x_\infty\) in that fibre such that each \((x_j, x_\infty) \in X_1\).

Then \(\|\epsilon_\bullet\|: \|X_\bullet\| \to X\) is a weak homotopy equivalence.
Definition (Palais ’60, Cerf ’61)

If $G$ is a (topological) group acting on $X$, we say that $X$ is $G$-locally retractile if, for each point $x \in X$, the orbit map $G \times \{x\} \rightarrow X$ that sends $g \mapsto g \cdot x$ has local sections (in the weak sense).
Techniques II: How to prove that something is a fibration

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**Lemma (Palais '60, Cerf '61)**

If $X$ and $Y$ are $G$-spaces, and $f : X \to Y$ is $G$-equivariant and $Y$ is $G$-locally retractile, then $f$ is a locally trivial fibration.
Techniques II: How to prove that something is a fibration

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*If $X$ and $Y$ are $G$-spaces, and $f : X \to Y$ is $G$-equivariant and $Y$ is $G$-locally retractile, then $f$ is a locally trivial fibration.*

**Proposition (Palais ’60, Cerf ’61, Lima ’63, Binz–Fischer ’81)**

*The space of embeddings of a compact manifold into a manifold $M$ and the space $\mathcal{E}_g(M)$ are $\text{Diff}(M)$-locally retractile.*
Lemma

If $X_\bullet \to X$ is an $m$-resolution, $X_i$ is homologically $(n - i)$-connected, and $m \geq n$, then $X$ is homologically $n$-connected.
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If $X_{\bullet} \rightarrow X$ is an $m$-resolution, $X_i$ is homologically $(n - i)$-connected, and $m \geq n$, then $X$ is homologically $n$-connected.

Lemma

If a bundle map over $B$

\[
\begin{array}{ccc}
F_p & \rightarrow & F'_p \\
\downarrow & & \downarrow \\
E & \rightarrow & E' \\
\downarrow & & \downarrow \\
B & \rightarrow & B
\end{array}
\]

satisfies that for each $p \in B$ the induced map of fibres $F_p \rightarrow F'_p$ is homologically $k$-connected, then the map between total spaces is also homologically $k$-connected.
Proof: The two steps

1. construct **resolutions** of the source and target of the scanning map

\[ \mathcal{F}_g(M) \longrightarrow \mathcal{E}_g^\nu(M), \quad \mathcal{G}_g(M) \longrightarrow \Gamma_c(S(TM) \rightarrow M)_g \]

and a resolution of the scanning map

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\[ \mathcal{E}_g^\nu(M) \rightarrow \Gamma_c(S(TM) \rightarrow M)_g. \]

2. Construct vertical maps (called approximations)

\[ \mathcal{E}_g^\nu(M \setminus \{p_1, \ldots, p_i\}) \rightarrow \Gamma_c(S(TM \setminus \{p_1, \ldots, p_i\}) \rightarrow M \setminus \{p_1, \ldots, p_i\})_g \]

\[ \mathcal{F}_g(M)_i \rightarrow \mathcal{G}_g(M)_i \]

from a scanning map for which Theorem A applies, and deduce that the bottom map is homologically \( \frac{2}{3}(g - 1) \)-connected.
Proof: Resolution of $\mathcal{E}_g^\nu(M)$

Let $\mathcal{F}_g(M)_i$ be the space of tuples $(W, a, d_0, \ldots, d_i)$ where

1. $(W, u) \in \mathcal{E}_g^\nu(M)$
2. $d_0, \ldots, d_i: D^n \to M$ are disjoint embeddings of discs such that $d_j(0) \notin U$ for all $j$.

These spaces form a semi-simplicial space $\mathcal{F}_g(M)_\bullet$ where the $j$th face map forgets the $j$th disc, and there is an augmentation to $\mathcal{E}_g^\nu(M)$ that forgets all the discs.
Proof: Resolution of $E^\nu_g(M)$

Let $F_g(M)_i$ be the space of tuples $(W, a, d_0, \ldots, d_i)$ where

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$F_g(M)$ is a resolution of $E^\nu_g(M)$.
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**Proposition**

$F_g(M)$ is a resolution of $E^\nu_g(M)$.

**Proof.**

Let $F'_g(M)_\bullet$, the semi-simplicial space defined as $F_g(M)_\bullet$, except that the embeddings are only required to be disjoint at the centers of the discs. Then
- the inclusion $F_g(M)_\bullet \subset F'_g(M)_\bullet$ is a levelwise equivalence.
- $F'_g(M)_\bullet$ is a topological flag complex augmented over $E^\nu_g(M)$.
- $F'_g(M)_\bullet$ satisfies the conditions of our lemma on topological flag complexes, hence is a resolution.
Proof: Resolution of $\Gamma^c_c(S(TM) \to M)_g$

Let $G_g(M)_i$ be the space of tuples $(f, d_0, \ldots, d_i, h_0, \ldots, h_i)$ where

1. $f \in \Gamma^c_c(S(TM) \to M)_g$;
2. $d_0, \ldots, d_i: D^n \to M$ are disjoint embeddings of discs such that $d_j(0) \notin U$ for all $j$.
3. $h_0, \ldots, h_i$ are smooth homotopies of sections of $d_j^*(S(TM))$, constant near the boundary, and such that

$$h_j(x, 0) = f \circ d_j, \quad h_j(0, 1) = \infty.$$ 

The $j$th face map forgets $d_j$ and $h_j$, and there is an augmentation to $\Gamma^c_c(S(TM) \to M)_g$ by forgetting all discs and homotopies.
Proof: Resolution of $\Gamma_c(S(TM) \to M)_g$

Let $G_g(M)_i$ be the space of tuples $(f, d_0, \ldots, d_i, h_0, \ldots, h_i)$ where

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Proposition

$G_g(M)_\bullet$ is a resolution of $\Gamma_c(S(TM) \to M)_g$.

Proof.
Proof: Resolution of the scanning map

We can extend the scanning map to a map of resolutions:

\[
\begin{array}{c}
\mathcal{F}_g(M) \quad \longrightarrow \quad \mathcal{G}_g(M) \\
\downarrow \quad \quad \downarrow \\
\mathcal{E}_g^\nu(M) \quad \longrightarrow \quad \Gamma_c(S(TM) \to M)_g
\end{array}
\]

by sending a tuple \((W, u, d_0, \ldots, d_i)\) to \((\mathcal{S}(W, u), d_0, \ldots, d_i, h_0, \ldots, h_i)\), where \(h_j\) are constant homotopies.
Proof: First step accomplished

1. construct resolutions of the source and target of the scanning map

\[ \mathcal{F}_g(M) \longrightarrow \mathcal{E}_g(M), \quad \mathcal{G}_g(M) \longrightarrow \Gamma_c(S(TM) \to M)_g \]

and a resolution of the scanning map

\[ \mathcal{E}_g(M) \longrightarrow \Gamma_c(S(TM) \to M)_g. \]

2. Construct vertical maps (called approximations)

\[ \mathcal{E}_g(M \setminus \{p_1, \ldots, p_i\}) \longrightarrow \Gamma_c(S(TM \setminus \{p_1, \ldots, p_i\}) \to M \setminus \{p_1, \ldots, p_i\})_g \]

from a scanning map for which Theorem A applies, and deduce that the bottom map is homologically \( \frac{2}{3}(g - 1) \)-connected.
Proof: The approximation maps

Forgetting the surface + tubular neighbourhood or the section defines a pair of maps

\[ \mathcal{F}_g(M)_i \quad \downarrow \quad \mathcal{G}_g(M)_i \]

\[ C_i(M) \quad \downarrow \quad C_i(M), \]

to the space \( C_i(M) := \text{Emb}([i] \times D^d, M). \)
Proof: The approximation maps

Forgetting the surface $\mathcal{W} +$ tubular neighbourhood or the section gives homotopy fibre sequences

$$
\begin{align*}
\mathcal{E}_g^\nu (M \setminus p) & \quad \rightarrow \quad \Gamma_c (S(TM \setminus p) \rightarrow M \setminus p)_g \\
\mathcal{F}_g (M)_i & \quad \rightarrow \quad \mathcal{G}_g (M)_i \\
C_i (M) & \quad \rightarrow \quad C_i (M),
\end{align*}
$$

\[
\xymatrix{ 
\mathcal{E}_g^\nu (M \setminus p) \ar[r] & \Gamma_c (S(TM \setminus p) \rightarrow M \setminus p)_g \\
\mathcal{F}_g (M)_i \ar[r] & \mathcal{G}_g (M)_i \\
C_i (M) \ar[u] & C_i (M), \quad \text{to the space } C_i (M) := \text{Emb}([i] \times D^d, M). \text{ The fibre is taken over the point } (d_0, \ldots, d_j) \text{ and } p = \{d_0(0), \ldots, d_i(0)\}.}
\]
Proof: The approximation maps

Forgetting the surface + tubular neighbourhood or the section defines a pair of maps

\[
\begin{align*}
E_\nu(M \setminus p) & \longrightarrow \Gamma_c(S(TM \setminus p) \to M \setminus p)_g \\
\downarrow & \quad \downarrow \\
F_g(M)_i & \longrightarrow G_g(M)_i \\
\downarrow & \quad \downarrow \\
C_i(M) & \longrightarrow C_i(M),
\end{align*}
\]

to the space \(C_i(M) := \text{Emb}([i] \times D^d, M)\). The fibre is taken over the point \((d_0, \ldots, d_j)\) and \(p = \{d_0(0), \ldots, d_i(0)\}\).
Proof: The approximation maps

Forgetting the surface + tubular neighbourhood or the section defines a pair of maps

\[ \mathcal{E}_g(M \setminus p) \longrightarrow \Gamma_c(S(TM \setminus p) \rightarrow M \setminus p)_g \]

\[ \mathcal{F}_g(M)_i \longrightarrow \mathcal{G}_g(M)_i \]

\[ C_i(M) \longrightarrow C_i(M), \]

to the space \( C_i(M) := \text{Emb}([i] \times D^d, M) \). The fibre is taken over the point \((d_0, \ldots, d_j)\) and \(p = \{d_0(0), \ldots, d_i(0)\}\).

The scanning map commutes with the map between spaces of \( i \)-simplices.
Proof: The approximation maps

Forgetting the surface + tubular neighbourhood or the section defines a pair of maps

\[
\begin{array}{ccc}
\mathcal{E}_g^\nu(M \setminus p) & \longrightarrow & \Gamma_c(S(TM \setminus p) \rightarrow M \setminus p)_g \\
\downarrow & & \downarrow \\
\mathcal{F}_g(M)_i & \longrightarrow & \mathcal{G}_g(M)_i \\
\downarrow & & \downarrow \\
C_i(M) & \longrightarrow & C_i(M),
\end{array}
\]

to the space \( C_i(M) := \text{Emb}([i] \times D^d, M) \). The fibre is taken over the point \((d_0, \ldots, d_i)\) and \( p = \{d_0(0), \ldots, d_i(0)\} \).

The scanning map commutes with the map between spaces of \( i \)-simplices.

Corollary

Since the scanning map on the fibres is a homology isomorphism in degrees \( * \leq \frac{2}{3}(g - 1) \), it follows from a previous lemma that the map between total spaces is a homology isomorphism in those degrees.
Proof: Second step accomplished

1. Construct resolutions of the source and target of the scanning map

\[ \mathcal{F}_g(M) \rightarrow \mathcal{E}_g(M), \quad \mathcal{G}_g(M) \rightarrow \Gamma_c(S(TM) \rightarrow M)_g \]

and a resolution of the scanning map

\[ \mathcal{F}_g(M) \rightarrow \mathcal{E}_g(M) \rightarrow \Gamma_c(S(TM) \rightarrow M)_g. \]

2. Construct a map of pairs (called *approximation*)

\[ \mathcal{E}_g(M \setminus \{p_1, \ldots, p_i\}) \rightarrow \Gamma_c(S(TM \setminus \{p_1, \ldots, p_i\}) \rightarrow M \setminus \{p_1, \ldots, p_i\}_g \]

\[ \mathcal{F}_g(M)_i \rightarrow \mathcal{G}_g(M)_i \]

from a scanning map for which Theorem A applies, and deduce that the bottom map is homologically \( \frac{2}{3}(g - 1) \)-connected.