Localizations of models of theories with arities

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Abstract

We give a necessary and sufficient condition for the existence of liftings of enriched localizations and colocalizations on a bicomplete closed symmetric monoidal category $\mathcal{V}$ to models of algebraic theories enriched in $\mathcal{V}$ with arbitrary arities. This condition is automatically fulfilled for single-sorted finite-product theories if $\mathcal{V}$ is additive and generated by multiples of the monoidal unit.

Introduction

It is well known that augmented or coaugmented idempotent functors preserve many kinds of algebraic structures. This phenomenon was first observed in algebraic topology when Bousfield found in [8] that homological localizations preserve products of Eilenberg–Mac Lane spaces, which are precisely the spaces homotopy equivalent to commutative topological groups. It was later proved that arbitrary homotopical localizations and cellularizations have the same property, and they also preserve loop spaces, i.e., spaces homotopy equivalent to topological groups [4, 9, 15]. In stable homotopy, localizations and cellularizations also preserve products of Eilenberg–Mac Lane spectra, and, more generally, spectra homotopy equivalent to modules over connective ring spectra [10, 12, 17].

Purely algebraic instances of the same phenomenon were studied in the category of groups, where abelian groups and nilpotent groups of class 2 are preserved by localizations [11] while nilpotent groups of any class are preserved by colocalizations [16]. As discussed in [13, 14], localizations and colocalizations of abelian groups lift to R-modules for any ring R; this means

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that they not only preserve the class of groups underlying R-modules for every given R (for example, uniquely divisible abelian groups when $R = \mathbb{Q}$), but they give rise to functors on the category of R-modules.

In Section 1 we generalize the latter fact to models of algebraic theories. It turns out that all localizations and colocalizations lift to models of Lawvere theories (i.e., single-sorted finite-product theories) enriched in abelian groups. This is due to the fact that localizations and colocalizations on additive categories commute with finite products. The same idea was used in [4] with models of Lawvere theories in simplicial sets, thanks to the fact that homotopical localizations and cellularizations also commute with finite products. A similar argument was used in [30, Proposition 5.1].

In fact we show that localizations and colocalizations lift to models of Lawvere theories enriched in any additive bicomplete closed symmetric monoidal category in which $\aleph_0$ is dense, meaning that the full subcategory with objects nI for $n \geq 1$ is a dense generator of $\mathcal{V}$, where nI denotes the coproduct of n copies of the monoidal unit of $\mathcal{V}$ (thus, $nI = \mathbb{Z}^n$ in abelian groups).

Since Lawvere theories are precisely the algebraic theories with arities $\aleph_0$, it is natural to ask for the validity of the same result for algebraic theories with arbitrary arities. We discuss this problem in Section 2, where we state a necessary and sufficient condition under which localizations and colocalizations lift to models of enriched algebraic theories with given arities $\mathcal{A}$. Specifically, a localization E lifts if and only if the functor $\mathcal{V}(A, -)$ preserves E-equivalences for every $A \in \mathcal{A}$, and a colocalization C lifts if and only if $\mathcal{V}(A, -)$ preserves C-colocal objects for every $A \in \mathcal{A}$. When $\mathcal{A} = \aleph_0$ we have that $\mathcal{V}(nI, X) \cong X^n$, so the condition holds automatically if $\mathcal{V}$ is additive.

The formalism of algebraic theories as a categorical version of universal algebra goes back to Lawvere [22, 23] and Linton [25]. Although it is not equivalent to the formalism of *operads* (for instance, the category of groups is not a category of algebras over any operad), it is more convenient in some situations, as in [4, 19]. A common feature of operads and algebraic theories is that they are associated with *monads.* It has long been known that models of Lawvere theories are equivalent to algebras over finitary monads, not only in sets (which is not a useful category for our purposes), but also with a suitable enrichment. Detailed accounts of the correspondence between algebraic theories with arities and monads with arities can be found in [5, 7, 29]. Liftings of localizations and colocalizations to categories of algebras over monads were discussed in [13], from which we translate a central result to the enriched context (Lemma 1.7 below).

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1 Additive Lawvere theories

1.1 Lawvere theories

We denote by $\aleph_0$ the full subcategory of the category of sets whose objects are the finite ordinals. Thus $\aleph_0$ has an initial object and a strictly associative coproduct given by ordinal sum. Consequently, its opposite category $\aleph_0^{\mathrm{op}}$ has a strictly associative product and a terminal object.

A *Lawvere theory* is a small category $\mathcal{L}$ with finite products and with the same objects as $\aleph_0$, equipped with a functor $(\aleph_0)^{\mathrm{op}} \to \mathcal{L}$ which is the identity on objects and strictly preserves finite products. For each finite ordinal n, the morphism set $\mathcal{L}(n, 1)$ is called the set of *n-ary operations*.

A *model* of a Lawvere theory $\mathcal{L}$ with values in a category $\mathcal{C}$ with finite products is a functor $M\colon \mathcal{L} \to \mathcal{C}$ preserving finite products (here and in what follows, preservation is meant up to a unique isomorphism commuting with projections). The category of such models —with natural transformations as morphisms— will be denoted by $\mathsf{Mod}(\mathcal{L}, \mathcal{C})$.

Each M in $\mathsf{Mod}(\mathcal{L}, \mathcal{C})$ is determined by its underlying object $X = M1$ together with a function $X^n \to X^m$ for each morphism $f \in \mathcal{L}(n, m)$, subject to the restrictions imposed by the composition and product structure of $\mathcal{L}$. There is a forgetful functor $U\colon \mathsf{Mod}(\mathcal{L}, \mathcal{C}) \to \mathcal{C}$ (also called *evaluation*) defined as $UM = M1$. If $\mathcal{C}$ is locally presentable, then this functor has a left adjoint F and hence there is a monad $T_{\mathcal{L}}$ on $\mathcal{C}$ defined as $T_{\mathcal{L}} = UF$.

If $\mathcal{C} = \mathsf{Set}$ (the category of sets), then the functor $F\colon \mathsf{Set} \to \mathsf{Mod}(\mathcal{L}, \mathsf{Set})$ is the left Kan extension of the functor $\aleph_0 \to \mathsf{Mod}(\mathcal{L}, \mathsf{Set})$ sending each finite ordinal n to the model Fn given on objects by $(Fn)k = \mathcal{L}(n, k)$ for all finite ordinals k, and on morphisms by composition in $\mathcal{L}$.

Since filtered colimits in Set commute with finite products, the evaluation functor $U\colon \mathsf{Mod}(\mathcal{L}, \mathsf{Set}) \to \mathsf{Set}$ creates filtered colimits. Therefore the monad $T_{\mathcal{L}}$ preserves filtered colimits and its values can be computed as coends:

$$T_{\mathcal{L}}X \cong \int^{n \in \aleph_0} T_{\mathcal{L}}(n) \times \mathsf{Set}(n, X) \cong \int^{n \in \aleph_0} \mathcal{L}(n, 1) \times X^n. \tag{1.1}$$

For each Lawvere theory $\mathcal{L}$, the category $\mathsf{Mod}(\mathcal{L}, \mathsf{Set})$ is equivalent to the category of algebras over $T_{\mathcal{L}}$, and the theory $\mathcal{L}$ is recovered as the opposite of the Kleisli category of $T_{\mathcal{L}}$, that is,

$$\mathcal{L}(n, m) \cong \mathsf{Mod}(\mathcal{L}, \mathsf{Set})(Fm, Fn) \cong \mathsf{Set}(m, T_{\mathcal{L}}(n)) \cong T_{\mathcal{L}}(n)^m.$$

In fact there is an equivalence of categories between Lawvere theories and finitary monads on Set, where a monad is called *finitary* if it preserves filtered colimits. For details and more general versions, see [5, 18, 22, 23, 25].

1.2 Enriched Lawvere theories

A small full subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is called *dense* [2, § 1.23] if every object of $\mathcal{C}$ is a canonical colimit of objects from $\mathcal{A}$; that is, every object X of $\mathcal{C}$ is a colimit of the forgetful functor $(\mathcal{A} \downarrow X) \to \mathcal{C}$ sending each morphism $A \to X$ to A, where $(\mathcal{A} \downarrow X)$ denotes the slice category over X. In other words, $\mathcal{A}$ is dense in $\mathcal{C}$ if the $\mathcal{A}$-cocones in $\mathcal{C}$ are colimit cocones. Every dense subcategory $\mathcal{A} \subseteq \mathcal{C}$ is a *generator*, meaning that two morphisms $f, g \colon X \to Y$ in $\mathcal{C}$ are distinct if and only if there is an object A in $\mathcal{A}$ and a morphism $h \colon A \to X$ such that $h \circ f \neq h \circ g$.

A category is called *preadditive* if it is enriched in abelian groups. In a preadditive category, finite products are coproducts and finite coproducts are products. A preadditive category with finite products is *additive*.

In this section, we let $\mathcal{V}$ be a closed symmetric monoidal category, which we assume complete, cocomplete and additive (including the assumption that the tensor product is additive in each variable). Such a category $\mathcal{V}$ is called an *additive cosmos* —this term is due to Bénabou [32]. The main example is the category Ab of abelian groups or, more generally, the category R-Mod of modules over a unitary commutative ring R. We view $\mathcal{V}$ as enriched in itself, and assume endofunctors on $\mathcal{V}$ and natural transformations between them to be enriched.

We will also impose that $\aleph_0$ be dense in $\mathcal{V}$, in the following sense. For a finite ordinal $n \in \aleph_0$, we denote by $nI \in \mathcal{V}$ a coproduct of n copies of the monoidal unit I of $\mathcal{V}$, and keep denoting by $\aleph_0$ the full subcategory of $\mathcal{V}$ with these objects (compare with [26, Example 3.7] or [6, Definition 2.1.1]).

We will consider Lawvere theories enriched in an additive cosmos $\mathcal{V}$ in which $\aleph_0$ is dense. This is a small category $\mathcal{L}$ enriched in $\mathcal{V}$ with finite cotensors together with an identity-on-objects functor $(\aleph_0)^{\mathrm{op}} \to \mathcal{L}$ strictly preserving finite cotensors [18, 29].

As a special case of [7], the category of $\mathcal{V}$-enriched Lawvere theories is equivalent to the category of $\aleph_0$-finitary $\mathcal{V}$-monads, where a monad T on $\mathcal{V}$ is called $\aleph_0$-*finitary* if it sends $\aleph_0$-cocones to colimit cocones. An $\aleph_0$-finitary $\mathcal{V}$-monad T on $\mathcal{V}$ yields a Lawvere theory enriched in $\mathcal{V}$ with morphisms

$$\mathcal{L}_T(n, m) = \mathcal{V}(mI, T(nI)) \cong T(nI)^m,$$

and the category of T-algebras is equivalent to the category of models of $\mathcal{L}_T$ with values in $\mathcal{V}$. Conversely, a Lawvere theory $\mathcal{L}$ enriched in $\mathcal{V}$ yields an $\aleph_0$-finitary $\mathcal{V}$-monad $T_{\mathcal{L}}$ whose value on an object X of $\mathcal{V}$ is

$$T_{\mathcal{L}} X \cong \int^{n \in \aleph_0} T_{\mathcal{L}}(nI) \otimes \mathcal{V}(nI, X) \cong \int^{n \in \aleph_0} \mathcal{L}(n, 1) \otimes X^n. \tag{1.2}$$

This coend formula comes from the fact that $\mathsf{Mod}(\mathcal{L}, \mathcal{V})$ is tensored over $\mathcal{V}$ objectwise, so the forgetful functor $\mathsf{Mod}(\mathcal{L}, \mathcal{V}) \to \mathcal{V}$ sends the tensoring to the monoidal product in $\mathcal{V}$. Hence (1.2) results from the assumption that $\aleph_0$ is dense in $\mathcal{V}$, by evaluating at 1 the left Kan extension formula for the free functor $F_{\mathcal{L}}$ given by $F_{\mathcal{L}}(nI)k = \mathcal{L}(n, k)$ for all finite ordinals k.

1.3 Lifting localizations and colocalizations

A monad (E, μ, η) on a category $\mathcal{C}$, where $\mu\colon EE \to E$ denotes the multiplication and $\eta\colon \mathrm{Id}_{\mathcal{C}} \to E$ the unit, is called *idempotent* if μ is an isomorphism. Idempotent monads are also called *localizations*. Thus, a localization on a category $\mathcal{C}$ consists of a functor $E\colon \mathcal{C} \to \mathcal{C}$ equipped with a natural transformation $\eta\colon \mathrm{Id}_{\mathcal{C}} \to E$ such that, for all X, the morphisms $\eta_{EX}\colon EX \to EEX$ and $E\eta_X\colon EX \to EEX$ are equal and they are isomorphisms (in fact, they are then inverses of μ_X, since $\mu_X \circ \eta_{EX} = \mu_X \circ E\eta_X = \mathrm{id}_{EX}$).

Objects in the essential image of E are called *E-local*. An *E-equivalence* is a morphism $f\colon X \to Y$ in $\mathcal{C}$ such that $Ef\colon EX \to EY$ is an isomorphism. The classes of E-local objects and E-equivalences determine each other by *orthogonality*, i.e., a morphism $f\colon X \to Y$ is an E-equivalence if and only if

$$\mathcal{C}(f, Z)\colon \mathcal{C}(Y, Z) \longrightarrow \mathcal{C}(X, Z) \tag{1.3}$$

is a bijection for every E-local object Z, and an object Z is E-local if and only if (1.3) is a bijection for every E-equivalence $f\colon X \to Y$. Consequently, the class of E-equivalences is closed under those colimits that exist in $\mathcal{C}$, while the class of E-local objects is closed under limits.

Dually, a *colocalization* on a category $\mathcal{C}$ is an idempotent comonad, that is, a functor $C\colon \mathcal{C} \to \mathcal{C}$ with a natural transformation $\varepsilon\colon C \to \mathrm{Id}_{\mathcal{C}}$ such that, for all X, the morphisms $\varepsilon_{CX}\colon CCX \to CX$ and $C\varepsilon_X\colon CCX \to CX$ are equal and they are isomorphisms. Objects in the essential image of C are called *C-colocal*. A *C-equivalence* is a morphism $f\colon X \to Y$ in $\mathcal{C}$ such that $Cf\colon CX \to CY$ is an isomorphism. The classes of C-colocal objects and C-equivalences determine each other by *coorthogonality*, meaning that a morphism $f\colon X \to Y$ is a C-equivalence if and only if

$$\mathcal{C}(A, f)\colon \mathcal{C}(A, X) \longrightarrow \mathcal{C}(A, Y) \tag{1.4}$$

is a bijection for every C-colocal object A, and an object A is C-colocal if and only if (1.4) is a bijection for every C-equivalence $f\colon X \to Y$. Accordingly, the class of C-equivalences is closed under limits and the class of C-colocal objects is closed under colimits.

Localizations and colocalizations on a preadditive category are automatically additive, since left adjoints and right adjoints of additive functors are additive [27, Ch. IV, Theorem 3].

In what follows we consider $\mathcal{V}$-localizations and $\mathcal{V}$-colocalizations on an additive cosmos $\mathcal{V}$. These are, respectively, idempotent $\mathcal{V}$-monads and idempotent $\mathcal{V}$-comonads. Thus, for a $\mathcal{V}$-localization $E\colon \mathcal{V} \to \mathcal{V}$, the E-equivalences and the E-local objects are related by enriched orthogonality, that is,

$$\mathcal{V}(f, Z)\colon \mathcal{V}(Y, Z) \longrightarrow \mathcal{V}(X, Z) \tag{1.5}$$

is an isomorphism in $\mathcal{V}$ whenever f is an E-equivalence and Z is E-local, and similarly for colocalizations.

Our motivating example is the following. For a unitary ring R (not necessarily commutative), consider the additive Lawvere theory with

$$\mathcal{L}(m, n) = \mathrm{Hom}_R(R^n, R^m) \cong \mathsf{Ab}(\mathbb{Z}^n, R^m),$$

whose models are precisely the left R-modules, both in sets and in abelian groups. The associated monad on Set sends each set X to the underlying set of the free R-module on X, and the associated additive monad on Ab sends each abelian group A to $R \otimes A$.

A localization $E\colon \mathsf{Ab} \to \mathsf{Ab}$ is said to *lift* to R-modules [13, § 4] if there is a localization $\tilde{E}$ on R-modules such that $EU \cong U\tilde{E}$, where U is the forgetful functor sending each R-module to the underlying abelian group —in other words, the underlying abelian group of $\tilde{E}M$ is naturally isomorphic to EUM for every R-module M. It was shown in [14] that every localization on abelian groups lifts uniquely to R-modules for every unitary ring R. What follows is a generalization of this fact.

Lemma 1.1. *Localizations and colocalizations on additive categories preserve finite products.*

Proof. For a localization $E\colon \mathcal{C} \to \mathcal{C}$ with unit η on an additive category $\mathcal{C}$ and a finite collection of objects $X_1, \ldots, X_n$ in $\mathcal{C}$, the product of the morphisms $\eta_{X_i}\colon X_i \to EX_i$ is an E-equivalence because it is a coproduct of E-equivalences, and its codomain is E-local because it is a product of E-local objects. Hence $E(X_1 \times \cdots \times X_n) \cong EX_1 \times \cdots \times EX_n$ naturally.

Similarly, every colocalization C preserves finite products as every product of C-equivalences is a C-equivalence and every coproduct of C-colocal objects is C-colocal. □

For a Lawvere theory $\mathcal{L}$ enriched in $\mathcal{V}$ and a $\mathcal{V}$-endofunctor $E\colon \mathcal{V} \to \mathcal{V}$, we say that a $\mathcal{V}$-endofunctor $\tilde{E}$ on $\mathsf{Mod}(\mathcal{L}, \mathcal{V})$ is a *lifting* of E if there is a natural isomorphism $EU \cong U\tilde{E}$, where $U\colon \mathsf{Mod}(\mathcal{L}, \mathcal{V}) \to \mathcal{V}$ is evaluation at 1.

Proposition 1.2. *Let $\mathcal{V}$ be an additive cosmos in which $\aleph_0$ is dense. Then every $\mathcal{V}$-localization on $\mathcal{V}$ lifts uniquely to a $\mathcal{V}$-localization on the category of models of any Lawvere theory enriched in $\mathcal{V}$.*

Proof. Suppose given a $\mathcal{V}$-localization $E\colon \mathcal{V} \to \mathcal{V}$ with unit η and a Lawvere theory $\mathcal{L}$ enriched in $\mathcal{V}$. We are going to prove that the $\mathcal{V}$-functor

$$\tilde{E}\colon \mathsf{Mod}(\mathcal{L}, \mathcal{V}) \longrightarrow \mathsf{Mod}(\mathcal{L}, \mathcal{V})$$

defined as $\tilde{E}M = EM$ is a lifting of E. This functor is well defined since, for every $M\colon \mathcal{L} \to \mathcal{V}$ preserving finite products, EM also preserves finite products because E preserves them by Lemma 1.1.

From the fact that E is idempotent with unit η it follows that $\tilde{E}$ is idempotent with unit $\tilde{\eta}$ given by

$$(\tilde{\eta}_M)_n = \eta_{Mn}\colon Mn \longrightarrow EMn$$

for all $n \in \aleph_0$ and $M \in \mathsf{Mod}(\mathcal{L}, \mathcal{V})$. The localization $\tilde{E}$ is a lifting of E since

$$EUM = EM1 = U\tilde{E}M$$

for all M. Uniqueness of $\tilde{E}$ up to isomorphism follows from the fact that if $EU \cong UE'$ naturally then $EM1 = EUM \cong UE'M = E'M1$ and therefore $EMn \cong E'Mn$ naturally for all n, since M preserves finite products. Naturality ensures that $E' \cong \tilde{E}$ as functors. □

Note that, in the proof of Proposition 1.2, the localization $\tilde{E}$ happens to be a strict lifting of E, since $EUM = U\tilde{E}M$ for all $M \in \mathsf{Mod}(\mathcal{L}, \mathcal{V})$. This is consistent with [13, Theorem 4.2], where it was shown that if there is a lifting of a localization E to a category of algebras over a monad then there is also a strict lifting of E. For a monad T on $\mathcal{V}$, a $\mathcal{V}$-endofunctor $\tilde{E}$ on T-algebras is a *lifting* of a $\mathcal{V}$-endofunctor E if there is a natural isomorphism $EU \cong U\tilde{E}$, where U is the forgetful functor from T-algebras to $\mathcal{V}$.

Example 1.3. If a monad T is idempotent (that is, T is itself a localization), then lifting a $\mathcal{V}$-endofunctor E to T-algebras is equivalent to restricting E to the essential image of T, that is, to the full subcategory of T-local objects. For example, rationalization restricts to torsion-free abelian groups.

Proposition 1.4. *Let $\mathcal{V}$ be an additive cosmos in which $\aleph_0$ is dense. Then every $\mathcal{V}$-colocalization on $\mathcal{V}$ lifts uniquely to a $\mathcal{V}$-colocalization on the category of models of any Lawvere theory enriched in $\mathcal{V}$.*

Proof. The proof is analogous to the proof of Proposition 1.2. □

Example 1.5. If $\mathcal{C}$ is any variety of groups (i.e., a class of groups closed under subgroups, quotients and products), then localizations and colocalizations on Ab preserve $\mathcal{C} \cap \mathsf{Ab}$. This follows from Proposition 1.2 and Proposition 1.4, since $\mathcal{C} \cap \mathsf{Ab}$ is equivalent to the category of models of an additive Lawvere theory, namely the one determined by the abelianized free objects in $\mathcal{C}$. Lifting an endofunctor to this category of models amounts to restricting it to $\mathcal{C}$. However, the only nontrivial examples are the classes of abelian groups of exponent m for some $m > 1$, and these are precisely the abelian groups underlying $\mathbb{Z}/m$-modules.

The question of which varieties are preserved by localizations or colocalizations on the category of groups is more substantial. Positive examples include the variety of abelian groups itself, the variety of nilpotent groups of class 2, and the variety of groups with a fixed exponent [11]. Nilpotent groups of any class are preserved by colocalizations [16]. On the other hand, it was proved in [28] that localization at primes does not restrict to the variety of metabelian groups.

Example 1.6. Proposition 1.2 and Proposition 1.4 apply to the category of R-modules if R is any unitary commutative ring. Hence every localization and every colocalization on R-modules lifts to models of any Lawvere theory enriched in R-modules. As a special case, if $R \to R'$ is a central ring homomorphism where R' is any unitary ring (not necessarily commutative), then every localization and every colocalization on R-modules lifts to left R'-modules. This appeared in [13, Examples 4.3 and 4.8]. Note that, as shown in [3, Corollary B8], if a category of models of a Lawvere theory $\mathcal{L}$ is *abelian*, then it is equivalent to the category of left R'-modules for a unitary ring R', namely $R' = \mathcal{L}(1, 1)$.

Every multiplicatively closed set S of elements of R yields a ring homomorphism $R \to S^{-1}R$ sending the elements of S to units, and hence an exact localization on R-modules by $S^{-1}M = S^{-1}R \otimes_R M$. If $\alpha \colon R \to R'$ is a central ring homomorphism and S is a multiplicatively closed set of elements of R, then αS is an Ore set [31] in R' and hence $(\alpha S)^{-1}N = (\alpha S)^{-1}R' \otimes_{R'} N$ is a well-defined extension of the former from R-modules to left R'-modules. Rationalization of abelian groups is an instance thereof.

The next two results (Proposition 1.8 and Proposition 1.9) are equivalent to Proposition 1.2 and Proposition 1.4, since the category of algebras over an $\aleph_0$-finitary $\mathcal{V}$-monad T is equivalent to the category of models in $\mathcal{V}$ of the associated Lawvere theory $\mathcal{L}_T$. However, the proofs are of a different nature and independent interest. They are based on the following fact.

Lemma 1.7. *For a $\mathcal{V}$-localization E and a $\mathcal{V}$-monad T on a category $\mathcal{C}$ enriched in a closed symmetric monoidal category $\mathcal{V}$, the following facts are equivalent:*

(a) *E lifts to a $\mathcal{V}$-localization on T-algebras.*

(b) *T preserves E-equivalences.*

Similarly, for a $\mathcal{V}$-colocalization C and a $\mathcal{V}$-monad T, the following facts are equivalent:

(a) *C lifts to a $\mathcal{V}$-colocalization on T-algebras.*

(b) *T preserves C-colocal objects.*

Moreover, liftings are unique up to isomorphism in both cases.

Proof. If T preserves E-equivalences for a $\mathcal{V}$-localization E, then a lifting $\tilde{E}$ of E to the category $\mathcal{C}^T$ of T-algebras can be constructed as in the proof of [13, Theorem 4.2]. In order to prove that $\tilde{E}$ is a $\mathcal{V}$-functor, it is necessary to use that E is itself a $\mathcal{V}$-functor and that a morphism object $\mathcal{C}^T((A,a),(B,b))$ is defined as the equalizer in $\mathcal{V}$ of the morphisms from $\mathcal{C}(A,B)$ to $\mathcal{C}(TA,B)$ given by $f \mapsto f \circ a$ and $f \mapsto b \circ Tf$. A similar argument is used for the lifting $\tilde{C}$ of a $\mathcal{V}$-colocalization C, by means of [13, Theorem 4.4].

For the converse, if a lifting $\tilde{E}$ of E exists, then one has to use the fact that the free-forgetful decomposition $T = UF$ is a $\mathcal{V}$-enriched adjunction to infer that F preserves E-equivalences. Since U also preserves E-equivalences because $U\tilde{E} \cong EU$, we may conclude that T preserves them. The same argument works for colocalizations. □

Proposition 1.8. *Let $\mathcal{V}$ be an additive cosmos in which $\aleph_0$ is dense. Then every $\mathcal{V}$-localization on $\mathcal{V}$ lifts uniquely to a $\mathcal{V}$-localization on the category of T-algebras for every $\aleph_0$-finitary $\mathcal{V}$-monad T on $\mathcal{V}$.*

Proof. Let T be an $\aleph_0$-finitary $\mathcal{V}$-monad on $\mathcal{V}$ and let $\mathcal{L}_T$ be the associated Lawvere theory enriched in $\mathcal{V}$. In view of Lemma 1.7, we need to prove that T preserves E-equivalences.

For every object X in $\mathcal{V}$, it follows from (1.2) that TX is a colimit of a diagram with values in $\mathcal{L}_T(n,1) \otimes X^n$ over all finite ordinals n. Since the class of E-equivalences is closed under colimits, it suffices to prove that

$$\mathcal{L}_T(n,1) \otimes X^n \longrightarrow \mathcal{L}_T(n,1) \otimes Y^n$$

is an E-equivalence whenever $X \to Y$ is an E-equivalence. Now Lemma 1.1 tells us that $X^n \to Y^n$ is an E-equivalence for all n, and we can then infer that $W \otimes X^n \to W \otimes Y^n$ is an E-equivalence for every $W \in \mathcal{V}$, since if Z is any E-local object we have

$$\mathcal{V}(W \otimes Y^n, Z) \cong \mathcal{V}(W, \mathcal{V}(Y^n, Z)) \cong \mathcal{V}(W, \mathcal{V}(X^n, Z)) \cong \mathcal{V}(W \otimes X^n, Z),$$

using the fact that $\mathcal{V}$ is closed monoidal and enriched over itself. □

Proposition 1.9. *Let $\mathcal{V}$ be an additive cosmos in which $\aleph_0$ is dense. Then every $\mathcal{V}$-colocalization on $\mathcal{V}$ lifts uniquely to a $\mathcal{V}$-colocalization on the category of T-algebras for every $\aleph_0$-finitary $\mathcal{V}$-monad T on $\mathcal{V}$.*

Proof. The proof is analogous to the proof of Proposition 1.8. Here we need to prove that T preserves C-colocal objects for a given colocalization C. For this, we use again the fact that, for every C-colocal object $X \in \mathcal{V}$, it follows from (1.2) that TX is a colimit of a diagram with values in $\mathcal{L}_T(n,1) \otimes X^n$ over all finite ordinals n. Hence TX is C-colocal since X^n is a finite product and therefore a finite coproduct of copies of X, and consequently $W \otimes X^n$ is C-colocal for all $W \in \mathcal{V}$. □

We end this section with examples showing that the conclusions of Propositions 1.8 and 1.9 need not hold for arbitrary monads; see also Examples 2.7 and 1.12 at the end of the next section.

Example 1.10. The Ext-p-completion monad for a prime p sends every abelian group A to $\mathrm{Ext}(\mathbb{Z}/p^\infty, A)$ with the natural unit map

$$A \longrightarrow \mathrm{Ext}(\mathbb{Z}/p^\infty, A)$$

coming from the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[1/p] \longrightarrow \mathbb{Z}/p^\infty \longrightarrow 0.$$

This monad is not $\aleph_0$-finitary, since the underlying abelian group in $\mathbb{Q}$ is a filtered colimit of a diagram with values $\mathbb{Z}$, and applying $\mathrm{Ext}(\mathbb{Z}/p^\infty, -)$ yields a diagram with values $\mathbb{Z}^\wedge_p$ (the p-adic integers) whose colimit is the p-adic field $\mathbb{Q}^\wedge_p$. However, $\mathrm{Ext}(\mathbb{Z}/p^\infty, \mathbb{Q}) = 0$. If E is the rationalization functor $EA = \mathbb{Q} \otimes A$, then the Ext-$p$-completion monad does not preserve E-equivalences, since the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an E-equivalence, yet $\mathbb{Z}^\wedge_p \to 0$ is not.

Example 1.11. The double dual monad, sending every abelian group A to

$$A^{**} = \mathrm{Hom}(\mathrm{Hom}(A, \mathbb{Z}), \mathbb{Z}),$$

is the codensity monad associated with the full subcategory of finitely generated free abelian groups, as in [1] or [24]. This monad is not $\aleph_0$-finitary either. Similarly as in the previous example, if E is the rationalization functor in the category of abelian groups, then the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an E-equivalence; however, $\mathbb{Z}^{**} \cong \mathbb{Z}$ while $\mathbb{Q}^{**} = 0$.

Example 1.12. It is not true that colocalizations on Ab lift to colocalizations on the category of unitary rings, since in general the colocalization of the abelian group underlying a unitary ring need not admit a compatible unit. There is a monad on Ab whose algebras are the unitary rings, namely the free monoid monad. However, this monad is not additive, as $f + g$ need not be a ring homomorphism if f and g are ring homomorphisms. Hence, unitary rings are not models of any *additive* Lawvere theory.

2 Theories with arities

Although in this section we focus on a (not necessarily additive) cosmos $\mathcal{V}$ enriched in itself, the preliminaries are stated, more generally, for a category $\mathcal{E}$ enriched in $\mathcal{V}$.

If $\mathcal{E}$ is a $\mathcal{V}$-category, the *nerve functor* [5, § 1.6] (also called *canonical functor* [2, § 1.25]) relative to a full $\mathcal{V}$-subcategory $\mathcal{A} \subseteq \mathcal{E}$ is the functor

$$N_{\mathcal{A}} \colon \mathcal{E} \longrightarrow \mathcal{V}^{\mathcal{A}^{\mathrm{op}}}$$

sending each object X to the $\mathcal{V}$-functor $A \mapsto \mathcal{E}(A, X)$, which we also denote by $\mathcal{E}(-, X)|_{\mathcal{A}}$. As shown in [2, Proposition 1.26] or in [5, Lemma 1.7], the subcategory $\mathcal{A}$ is dense if and only if $N_{\mathcal{A}}$ is fully faithful. By the argument given in [2, Proposition 1.26], if λ is a regular cardinal and every object in $\mathcal{A}$ is λ-presentable, then $N_{\mathcal{A}}$ preserves λ-filtered colimits if such colimits exist in $\mathcal{E}$.

A $\mathcal{V}$-monad T on a $\mathcal{V}$-category $\mathcal{E}$ (not necessarily cocomplete) with a dense generator $\mathcal{A}$ is called a *monad with arities* $\mathcal{A}$ if the composite functor

$$\mathcal{E} \xrightarrow{\ T\ } \mathcal{E} \xrightarrow{\ N_{\mathcal{A}}\ } \mathcal{V}^{\mathcal{A}^{\mathrm{op}}}$$

takes $\mathcal{A}$-cocones in $\mathcal{E}$ to colimit cocones in $\mathcal{V}^{\mathcal{A}^{\mathrm{op}}}$. In other words, T is a monad with arities $\mathcal{A}$ if and only if the functor

$$(\mathcal{A} \downarrow X) \longrightarrow \mathcal{E} \xrightarrow{\ T\ } \mathcal{E} \xrightarrow{\ N_{\mathcal{A}}\ } \mathcal{V}^{\mathcal{A}^{\mathrm{op}}}$$

sending $A \to X$ to $\mathcal{E}(-, TA)|_{\mathcal{A}}$ has colimit $\mathcal{E}(-, TX)|_{\mathcal{A}}$ for every object X in $\mathcal{E}$.

Lemma 2.1. *For a regular cardinal λ, let $\mathcal{E}$ be a locally λ-presentable $\mathcal{V}$-category and let $\mathcal{A}$ be a skeleton of the full subcategory of the λ-presentable objects in $\mathcal{E}$. Let T be a $\mathcal{V}$-monad on $\mathcal{E}$. Then T is a monad with arities $\mathcal{A}$ if and only if T takes $\mathcal{A}$-cocones to colimit cocones.*

Proof. According to [2, Proposition 1.22], the full subcategory $\mathcal{A}$ is dense and λ-filtered. Since $N_{\mathcal{A}}$ preserves λ-filtered colimits, every $\mathcal{V}$-monad that takes $\mathcal{A}$-cocones to colimit cocones is a monad with arities $\mathcal{A}$. The converse implication holds since $N_{\mathcal{A}}$ is fully faithful and therefore it reflects isomorphisms. □

If $\aleph_0$ is dense in $\mathcal{V}$, then a $\mathcal{V}$-monad T on $\mathcal{V}$ is $\aleph_0$-finitary if and only if it is a monad with arities $\aleph_0$.

To every $\mathcal{V}$-monad $T\colon \mathcal{E} \to \mathcal{E}$ we may associate the Eilenberg–Moore $\mathcal{V}$-category $\mathcal{E}^T$ of T-algebras and the Kleisli $\mathcal{V}$-category $\mathcal{E}_T$, which is given by factoring the free functor $\mathcal{E} \to \mathcal{E}^T$ into a functor that is the identity on objects and a fully faithful functor

$$\mathcal{E} \longrightarrow \mathcal{E}_T \longrightarrow \mathcal{E}^T.$$

Thus, there are natural isomorphisms $\mathcal{E}_T(X,Y) \cong \mathcal{E}(X,TY)$ in $\mathcal{V}$.

According to the *nerve theorem* [5, Theorem 1.10] (enhanced in [7] to cover the enriched case), if $\mathcal{E}$ is a $\mathcal{V}$-category with a dense generator $\mathcal{A}$, then, for every monad T with arities $\mathcal{A}$, the full subcategory Θ_T spanned by the free T-algebras on the arities is a dense generator of the Eilenberg–Moore category $\mathcal{E}^T$.

As a special case, let T be any finitary monad on sets and let $\mathcal{L}_T$ be the associated Lawvere theory. Then $\aleph_0$ is dense in $\mathsf{Mod}(\mathcal{L}_T, \mathsf{Set})$ by the nerve theorem. Hence, for example, $\{\mathbb{Z}^n\}_{n\in\aleph_0}$ is dense in Ab and, more generally, $\{R^n\}_{n\in\aleph_0}$ is dense in R-Mod for any unitary ring R. However, these collections are not filtered, since parallel arrows need not have coequalizers.

Let $\mathcal{E}$ be a $\mathcal{V}$-category with a dense generator $\mathcal{A}$. A *theory with arities* $\mathcal{A}$ on $\mathcal{E}$ is a small $\mathcal{V}$-category Θ equipped with a $\mathcal{V}$-functor $J\colon \mathcal{A} \to \Theta$ which is bijective on objects and such that the monad $J^*J_!$ on $\mathcal{V}^{\mathcal{A}^{\mathrm{op}}}$ obtained by composing the restriction functor with its left adjoint

$$\mathcal{V}^{\mathcal{A}^{\mathrm{op}}} \xrightarrow{J_!} \mathcal{V}^{\Theta^{\mathrm{op}}} \xrightarrow{J^*} \mathcal{V}^{\mathcal{A}^{\mathrm{op}}}$$

preserves the essential image of the nerve functor $N_{\mathcal{A}}\colon \mathcal{E} \to \mathcal{V}^{\mathcal{A}^{\mathrm{op}}}$ (which consists of filtered colimits of representable presheaves). If $A \in \mathcal{A}$ then the value of $J_!$ on a representable presheaf $\mathcal{E}(-,A)|_{\mathcal{A}} = \mathcal{A}(-,A)$ is its left Kan extension along J, namely the functor $\Theta(-,A)$.

A *model* of Θ is a $\mathcal{V}$-functor $M\colon \Theta^{\mathrm{op}} \to \mathcal{V}$ whose restriction along J belongs to the essential image of $N_{\mathcal{A}}$. We denote by $\mathsf{Mod}(\Theta, \mathcal{V})$ the full subcategory of $\mathcal{V}^{\Theta^{\mathrm{op}}}$ whose objects are models of Θ and whose morphisms are natural transformations.

For a $\mathcal{V}$-monad T on a locally presentable $\mathcal{V}$-category $\mathcal{E}$ with arities $\mathcal{A}$, the functor $J_T\colon \mathcal{A} \to \Theta_T$ sending each arity A to the free T-algebra on A is a theory with arities with the property that the category $\mathcal{E}^T$ of T-algebras is equivalent to the category $\mathsf{Mod}(\Theta_T, \mathcal{V})$ of models of Θ_T; see [5, Proposition 3.2]. The theory Θ_T embeds into the Eilenberg–Moore category $\mathcal{E}^T$ through the Kleisli category $\mathcal{E}_T$, and this implies functoriality of the assignment $T \mapsto \Theta_T$. Furthermore, as shown in [5, Theorem 3.4] and [7, Theorem 17], the assignment $T \mapsto \Theta_T$ sets up an adjoint equivalence between the category of $\mathcal{V}$-monads with arities $\mathcal{A}$ and the category of theories enriched in $\mathcal{V}$ with arities $\mathcal{A}$.

For a $\mathcal{V}$-monad T with arities $\mathcal{A}$, the associated theory Θ_T is the Kleisli category of T; thus, $\Theta_T(B, A) = \mathcal{E}(B, TA)$ for all A, B in $\mathcal{A}$. For a theory Θ, the associated monad T_Θ is recovered as follows. By definition, the monad $J^* J_!$ restricts to the essential image of $N_{\mathcal{A}}$. The choice of a right adjoint $\rho_{\mathcal{A}}\colon \mathrm{EssIm}(N_{\mathcal{A}}) \to \mathcal{E}$ to the equivalence $N_{\mathcal{A}}\colon \mathcal{E} \to \mathrm{EssIm}(N_{\mathcal{A}})$ induces a monad $T_\Theta = \rho_{\mathcal{A}} J^* J_! N_{\mathcal{A}}$ on $\mathcal{E}$, which has arities $\mathcal{A}$. In particular, for each $A \in \mathcal{A}$ we have that $N_{\mathcal{A}} T_\Theta A \cong J^* J_! N_{\mathcal{A}} A$, that is,

$$\mathcal{E}(-, T_\Theta A)|_{\mathcal{A}} \cong J^* J_!\, \mathcal{E}(-, A)|_{\mathcal{A}} \cong \Theta(-, A)|_{\mathcal{A}}$$

as functors on $\mathcal{A}^{\mathrm{op}}$. Moreover, since T_Θ takes $\mathcal{A}$-cocones to colimit cocones, if $\mathcal{E}$ is $\mathcal{V}$-tensored then for every object $X \in \mathcal{E}$ we have

$$T_\Theta X \cong \int^{A \in \mathcal{A}} T_\Theta A \otimes \mathcal{E}(A, X). \tag{2.1}$$

As explained in [5, § 3.5], a theory Θ with arities $\aleph_0$ can be viewed as a functor $\aleph_0 \to \Theta$ that preserves coproducts. Hence Θ^{op} is a Lawvere theory.

2.1 Lifting localizations and colocalizations

We next aim to generalize Propositions 1.2, 1.4, 1.8 and 1.9 to theories with arities. Let $\mathcal{V}$ be a cosmos, not necessarily additive, which we view as enriched over itself, and let $\mathcal{A}$ be a dense full subcategory of $\mathcal{V}$, for instance a skeleton of the full subcategory of λ-presentable objects if $\mathcal{V}$ is locally λ-presentable. All functors (including localizations and colocalizations) will be enriched in $\mathcal{V}$. For local presentability in the enriched sense, see [21]. A $\mathcal{V}$-category is *locally*

λ-*presentable* if it is cocomplete as a $\mathcal{V}$-category and its underlying category is locally λ-presentable.

Let Θ be a theory enriched in $\mathcal{V}$ with arities $\mathcal{A}$ and let $J\colon \mathcal{A} \to \Theta$ be the associated bijective-on-objects functor. A $\mathcal{V}$-endofunctor $\tilde{E}$ on $\mathsf{Mod}(\Theta, \mathcal{V})$ is called a *lifting* of a $\mathcal{V}$-endofunctor E on $\mathcal{V}$ if $J^*(\tilde{E}M) \cong EMJ^{\mathrm{op}}$ for all $M \in \mathsf{Mod}(\Theta, \mathcal{V})$.

Theorem 2.2. *Let E be a $\mathcal{V}$-localization on a cosmos $\mathcal{V}$ and let $\mathcal{A}$ be a dense full subcategory of $\mathcal{V}$. Then* (a) *implies both* (b) *and* (c)*:*

- (a) *The functor $\mathcal{V}(A, -)$ preserves E-equivalences for every $A \in \mathcal{A}$.*
- (b) *E lifts to a $\mathcal{V}$-localization on the category of models of any theory enriched in $\mathcal{V}$ with arities $\mathcal{A}$.*
- (c) *E lifts to a $\mathcal{V}$-localization on the category of T-algebras for any $\mathcal{V}$-monad T on $\mathcal{V}$ with arities $\mathcal{A}$.*

Moreover, if $\mathcal{A}$ contains the unit of the monoidal structure, then (a), (b) *and* (c) *are equivalent. If these statements hold, then the lifting of E is unique up to isomorphism.*

Proof. Suppose that (a) holds. Let $J\colon \mathcal{A} \to \Theta$ be a theory with arities $\mathcal{A}$, and define

$$\tilde{E}\colon \mathsf{Mod}(\Theta, \mathcal{V}) \longrightarrow \mathsf{Mod}(\Theta, \mathcal{V})$$

by sending each $M\colon \Theta^{\mathrm{op}} \to \mathcal{V}$ to the composite EM, as in Proposition 1.2. We need to check that if M restricts to the essential image of $N_{\mathcal{A}}$ then EM also does. Thus suppose that $J^*M \cong N_{\mathcal{A}}X = \mathcal{V}(-, X)|_{\mathcal{A}}$ for some $X \in \mathcal{V}$. It follows from (a) that the morphism

$$\mathcal{V}(A, X) \longrightarrow \mathcal{V}(A, EX)$$

induced by the unit map $\eta_X\colon X \to EX$ is an E-equivalence. The adjunction between $\mathcal{V}(A, -)$ and $- \otimes A$ is used to infer that $\mathcal{V}(A, EX)$ is E-local. Hence we obtain a natural isomorphism

$$E\mathcal{V}(A, X) \cong \mathcal{V}(A, EX)$$

for all $X \in \mathcal{V}$ and all $A \in \mathcal{A}$. Consequently,

$$J^*(EM) = EMJ^{\mathrm{op}} = EJ^*M \cong E\mathcal{V}(-, X)|_{\mathcal{A}} \cong \mathcal{V}(-, EX)|_{\mathcal{A}} = N_{\mathcal{A}}(EX),$$

as needed. Hence (a) implies (b).

The equivalence of (b) and (c) follows from the general fact that the category of algebras over a $\mathcal{V}$-monad T with arities $\mathcal{A}$ is equivalent to the category of models of the associated theory Θ_T. Nevertheless, a direct argument for the implication (a) $\Rightarrow$ (c) can be given as in the proof of Proposition 1.8, as follows. Let T be a $\mathcal{V}$-monad on $\mathcal{V}$ with arities $\mathcal{A}$. Let $f\colon X \to Y$ be a morphism in $\mathcal{V}$ such that Ef is an isomorphism, and consider $Tf\colon TX \to TY$. We need to prove that ETf is also an isomorphism. For this, use the fact that, since T is a monad with arities $\mathcal{A}$, we have

$$TX \cong \int^{A\in\mathcal{A}} TA \otimes \mathcal{V}(A, X), \tag{2.2}$$

as in (2.1). Since the class of E-equivalences is closed under arbitrary colimits, we need to look at the morphisms

$$TA \otimes \mathcal{V}(A, X) \longrightarrow TA \otimes \mathcal{V}(A, Y)$$

induced by f. It suffices to prove that

$$\mathcal{V}(A, X) \longrightarrow \mathcal{V}(A, Y)$$

is an E-equivalence, which is guaranteed by (a). Hence (a) implies (c).

Conversely, if the unit I of the monoidal structure belongs to $\mathcal{A}$, assuming (b) we may choose $\Theta = \mathcal{A}$ with the identity functor, and the fact that E lifts to models of Θ implies that for every $X \in \mathcal{V}$ there is a natural isomorphism $E\mathcal{V}(A, X) \cong \mathcal{V}(A, Y)$ for some $Y \in \mathcal{V}$. By choosing $A = I$ we infer that $Y \cong EX$, and if $\mathcal{V}(A, X) \to \mathcal{V}(A, EX)$ is an E-equivalence for all X then $\mathcal{V}(A, -)$ preserves all E-equivalences, so (a) holds.

Uniqueness of liftings is a consequence of the fact that $\mathcal{A}$ is dense in $\mathcal{V}$. □

Condition (a) in Theorem 2.2 is fulfilled for every localization E on any additive cosmos $\mathcal{V}$ in which $\aleph_0$ is dense, since $\mathcal{V}(nI, X) \cong X^n$ for all n and every object X of $\mathcal{V}$. By the nerve theorem [5, 7], this includes all cases in which $\mathcal{V}$ is itself a category of models of an additive Lawvere theory, such as R-Mod for any unitary commutative ring R. Hence Proposition 1.2 is implied by Theorem 2.2.

Corollary 2.3. *For a regular cardinal λ, let $\mathcal{V}$ be a locally λ-presentable closed symmetric monoidal category and let $\mathcal{A}$ be a skeleton of the full subcategory of λ-presentable objects in $\mathcal{V}$. Let E be a $\mathcal{V}$-localization on $\mathcal{V}$. Then the following statements are equivalent:*

(a) *The functor $\mathcal{V}(A, -)$ preserves E-equivalences for every $A \in \mathcal{A}$.*

(b) *E lifts to a $\mathcal{V}$-localization on the category of models of any theory enriched in $\mathcal{V}$ with arities $\mathcal{A}$.*

(c) *E lifts to a $\mathcal{V}$-localization on the category of T-algebras for any $\mathcal{V}$-monad T on $\mathcal{V}$ with arities $\mathcal{A}$.*

Moreover, if any of these conditions hold, then the lifting of E is unique up to isomorphism.

Proof. Theorem 2.2 applies under these assumptions. The unit I of the monoidal structure is finitely presentable since

$$\mathcal{V}(I, \operatorname{colim} D) \cong \operatorname{colim} D \cong \operatorname{colim} \mathcal{V}(I, D)$$

for every diagram D in $\mathcal{V}$, as $\mathcal{V}$ is closed monoidal. □

The version of Theorem 2.2 and Corollary 2.3 for colocalizations reads as follows. Proofs are omitted.

Theorem 2.4. *Let C be a $\mathcal{V}$-colocalization on a cosmos $\mathcal{V}$ and let $\mathcal{A}$ be a dense full subcategory of $\mathcal{V}$. Then* (a) *implies both* (b) *and* (c)*:*

(a) *The functor $\mathcal{V}(A, -)$ preserves C-colocal objects for every $A \in \mathcal{A}$.*

(b) *C lifts to a $\mathcal{V}$-colocalization on the category of models of any theory enriched in $\mathcal{V}$ with arities $\mathcal{A}$.*

(c) *C lifts to a $\mathcal{V}$-colocalization on the category of T-algebras for any $\mathcal{V}$-monad T on $\mathcal{V}$ with arities $\mathcal{A}$.*

Moreover, if $\mathcal{A}$ contains the unit of the monoidal structure, then (a), (b) *and* (c) *are equivalent. If these statements hold, then the lifting of C is unique up to isomorphism.*

Corollary 2.5. *For a regular cardinal λ, let $\mathcal{V}$ be a locally λ-presentable closed symmetric monoidal category and let $\mathcal{A}$ be a skeleton of the full subcategory of λ-presentable objects in $\mathcal{V}$. Let C be a $\mathcal{V}$-colocalization on $\mathcal{V}$. Then the following statements are equivalent:*

(a) *The functor $\mathcal{V}(A, -)$ preserves C-colocal objects for every $A \in \mathcal{A}$.*

(b) *C lifts to a $\mathcal{V}$-colocalization on the category of models of any theory enriched in $\mathcal{V}$ with arities $\mathcal{A}$.*

(c) *C lifts to a $\mathcal{V}$-colocalization on the category of T-algebras for any $\mathcal{V}$-monad T on $\mathcal{V}$ with arities $\mathcal{A}$.*

Moreover, if any of these conditions hold, then the lifting of C is unique up to isomorphism.

Example 2.6. For a unitary commutative ring R, consider the category $\mathsf{Ch}(R)$ of unbounded chain complexes of R-modules with its standard closed symmetric monoidal structure:

$$(X \otimes Y)_n = \bigoplus_{i \in \mathbb{Z}} X_i \otimes_R Y_{n-i}; \qquad \mathrm{Hom}(X,Y)_n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(X_i, Y_{n+i}).$$

Limits and colimits are computed degreewise, and every localization E on R-modules can be extended to an enriched localization $\tilde{E}$ on $\mathsf{Ch}(R)$ degreewise. If so done, then $\tilde{E}$ lifts to the category of models of any theory enriched in $\mathsf{Ch}(R)$ with arities

$$\mathcal{A} = \{\Sigma^k R^n \mid k \in \mathbb{Z},\, n \geq 1\}.$$

Such theories include change of rings from $\mathsf{Ch}(R)$ to $\mathsf{Ch}(R')$ for every central ring homomorphism $R \to R'$, as in Example 1.6. Condition (a) from Theorem 2.2 holds because

$$\mathrm{Hom}(\Sigma^k R^n, X) \cong \Sigma^{-k} X^n$$

for all k and n and every chain complex X, where X^n denotes n-fold direct sum of X with itself. The same fact is true for degreewise colocalizations, by Theorem 2.4.

Example 2.7. If we choose $\mathcal{A}$ to be a skeleton of the full subcategory of finitely generated abelian groups, then condition (a) in Theorem 2.2 or Corollary 2.3 is no longer fulfilled for all localizations. An explicit counterexample follows. Let $EG = G/mG$ where $m > 1$. This is a localization on Ab and $\mathrm{Hom}(\mathbb{Z}/m, -)$ does not preserve E-equivalences, since the projection $\mathbb{Z} \to \mathbb{Z}/m$ is an E-equivalence while the induced homomorphism $\mathrm{Hom}(\mathbb{Z}/m, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/m, \mathbb{Z}/m)$ is not. Accordingly, E does not lift to T-algebras if T sends each abelian group to its maximal torsion-free quotient, since the T-algebras are the torsion-free abelian groups. This monad T is not associated with any additive Lawvere theory, although it is associated with an additive theory with arities the finitely generated abelian groups.

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