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Anderson Localization from a Modern Point of View

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Abstract. Anderson localization at a set of primes P is a functor which is defined on arbitrary CW-complexes and is naturally equivalent to Sullivan's P-localization when restricted to the homotopy category of simply-connected spaces. However, its properties have never been described beyond the class of simple spaces. In the present note, we use recent developments in homotopy theory (mainly Dror Farjoun's localization with respect to a map) in order to discuss the effect of Anderson localization on the homotopy groups of arbitrary spaces. In particular, we point out its close connection with both Quillen's plus-construction and localization with respect to self-maps of the circle.

1. Introduction

Our main purpose is to update and supplement an old paper of Anderson [**2**], by using the machinery presented by Dror Farjoun in [**12**], together with the special case discussed by Peschke and the author in [**10**].

Anderson's paper was published shortly after the work of Sullivan on localization of simply-connected spaces at a set of primes [**18**]. While Sullivan's construction suffered from the fact that it was only functorial up to homotopy, Anderson's construction was functorial on the topological category; moreover, it did not require any restriction whatsoever on the homotopy groups of the spaces on which it could be applied. However, most of its basic properties, as described in [**2**], only hold under the assumption that the spaces be simple, and its behaviour on more general spaces has not been discussed so far.

In the present note, we show that for every connected space X, the 1-connected cover of the Anderson localization of X at a set of primes P is homotopy equivalent to the Bousfield–Kan $\mathbb{Z}_P$-completion [**9**]

$$(\mathbb{Z}_P)_\infty \widehat{X}$$

(where $\mathbb{Z}_P$ denotes the ring of integers localized at P) of a regular cover $\widehat{X}$ of X. This cover $\widehat{X}$ is classified by a certain subgroup of $\pi_1(X)$, which we call the P'-*radical* (the precise definition is given in Section 3). Hence, the effect of Anderson

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localization on the higher homotopy groups of non-simple spaces can be as drastic as the effect of homology localization with $\mathbb{Z}_P$ coefficients [**6**].

There is another well-known construction in homotopy theory for which something very similar to the above holds. Namely, if X is any connected space and we denote by $\widehat{X}$ the regular cover of X classified by the *perfect radical* [**4**] of $\pi_1(X)$, then $\mathbb{Z}_\infty \widehat{X}$ is homotopy equivalent to the 1-connected cover of Quillen's plus-construction X^+. Moreover, by replacing $\mathbb{Z}$ by any ring $R \subseteq \mathbb{Q}$, one obtains the same result, but involving an analog with R coefficients of the plus-construction, which was called *partial R-completion* in §6 of ch. VII in [**9**].

The common pattern in the two situations considered is better understood if formulated in the following language. Given any map $g\colon W \to V$ between CW-complexes, the g-localization functor L_g (in the sense of [**12**]) is to be compared with the Σg-localization functor, which we call the *semilocalization* relative to g, and if M is the mapping cone of g, then the M-periodization functor (i.e., localization with respect to $M \to \mathrm{pt}$) is an intermediate device between L_g and $L_{\Sigma g}$. We call it the *Anderson localization* or *generalized plus-construction* relative to g. The usual plus-construction occurs when g is any map for which L_g is homology localization with $\mathbb{Z}$ coefficients, and Anderson's original construction arises when L_g is the "localization with respect to self-maps of the circle" developed in [**10**].

One case of major interest, in view of the results of [**8**] and [**12**], is Anderson localization relative to a v_n self-map of a finite p-local CW-complex. In fact, the present paper may be viewed as the discussion of the case $n = 0$. Analysis of the case $n = 1$ might give relevant information on K-acyclic spaces.

I am indebted to Joe Neisendorfer for several illuminating comments, and to my colleagues in Barcelona for the joint work done on this topic during our 1993 seminar.

2. Homotopy localization with respect to a map

Overall, by "space" we mean a CW-complex with basepoint. Let $f\colon W \to V$ be any (based) map. As in [**12**], we call a space X *f-local* if the map of based function spaces

$$f^*\colon \mathrm{map}_*(V, X) \to \mathrm{map}_*(W, X)$$

is a homotopy equivalence. This terminology is motivated by the existence of an *f-localization* functor, turning f into a homotopy equivalence in a universal way. More precisely, as shown in [**12**],

Theorem 2.1. *For every map $f\colon W \to V$ there is a functor L_f on the pointed category of spaces, together with a natural transformation $\mathrm{Id} \to L_f$, such that, for every space X, the map $l\colon X \to L_f X$ is homotopy initial among all maps from X to f-local spaces.*

We emphasize the fact that L_f preserves commutativity of diagrams of spaces and maps, not only up to homotopy. On the other hand, L_f also defines a functor on the pointed homotopy category $\mathcal{H}$, and, if so viewed, then it is part of an idempotent monad on $\mathcal{H}$. Thus, L_f is left adjoint to the inclusion of the full subcategory of f-local spaces in $\mathcal{H}$.

Accordingly, a map $X \to Y$ is said to be an *f-equivalence* if the induced map $L_f X \to L_f Y$ is a homotopy equivalence. This is the same as imposing that $[Y, L] \to [X, L]$ be bijective for all f-local spaces L; see [**1**].

Particular instances of L_f are well-known; see [**12**] for an extensive list. In the special case when f is of the form $f\colon W \to \mathrm{pt}$, the functor L_f has also been called *W-localization* or *W-periodization* [**8**], [**13**], [**16**]. An interesting source of examples of this kind can be obtained as follows; cf. [**2**]. Let A be any abelian group. Choose a free abelian presentation

$$0 \to F_0 \to F_1 \to A \to 0$$

and a map $w\colon W_0 \to W_1$ between suitable wedges of circles inducing the inclusion $F_0 \to F_1$ on homology. Let M be the mapping cone of w. We consider localization with respect to the map

$$f\colon M \to \mathrm{pt}. \tag{2.1}$$

Then a space X is f-local if and only if $\mathrm{Hom}(\pi_1(M), \pi_1(X))$ is trivial, and

$$\mathrm{Hom}(A, \pi_k(X)) = \mathrm{Ext}(A, \pi_k(X)) = 0 \qquad \text{for } k \geq 2, \tag{2.2}$$

as one can check by applying $[\quad, X]$ to the cofibre sequence associated with w.

Since A is the abelianization of $\pi_1(M)$, this shows that the class of f-local spaces whose fundamental group is abelian does not depend on the choices made, but it only depends on A.

Example 2.2. Let $A = \mathbb{Z}/p$, where p is a prime. Choose w to be the degree p self-map of S^1. Thus, M is a Moore space of type $(\mathbb{Z}/p, 1)$. A space X is f-local if and only if $\pi_1(X)$ does not contain any p-torsion elements, and $\pi_k(X)$ is a $\mathbb{Z}[1/p]$-module for $k \geq 2$ (this follows from (2.2)). These are precisely the target spaces of Anderson localization away from the prime p; cf. [**2**]. But two idempotent monads on the same category with the same class of targets are necessarily isomorphic [**1**].

More generally, let P be any set of primes, and

$$A = \bigoplus_{q \notin P} \mathbb{Z}/q. \tag{2.3}$$

Let W be a wedge of circles, one for each prime $q \notin P$; define w to be the self-map of W which maps the circle labelled by q to itself by a map of degree q. Then a space X is f-local if and only if $\pi_1(X)$ does not contain any P'-torsion elements (P' denotes the complement of P), and $\pi_k(X)$ is a $\mathbb{Z}_P$-module for $k \geq 2$, where $\mathbb{Z}_P$ is the ring of integers localized at P. Thus, L_f is equivalent to Anderson localization at P.

Note that, if $\pi_1(X)$ is trivial, then $L_f X$ is homotopy equivalent to X_P, the P-localization of X in the sense of Sullivan [**18**], [**9**], [**14**] (again, because two idempotent monads on the same category with the same class of local objects have to be isomorphic).

On the other hand, L_f is *not* P-localization in general on spaces outside the category of simply-connected spaces; not even on simple spaces, since, for example, $L_f S^1 = S^1$. In the next two sections, we analyze the effect of L_f on spaces which are not simply-connected.

Example 2.3. This example was pointed out to us by Neisendorfer. If we pick $A = \mathbb{Z}[1/p]$ and choose a map f as above, then a simply-connected space X is f-local if and only if $\pi_k(X)$ is Ext-p-complete for $k \geq 2$, in the sense of [**9**]. Therefore, if we restrict ourselves to the homotopy category of simply-connected spaces, then L_f coincides with the p-completion functor $(\mathbb{F}_p)_\infty$, since $(\mathbb{F}_p)_\infty$ is homotopy idempotent on simply-connected spaces. Again, however, $L_f S^1 = S^1$, since $\mathrm{Hom}\,(\mathbb{Z}[1/p], \mathbb{Z}) = 0$, and therefore L_f is distinct from $(\mathbb{F}_p)_\infty$ on more general spaces.

3. Localizing with respect to a group homomorphism

In this section we introduce an auxiliary algebraic tool whose usefulness in our context will be clear; a detailed study of its properties will be undertaken elsewhere. Namely, we consider localization with respect to a given homomorphism $\varphi\colon G \to K$ of (discrete) groups. A group π will be called *φ-local* if the induced map of sets

$$\varphi^*\colon \mathrm{Hom}(K, \pi) \to \mathrm{Hom}(G, \pi)$$

is bijective.

THEOREM 3.1. *For every group homomorphism $\varphi\colon G \to K$ there is a functor L_φ on the category of groups, together with a natural transformation* $\mathrm{Id} \to L_\varphi$*, such that, for every group π, the map $l\colon \pi \to L_\varphi \pi$ is initial among all homomorphisms from π to φ-local groups.*

This can be proved by paralleling the arguments of [**12**], or, alternatively, by resorting to [**7**] or [**11**], where more general results are given ensuring the existence of certain left adjoints in complete or cocomplete categories. We call L_φ *φ-localization* of groups. In the special case when φ is of the form $\varphi\colon G \to \{1\}$, we also use the terms *G-localization* or *G-reduction* (the latter is borrowed from [**8**]). A homomorphism $\pi \to \nu$ is a *φ-equivalence* if $L_\varphi \pi \to L_\varphi \nu$ is an isomorphism.

THEOREM 3.2. *Let G be any group, and $\varphi\colon G \to \{1\}$. Then, for any group π, the G-reduction map $l\colon \pi \to L_\varphi \pi$ is surjective.*

(In fact, for a more general homomorphism $\varphi\colon G \to K$, the φ-localization map $l\colon \pi \to L_\varphi \pi$ is surjective whenever φ is surjective, but we will not prove this here. Instead, we give an explicit description of $L_\varphi \pi$ in the case when φ is of the form $G \to \{1\}$; this description will be useful in the sequel.)

PROOF. Set $T_0 = \{1\}$, and define a (possibly transfinite) sequence $\{T_\alpha\}$ as follows. If α is a successor ordinal and $T_{\alpha-1}$ is normal in π, take $T_\alpha / T_{\alpha-1}$ to be the subgroup of $\pi / T_{\alpha-1}$ generated by all the images of homomorphisms $G \to \pi / T_{\alpha-1}$ (then T_α is normal in π). For a limit ordinal λ, set $T_\lambda = \varinjlim T_\alpha$, where the direct limit ranges over all ordinals $\alpha < \lambda$. Since each T_α is a subset of π, the system $\{T_\alpha\}$ will eventually stabilize. We denote by $T_G(\pi)$ the direct limit, and call it the *G-radical* of π. It has the property that $\pi / T_G(\pi)$ is the largest quotient of π admitting no nontrivial homomorphism from G. Observe also that every homomorphism $G \to \pi$ maps in fact into $T_G(\pi)$, and therefore

$$T_G(T_G(\pi)) = T_G(\pi). \tag{3.1}$$

(Bousfield has used the notation $\pi /\!/ G$ for $\pi / T_G(\pi)$ in the case when both π and G are commutative [**8**].)

We now claim that the projection $\pi \twoheadrightarrow \pi / T_G(\pi)$ is the G-reduction of π. By construction, the group $\pi / T_G(\pi)$ is G-reduced (i.e., φ-local). Moreover, given any homomorphism $\psi\colon \pi \to L$ where L is G-reduced, one proves by transfinite induction that $\psi(T_\alpha) = \{1\}$ for all ordinals α, and hence ψ factors uniquely to a homomorphism $\overline{\psi}\colon \pi/T_G(\pi) \to L$. This property characterizes the G-reduction up to a unique isomorphism. Hence, the proof of Theorem 3.2 is complete.

The terminology "G-radical" comes from the fact that, if $G = \mathbb{Z}/p$ for a prime p, then $T_G(\pi)$ —which we abbreviate to $T_p(\pi)$— has been called elsewhere the *p-radical* of π, or also the *p-isolator*. This notion is generalized in an obvious way to an arbitrary set of primes P. Of course, if π is nilpotent, then $T_P(\pi)$ is just the P-torsion subgroup of π.

Another instance is the *perfect radical* [**4**], i.e., the maximal perfect subgroup, which is the G-radical if G is chosen to be the free product of a set of representatives of all isomorphism classes of countable perfect groups [**5**].

Proposition 3.3. *Let $f\colon W \to V$ be any map between connected spaces, and let $f_*\colon \pi_1(W) \to \pi_1(V)$ be the induced homomorphism of fundamental groups. Then*
(1) *A discrete group π is f_*-local if and only if its classifying space $B\pi$ is f-local.*
(2) *If $g\colon X \to Y$ is any f-equivalence of connected spaces, then the homomorphism $g_*\colon \pi_1(X) \to \pi_1(Y)$ is an f_*-equivalence of groups.*

Proof. If π is f_*-local, then f induces a bijection $[V, B\pi] \cong [W, B\pi]$. This ensures that $\mathrm{map}_*(V, B\pi) \simeq \mathrm{map}_*(W, B\pi)$, since all connected components of these function spaces are contractible. The converse is obvious.

To prove (2), let L be any f_*-local group. Then, by part (1), BL is f-local, and hence $g^*\colon [Y, BL] \to [X, BL]$ is bijective. This tells us precisely that the map $\mathrm{Hom}(\pi_1(Y), L) \to \mathrm{Hom}(\pi_1(X), L)$ is bijective, as required.

Now, for any space X and every map $f\colon W \to V$, the f-localization map $l\colon X \to L_f X$ is an f-equivalence. Therefore, if X, W, V are connected, then the homomorphism

$$l_*\colon \pi_1(X) \to \pi_1(L_f X)$$

is an f_*-equivalence of groups. This implies that there is a natural homomorphism

$$\pi_1(L_f X) \to L_{f_*}\pi_1(X), \tag{3.2}$$

which is an epimorphism if $V = \mathrm{pt}$, by Theorem 3.2. However, $\pi_1(L_f X)$ need not be f_*-local in general. Thus, (3.2) may fail to be an isomorphism. Here is an explicit counterexample:

Example 3.4. Let $W = B(\mathbb{Z}/p)$ and $f\colon W \to \mathrm{pt}$; then, by Miller's main theorem in [**15**], all finite-dimensional spaces are f-local. Therefore, if X is any finite space for which $\pi_1(X)$ contains p-torsion, then $\pi_1(L_f X)$ fails to be f_*-local.

The moral in the above example is that the fundamental group of a space X may contain p-torsion elements not detectable by any map $B(\mathbb{Z}/p) \to X$. In the light of this observation, the following result should be obvious:

Theorem 3.5. *Suppose given a map $f\colon W \to \mathrm{pt}$ where W is a connected space. Then, for a space X, we have*

$$\pi_1(L_f X) \cong L_{f_*}\pi_1(X)$$

if and only if every homomorphism $\pi_1(W) \to \pi_1(L_f X)$ is induced by some map $W \to L_f X$ (and hence is trivial).

Note that, if W is a CW-complex of dimension 1 or 2, then the condition stated in Theorem 3.5 is automatically satisfied.

4. Homotopy groups of Anderson localization

In this section, we restrict attention to Anderson localization L_f with respect to a set of primes P. Thus, we consider a self-map $w\colon W \to W$ of a wedge of circles, as in Example 2.2, and $f\colon M \to \mathrm{pt}$, where M is the mapping cone of w.

Since M is two-dimensional, Theorem 3.5 yields

Theorem 4.1. *Let L_f be Anderson localization. Then, for every space X,*

$$\pi_1(L_f X) \cong L_{f_*}\pi_1(X) \cong \pi_1(X)/T_{P'}(\pi_1(X)),$$

where $T_{P'}$ denotes the P'-radical.

This is the way in which Theorem 2.2 of [**2**] is to be understood in the case when the fundamental group $\pi_1(X)$ is not commutative.

The effect of L_f on the higher homotopy groups of arbitrary spaces can be better analyzed if compared with the w-localization functor L_w, which has been extensively studied in [**10**]. Among other things, it was shown that $L_w X$ is homotopy equivalent to the P-localization X_P when X is nilpotent. For this reason, L_w was called *P-localization* in [**10**]. The Anderson localization functor L_f is "weaker" than L_w, in the sense that every w-local space is also f-local, but not conversely (the space S^1 provides the easiest counterexample).

The functor $L_{\Sigma w}$ is also significant. It was first considered by Bendersky [**3**], who called it *semilocalization* at the set P. Its effect on arbitrary spaces is the following. The map $l\colon X \to L_{\Sigma w}X$ induces isomorphisms

$$\pi_1(X) \cong \pi_1(L_{\Sigma w}X)$$

and

$$\mathbb{Z}_P \otimes \pi_k(X) \cong \pi_k(L_{\Sigma w}X) \qquad \text{for } k \geq 2.$$

In other words, the 1-connected cover of $L_{\Sigma w}X$ is the P-localization of the 1-connected cover of X. Every f-local space is also Σw-local, but not conversely. Hence, there are natural transformations of functors

$$L_{\Sigma\omega} \to L_f \to L_w.$$

Of course, this situation is much more general. Starting from any map $g\colon W \to V$ one obtains a natural *g-tower* of localizations for every space X, by resorting to the cofibre sequence

$$W \to V \to C_g \to \Sigma W \to \Sigma V \to \Sigma C_g \to \cdots .$$

In the special case $g\colon S^0 \to \mathrm{pt}$, this produces the Postnikov tower. We plan to continue the study of g-towers elsewhere.

Theorem 4.2. *Let w, f be as defined above. Then the Anderson localization $L_f X$ and the semilocalization $L_{\Sigma w}X$ are homotopy equivalent if and only if $\pi_1(X)$ does not contain P'-torsion.*

Proof. One implication is clear. To prove the converse, note that both L_f and $L_{\Sigma w}$ give rise to idempotent monads on the full subcategory of $\mathcal{H}$ of spaces whose fundamental group is P'-torsion-free. In this category, the classes of local objects associated to L_f and $L_{\Sigma w}$ coincide. Hence, the two functors are naturally equivalent.

THEOREM 4.3. *Let w, f be as defined above. If $T_{P'}(\pi_1(X)) = \pi_1(X)$, then $L_f X$ and $L_w X$ are homotopy equivalent.*

PROOF. Consider the full subcategory of $\mathcal{H}$ of spaces whose fundamental group is killed by L_{f_*}, and argue as in the previous proof. It is important to observe that, if a group G satisfies $L_{f_*}G = \{1\}$, then $L_{w_*}G = \{1\}$ as well. (The functor L_{w_*} is the usual P-localization of groups [**17**].)

Now, if X is any connected space, we may consider the homotopy fibration

$$\widehat{X} \to X \to B\pi,$$

where $\pi = \pi_1(X)/T_{P'}(\pi_1(X)) = L_{f_*}\pi_1(X)$. Then $B\pi$ is f-local, and hence, by Theorem C in [**13**], the following sequence is again a homotopy fibration

$$L_f\widehat{X} \to L_f X \to B\pi.$$

But from (3.1) we obtain

$$T_{P'}(\pi_1(\widehat{X})) = T_{P'}(T_{P'}(\pi_1(X))) = T_{P'}(\pi_1(X)) = \pi_1(\widehat{X}), \tag{4.2}$$

and hence, by Theorem 4.3, $L_f\widehat{X} \simeq L_w\widehat{X}$. Therefore, determining the groups $\pi_k(L_f X)$ amounts to knowing the groups $\pi_k(L_w\widehat{X})$, about which [**10**] contains some information; for example, it is shown in §8 that the property $L_{w_*}\pi_1(\widehat{X}) = \{1\}$ ensures that the space $L_w\widehat{X}$ is homotopy equivalent to the Bousfield–Kan $\mathbb{Z}_P$-completion $(\mathbb{Z}_P)_\infty\widehat{X}$, and, moreover, $\widehat{X}$ is $\mathbb{Z}_P$-good. Therefore, we have proved the following:

THEOREM 4.4. *Let L_f be Anderson localization at a set of primes P. Then, for every connected space X there is a homotopy fibration*

$$(\mathbb{Z}_P)_\infty\widehat{X} \to L_f X \to B\pi,$$

where π is the quotient of $\pi_1(X)$ by its P'-radical, and $\widehat{X}$ is the homotopy fibre of the map $X \to B\pi$. Moreover, the space $(\mathbb{Z}_P)_\infty\widehat{X}$ is simply-connected.

5. Anderson localization as a partial homology localization

As commented in the Introduction, Theorem 4.4 suggests that Anderson localization at P is closely related to the plus-construction with $\mathbb{Z}_P$ coefficients. We next make more precise the role of homology in our setting.

In this section, $w\colon W \to W$ and $f\colon M \to \mathrm{pt}$ have the same meaning as in Section 4. If X is simply-connected, we know that

$$l_*\colon H_*(X;\mathbb{Z}_P) \cong H_*(L_f X;\mathbb{Z}_P), \tag{5.1}$$

since $L_f X$ is precisely the P-localization of X. Moreover, from the fact that $f\colon M \to \mathrm{pt}$ is an $H_*(\ \ ;\mathbb{Z}_P)$-equivalence it follows, as already observed in [**2**], that $l\colon X \to L_f X$ is also an $H_*(\ \ ;\mathbb{Z}_P)$-equivalence for every space X; that is, the isomorphism (5.1) holds for all spaces X, not necessarily simply-connected. But even more is true:

THEOREM 5.1. *Let $f\colon M \to \mathrm{pt}$ be as defined above. Then, for every space X, the f-localization map $l\colon X \to L_f X$ induces isomorphisms*

$$l_*\colon H_*(X;A) \cong H_*(L_f X;A)$$

and

$$l^*\colon H^*(L_f X;A) \cong H^*(X;A)$$

for every $\mathbb{Z}_P[\pi_1(L_f X)]$-module A.

PROOF. This follows from the fact that $\tilde{H}_*(M;A) = 0$ and $\tilde{H}^*(M;A) = 0$, where the action of $\pi_1(M)$ on A is defined via the map $f_*\colon \pi_1(M) \to \{1\}$, and hence is trivial.

In [**10**] it was shown that the P-localization functor L_w may be viewed as homology localization with certain $\mathbb{Z}_P$-module *twisted* coefficients. In order that such an assertion makes sense, it is necessary to move provisionally outside from the category $\mathcal{H}$. Thus, an alternative construction of $L_w X$, for a given space X, proceeds by considering first another category —which will depend on X—, whose objects are spaces Y equipped with a group homomorphism from $\pi_1(Y)$ to $L = L_{w_*}\pi_1(X) = \pi_1(X)_P$. In this category, one may localize with respect to homology with coefficients in a certain $\mathbb{Z}[L]$-module F; see Theorem 5.1 of [**10**]. Namely, F is the localization of the group ring $\mathbb{Z}[L]$ at the multiplicative system generated by the elements $1+x+x^2+\cdots+x^{n-1}$ with $x \in L$ and $n \in P'$. After that, one returns to $\mathcal{H}$ by forgetting the group homomorphisms attached to spaces. This procedure yields an idempotent monad on $\mathcal{H}$, which is said to be *spliceable* (i.e., obtained by "splicing" all these twisted homology localization functors from various categories depending on the spaces considered).

The Anderson localization functor L_f is also spliceable. The proof is analogous as the one given in [**10**] for L_w. Namely, one checks that the map $l\colon X \to L_f X$ is a localization in the homotopy category over the group $L = L_{f_*}\pi_1(X)$, with respect to homology with coefficients in $F = \mathbb{Z}_P[L]$. Thus the functors

$$L\pi = \pi/T_{P'}(\pi) \qquad \text{and} \qquad F\mathbb{Z}[L] = \mathbb{Z}_P[L]$$

match in the sense of §6 of [**10**]. Now, by resorting to Theorem 7.2 of [**10**], we can extend Theorem 5.1 as follows (thus characterizing the family of maps which are rendered invertible by Anderson localization at P).

THEOREM 5.2. *Let $f\colon M \to \mathrm{pt}$ be as defined above. A map $g\colon X \to Y$ of connected spaces is an f-equivalence if and only if $g_*\colon \pi_1(X) \to \pi_1(Y)$ is an f_*-equivalence of groups, and $g_*\colon H_*(X;A) \to H_*(Y;A)$ is an isomorphism for every $\mathbb{Z}_P[L_{f_*}\pi_1(Y)]$-module A.*

As already pointed out in [**10**], Bendersky's semilocalization $L_{\Sigma w}$ is spliceable too. The corresponding pair of matching functors is $L\pi = \pi$, $F\mathbb{Z}[L] = \mathbb{Z}_P[L]$. However, $L_{\Sigma^2 w}$ is not spliceable; hence, it is not true that the semilocalization associated to a spliceable functor is again spliceable in general.

Finally, let g be the wedge of a set of representatives of all isomorphism classes of integral homology equivalences between spaces with countably many cells. Thus, L_g is the usual homology localization with $\mathbb{Z}$ coefficients. This functor is obviously spliceable, with $L = \{1\}$ and $F = \mathbb{Z}$. The Anderson localization relative to g is the usual plus-construction, which is again spliceable: Take $L\pi = \pi/\mathcal{P}\pi$, where $\mathcal{P}$

denotes the perfect radical, and $F\mathbb{Z}[L] = \mathbb{Z}[L]$. The semilocalization relative to g is just the identity, since Σg is a homotopy equivalence.

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