The behaviour of homology in the localization of finite groups

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Abstract

We show that, for a finite group G and a prime p, the following facts are equivalent: (i) the p-localization homomorphism $l\colon G \to G_p$ induces p-localization on integral homology; (ii) the higher homotopy groups of the Bousfield-Kan $\mathbf{Z}_p$-completion of a $K(G,1)$ vanish; (iii) the group G is p-nilpotent.

0 Introduction

We deal with P-localization of groups in the sense of [2, 13, 19, 29]. That is, given a set of primes P, a group G is called *P-local* if every element $x \in G$ has a unique nth root in G for every P'-number n (as customary, we denote by P' the set of primes not in P and say that $n \in P'$, or that n is a *P'-number*, if all prime divisors of n belong to P'). A group homomorphism $l\colon G \to G_P$ is said to be a *P-localization* if it is universal (initial) among all group homomorphisms from G to P-local groups. Every group G admits a P-localization, which is unique up to isomorphism and functorial; see for example [19].

Since many other notions of localization and completion have been defined in group theory and homotopy theory, we have supplied an appendix recalling

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the definition of all functors referred to in this paper and detailing how they are related among themselves.

For a nilpotent group G, the induced homomorphisms

$$l_*\colon H_k(G;\mathbf{Z}_P) \to H_k(G_P;\mathbf{Z}_P) \tag{0.1}$$

are isomorphisms for all k, where $\mathbf{Z}_P$ denotes the ring of integers localized at P (viewed as a trivial coefficient module). This is false in general for nonnilpotent groups. For example, if Σ_3 denotes the symmetric group on three elements, then $(\Sigma_3)_3$ is trivial, while $H_3(\Sigma_3;\mathbf{Z}_3) \neq 0$.

We have been interested in finding examples of nonnilpotent groups for which (0.1) is still an isomorphism for all k and all sets of primes P. Locally nilpotent and locally free groups have this property [11, 23]. Other (finitely generated) examples are the infinite dihedral group and the fundamental group of the Klein bottle [7].

In this note, we point out that no such examples are to be found in the class of *finite* groups, for if G is finite and (0.1) is an isomorphism for all k, then G must be P-nilpotent; cf. Theorem 1.5 below. Recall that a finite group G is called *P-nilpotent* if the subgroup of G generated by all P'-torsion elements does not contain P-torsion. Thus, a finite group G is nilpotent if and only if it is p-nilpotent for all primes p.

Our characterization of P-nilpotence is very much in the spirit of [15, 16, 18, 24, 25, 26], where cohomological criteria for the p-nilpotence of a finite group G (p stands for a single prime) were given in terms of the inclusion of a p-Sylow subgroup $i : S \hookrightarrow G$. Recent progress on that topic [12, 17] has provided generalizations of the results cited to compact Lie groups, using the solution of the Segal conjecture [6].

In Section 2, we give a homotopy-theoretic interpretation of our previous result by showing that, for any finite group G and each single prime p, the higher homotopy groups of the Bousfield-Kan $\mathbf{Z}_p$-completion [5] of a $K(G,1)$

$$\pi_k(\mathbf{Z}_p)_\infty K(G,1), \quad k \geq 2,$$

can be precisely interpreted as the obstruction to G being p-nilpotent or, equivalently, to $l\colon G \to G_p$ inducing p-localization on the integral homology groups.

A simple, illustrative example is the symmetric group Σ_3, for which

$$(\mathbf{Z}_2)_\infty K(\Sigma_3,1) \simeq K(\mathbf{Z}/2,1),$$

while $(\mathbf{Z}_3)_\infty K(\Sigma_3,1)$ is a simply-connected space with complicated homotopy [5, VII, 4.4].

As explained in Section 3, our theorem can be extended to an arbitrary set of primes P, provided we replace $\mathbf{Z}_p$-completion by a certain idempotent functor, developed in [8, 9, 10], which induces P-localization of fundamental groups in the above sense.

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1 Homological localization and P-nilpotence

In what follows, we restrict ourselves to the class of finite groups, in which the effect of P-localization on homology is particularly easy to analyze. First of all, the following facts are readily checked.

Proposition 1.1 [20] *If G is a finite group and P a set of primes, then the following assertions are equivalent:*

(a) *G is P-local;*

(b) *the order of G is a P-number;*

(c) *each $H_k(G;\mathbf{Z})$, $k \geq 1$, is a $\mathbf{Z}_P$-module.* $\square$

Proposition 1.2 [20] *Let G be a finite group and P a set of primes. Then the P-localization homomorphism $l\colon G \to G_P$ is surjective and* $\ker l$ *is the subgroup generated by all P'-torsion elements of G.* $\square$

Theorem 1.3 *Let G be a finite group and P a set of primes. Then P-localization induces an isomorphism $l_*\colon H_1(G;\mathbf{Z}_P) \cong H_1(G_P;\mathbf{Z}_P)$ and an epimorphism $l_*\colon H_2(G;\mathbf{Z}_P) \twoheadrightarrow H_2(G_P;\mathbf{Z}_P)$.*

Proof. Consider the extension

$$\ker l \rightarrowtail G \overset{l}{\twoheadrightarrow} G_P \tag{1.1}$$

given by Proposition 1.2 and look at the associated five-term exact sequence [14] tensored with $\mathbf{Z}_P$:

$$H_2(G;\mathbf{Z}_P) \to H_2(G_P;\mathbf{Z}_P) \to \mathbf{Z}_P \otimes (\ker l/[G,\ker l]) \to H_1(G;\mathbf{Z}_P) \to H_1(G_P;\mathbf{Z}_P) \to 0.$$

Our assertion is a consequence of the vanishing of the middle term, which follows from Lemma 1.4 below. □

Lemma 1.4 *Let G be a finite group, P a set of primes, and $l\colon G \to G_P$ the P-localization homomorphism. If A is any abelian epimorphic image of* $\ker l$*, then $\mathbf{Z}_P \otimes A = 0$.*

Proof. Since $\ker l$ is generated by P'-torsion elements, A is necessarily a P'-torsion group. □

If the group G is nilpotent, then $l\colon G \to G_P$ in fact induces isomorphisms

$$l_*\colon H_k(G;\mathbf{Z}_P) \cong H_k(G_P;\mathbf{Z}_P) \quad \text{for all } k. \tag{1.2}$$

It is natural to ask for which—possibly larger—class of finite groups (1.2) still holds. This question has a precise answer:

Theorem 1.5 *Let G be a finite group, P a set of primes, and $l\colon G \to G_P$ the P-localization homomorphism. The following statements are equivalent:*

(a) *G is P-nilpotent;*

(b) $\ker l$ *is P'-torsion;*

(c) *$l_*\colon H_k(G;A) \cong H_k(G_P;A)$ for all k and every P-local abelian group A with an action of G_P (here we consider homology with twisted coefficients);*

(d) *$l_*\colon H_k(G;\mathbf{Z}_P) \cong H_k(G_P;\mathbf{Z}_P)$ for all k;*

(e) *$l_*\colon H_k(G;\mathbf{Z}_P) \to H_k(G_P;\mathbf{Z}_P)$ is a monomorphism for all k;*

(f) *$l_*\colon H_k(G;\mathbf{Z}) \to H_k(G_P;\mathbf{Z})$ is a P-localization for all $k \geq 1$.*

Proof. The equivalence of (a) and (b) is immediate from Proposition 1.2. To prove that (b)⇒(c), assume given an action $\omega\colon G_P \to \mathrm{Aut}(A)$, where A is abelian and P-local. Then the induced action of $\ker l$ on A is trivial. Since we are assuming that $\ker l$ is P'-torsion and A is a $\mathbf{Z}_P$-module, we have $H_k(\ker l; A) = 0$ for $k \geq 1$, and $H_0(\ker l; A) \cong A$. Therefore the Lyndon-Hochschild-Serre spectral sequence [14] associated to the extension (1.1) collapses and gives isomorphisms $l_*\colon H_k(G; A) \cong H_k(G_P; A)$ for all k, as stated. The implications (c)⇒(d) and (d)⇒(e) are trivial. We next show that (e)⇒(b). Thus assume that l_* is a monomorphism for all k, and assume further that $\ker l$ contains an element $x \neq 1$ whose order is a P-number. The next argument is essentially contained in [16]: Let $C = \langle x \rangle$ be the cyclic group generated by x and $j\colon C \hookrightarrow G$ the corresponding embedding. By Corollary 2 in [28], the homomorphism

$$j_*\colon H_k(C; \mathbf{Z}) \to H_k(G; \mathbf{Z})$$

is nonzero for infinitely many values of k. Choose one such value of $k \geq 1$. Since $H_k(C; \mathbf{Z})$ is a $\mathbf{Z}_P$-module, the image of j_* is contained in the P-torsion part of $H_k(G; \mathbf{Z})$. Therefore the composition

$$H_k(C; \mathbf{Z}_P) \xrightarrow{j_*} H_k(G; \mathbf{Z}_P) \xrightarrow{l_*} H_k(G_P; \mathbf{Z}_P)$$

is nonzero. This is absurd, because C is contained in $\ker l$ and hence lj is trivial.

Finally, the equivalence of (f) and (d) is obvious, for $H_k(G; \mathbf{Z})_P \cong \mathbf{Z}_P \otimes H_k(G; \mathbf{Z}) \cong H_k(G; \mathbf{Z}_P)$ and $H_k(G_P; \mathbf{Z}) \cong H_k(G_P; \mathbf{Z}_P)$ by Proposition 1.1. □

Corollary 1.6 *If G is a finite group for which the homomorphisms*

$$l_*\colon H_k(G; \mathbf{Z}) \to H_k(G_p; \mathbf{Z})$$

are p-localizations for all $k \geq 1$ and each prime p, then G is nilpotent. □

2 Topological interpretation

If we restrict ourselves to the case when the set P consists of a single prime p, then Theorem 1.5 can be expanded and partially reproved using the machinery of [5]. Specifically, the failure of $l\colon G \to G_p$ to induce p-localization on integral homology is detected by the appearance of higher homotopy in the $\mathbf{Z}_p$-completion of a $K(G,1)$. We recall from [5] (see also the Appendix below) that $\mathbf{Z}_p$-completion, $H_*(\quad;\mathbf{Z}_p)$-localization and p-profinite completion all coincide on spaces $K(G,1)$ with G finite.

If the group G is nilpotent, then the result of applying any of these functors to a $K(G,1)$ is a $K(G_p,1)$. We next prove that this still holds if we only assume G p-nilpotent, and that this property in fact characterizes p-nilpotence.

Lemma 2.1 *Let G be a finite group and $l\colon G \to G_p$ its localization at a given prime p. Then the sequence*

$$(\mathbf{Z}_p)_\infty K(\ker l,1) \to (\mathbf{Z}_p)_\infty K(G,1) \to (\mathbf{Z}_p)_\infty K(G_p,1)$$

induced by the extension (1.1) *is a homotopy fibration. Moreover, the fibre is simply-connected and $(\mathbf{Z}_p)_\infty K(G_p,1) \simeq K(G_p,1)$.*

Proof. For each $k \geq 1$, the induced action of G_p on $H_k(\ker l;\mathbf{Z}_p)$ is nilpotent because they are both finite p-groups. Thus our first assertion follows from [5, II, 5.1]; cf. also [4, 14.4]. To prove that $\pi_1(\mathbf{Z}_p)_\infty K(\ker l,1)$ is trivial, use [5, I, 6.1] after observing that $H_1(\ker l;\mathbf{Z}_p)=0$ by Lemma 1.4. The last assertion is deduced from the fact that G_p is a finite p-group and hence nilpotent p-local. □

Theorem 2.2 *Let G be a finite group and $l\colon G \to G_p$ its localization at a given prime p. The following statements are equivalent:*

(a) *G is p-nilpotent;*

(b) $l_*\colon H_k(G;\mathbf{Z}_p) \cong H_k(G_p;\mathbf{Z}_p)$ *for all k;*

(c) *l induces a homotopy equivalence $(\mathbf{Z}_p)_\infty K(G,1) \simeq K(G_p,1)$;*

(d) $\pi_k(\mathbf{Z}_p)_\infty K(G,1) = 0$ *for all $k \geq 2$.*

Proof. The equivalence of (a) and (b) has been proved in Section 1 (in fact, we only use here the easier implication (a)⇒(b)). To check that (b)⇒(c), observe that from (b) it follows that l induces a homotopy equivalence

$$(\mathbf{Z}_p)_\infty K(G,1) \simeq (\mathbf{Z}_p)_\infty K(G_p,1),$$

and $(\mathbf{Z}_p)_\infty K(G_p,1) \simeq K(G_p,1)$ by Lemma 2.1. The implication (c)⇒(d) is trivial. To prove that (d)⇒(a), consider the fibration given by Lemma 2.1. From assumption (d) it follows that the fiber is contractible. Now, since a $K(G,1)$ with G finite is always $\mathbf{Z}_p$-good [5, VII, 4.3], the homology groups

$$H_k(\ker l; \mathbf{Z}_p) \cong H_k((\mathbf{Z}_p)_\infty K(\ker l, 1); \mathbf{Z}_p)$$

vanish for $k \geq 1$. This implies, by [28], that p does not divide the order of $\ker l$ and therefore G is p-nilpotent. □

Although the space $(\mathbf{Z}_p)_\infty K(G,1)$ is far from being a $K(G_p,1)$ in general, it is important to point out the following corollary of Lemma 2.1:

Corollary 2.3 *For every finite group G and every single prime p, the fundamental group of $(\mathbf{Z}_p)_\infty K(G,1)$ is isomorphic to G_p.* □

If we consider a set P containing more than one prime, then Corollary 2.3 is false and statements (c) and (d) can no longer be included in Theorem 2.2. For example, let G be a nontrivial perfect finite group and P be the set of all primes dividing its order. Then G is P-local and P-nilpotent, yet $(\mathbf{Z}_P)_\infty K(G,1)$ is a simply-connected space (by [5, I, 6.1]) which is not contractible because G is not $\mathbf{Z}_P$-acyclic. A justification of this somehow disappointing feature is given in the next section.

3 On P-localization of spaces

The techniques developed in [8, 9, 10] actually allow improvement of some of the above results. These papers contain a proof of the existence of an idempotent functor [1] in the pointed homotopy category of CW-complexes extending P-localization of nilpotent spaces and inducing P-localization on the fundamental group; cf. Appendix. We denote this functor by $(\ \ )_P$ and call it *P-localization*.

The associated P-equivalences can be described as maps $f\colon X \to Y$ inducing a P-equivalence of fundamental groups and isomorphisms $f_* : H_k(X; A) \to$

$H_k(Y; A)$ for all k, with certain (twisted) coefficients A whose underlying abelian group is P-local [8, 9], which include trivial $\mathbf{Z}_P$ coefficients.

Here is an application of the existence of this functor (showing that the finiteness assumption can be removed from Theorem 1.3):

Theorem 3.1 *For each group G and each set of primes P, the P-localization homomorphism $l\colon G \to G_P$ induces an isomorphism $l_*\colon H_1(G; \mathbf{Z}_P) \cong H_1(G_P; \mathbf{Z}_P)$ and an epimorphism $l_*\colon H_2(G; \mathbf{Z}_P) \twoheadrightarrow H_2(G_P; \mathbf{Z}_P)$.*

Proof. The P-localization map $K(G, 1) \to K(G, 1)_P$ is an $H_*(\quad; \mathbf{Z}_P)$-equivalence and induces $l\colon G \to G_P$ on fundamental groups. Our assertion follows, as in [3, Lemma 6.1], from the fact that, for every space X, the natural map $X \to K(\pi_1 X, 1)$ induces an isomorphism on H_1 and an epimorphism on H_2 with arbitrary coefficients. □

If the set P consists of a single prime p, then the effect of p-localization on a $K(G, 1)$ with G finite turns out to be precisely $\mathbf{Z}_p$-completion, i.e.

$$K(G, 1)_p \simeq (\mathbf{Z}_p)_\infty K(G, 1).$$

The proof, based on Corollary 2.3, is provided in [8]. Thus, the following theorem may be viewed as a generalization of Theorem 2.2 to an arbitrary set of primes P, and this makes it clear that the difficulty in the example at the end of Section 2 lies in the non-coincidence of the functors $(\quad)_P$ and $(\mathbf{Z}_P)_\infty$ in general.

Theorem 3.2 *Let G be a finite group and P a set of primes. Then G is P-nilpotent if and only if $\pi_k K(G, 1)_P = 0$ for all $k \geq 2$.*

Proof. If $\pi_k K(G, 1)_P = 0$ for $k \geq 2$, then $K(G, 1)_P \simeq K(G_P, 1)$ and hence the homomorphism $l\colon G \to G_P$ is an $H_*(\quad; \mathbf{Z}_P)$-equivalence. Now the P-nilpotence of G is deduced from Theorem 1.5. Conversely, assume that G is P-nilpotent. Then, since $l\colon G \to G_P$ is certainly a P-equivalence of groups and moreover, by Theorem 1.5, it is an $H_*(\quad; A)$-equivalence for every coefficient module A whose underlying abelian group is P-local, it follows that the induced map $K(G, 1) \to K(G_P, 1)$ is a P-equivalence of spaces and hence a P-localization. This proves that the higher homotopy groups of $K(G, 1)_P$ vanish. □

4 Appendix: A roadmap on localization and completion

A good method to understand the relationship between the various localization functors existing in the literature is to compare the respective classes of "local objects" and "equivalences" associated to them. Following [1], if E is an idempotent functor in a category $\mathcal{C}$, we call *E-local* the objects X of $\mathcal{C}$ such that $X \cong EX$ and *E-equivalences* the maps $f\colon A \to B$ such that $Ef\colon EA \cong EB$. These two classes determine each other by means of a simple rule: An object X is E-local if and only if each E-equivalence $f: A \to B$ induces a bijection $f^*: \mathrm{Mor}(B, X) \cong \mathrm{Mor}(A, X)$, and a map $f: A \to B$ is an E-equivalence if and only if it induces a bijection $f^*\colon \mathrm{Mor}(B, X) \cong \mathrm{Mor}(A, X)$ for each E-local object X.

If we have two idempotent functors E_1, E_2 in the same category $\mathcal{C}$, and the class of E_1-local objects is contained in the class of E_2-local objects (or, equivalently, the class of E_2-equivalences is contained in the class of E_1-equivalences), then there is a natural transformation of functors $E_2 \to E_1$.

On the other hand, if $\mathcal{C}'$ is a subcategory of $\mathcal{C}$, and if E, E' are idempotent functors in $\mathcal{C}$, $\mathcal{C}'$ respectively, then we say that E *extends* E' if both the class of E'-local objects is contained in the class of E-local objects, and the class of E'-equivalences is contained in the class of E-equivalences. If this is the case, then for every object X of the subcategory $\mathcal{C}'$, the objects $E'X$ and EX are naturally isomorphic. This approach is the starting point of the more detailed discussion on localization functors in categories contained in [9, 10].

(Gr 1) P-localization of nilpotent groups. In the category of nilpotent groups, the local objects associated to the P-localization functor developed by Bousfield-Kan [5], Hilton-Mislin-Roitberg [13], and Warfield [29] are the nilpotent groups in which P'-roots exist and are unique, and the equivalences are the P'-bijections [13]. These can alternatively be described as being homomorphisms $\varphi\colon G \to K$ such that $\varphi_*\colon H_k(G; \mathbf{Z}_P) \cong H_k(K; \mathbf{Z}_P)$ for all k.

(Gr 2) P-localization of groups. The localization functor in the category of all groups considered by Ribenboim [19], after earlier work of other authors on radicability in groups (see [2] and the references there), has as local objects —called P-*local groups*—the groups in which P'-roots exist and are unique. As far as we know, the class of associated P-equivalences has not explicitly been characterized in any useful form. However, it has been proved [11, 20] that this functor, which we denote by $(\ \ )_P$, indeed *extends* P-localization of nilpotent groups to the category of all groups.

(Gr 3) HR-localization of groups. Let R be a subring of the rationals or a finite cyclic ring. Bousfield defined in [3, 4] a class of groups called HR-*local* as follows (in fact, this definition makes sense for R an arbitrary abelian group). A group homomorphism $\varphi\colon K \to L$ inducing an isomorphism $\varphi_*\colon H_1(K;R) \cong H_1(L;R)$ and an epimorphism $\varphi_*\colon H_2(K;R) \twoheadrightarrow H_2(L;R)$ is called an HR-*map*, and a group G is said to be HR-*local* if each HR-map $\varphi\colon K \to L$ induces a bijection $\varphi^*\colon \mathrm{Hom}(L,G) \cong \mathrm{Hom}(K,G)$. There is an idempotent functor E_R in the category of groups, called HR-*localization*, whose local objects are the HR-local groups. Warning: the class of equivalences associated to E_R is *strictly bigger* than the class of HR-maps in general.

In the case $R = \mathbf{Z}_P$, this functor—which we denote by E_P for simplicity—extends P-localization of nilpotent groups [4]. Since multiplication by a P'-number $\mathbf{Z} \xrightarrow{\cdot n} \mathbf{Z}$ is an $H\mathbf{Z}_P$-map, it follows that $H\mathbf{Z}_P$-local groups are P-local. Hence, there is a natural transformation of functors $(\ \ )_P \to E_P$ in the category of groups. The homomorphism $G_P \to E_P G$ is an isomorphism in some cases; for example, whenever G_P is nilpotent.

Of course, there are many other functors extending P-localization of nilpotent groups to the category of all groups. The family of all them is partially ordered by inclusion of the respective classes of local objects. Moreover, it is easy to see [10] that the functor $(\ \ )_P$ is *initial* in this family. That is, for any other functor E extending P-localization of nilpotent groups to all groups, there is a natural transformation of functors $(\ \ )_P \to E$.

(Gr 4) R-completion of groups. Let R be a ring with 1. A group N is called R-*nilpotent* [5] if it has a finite central series in which the factors admit an R-module structure. For example, $\mathbf{Z}_P$-nilpotent groups are precisely P-local nilpotent groups. The *R-completion* of a group G, denoted by $G^\wedge_R$, is the inverse limit of a cofinal diagram in the system of all targets of homomorphisms from G to R-nilpotent groups [4, 5]. In general, $G^\wedge_R$ need not be R-nilpotent itself, but it is always HR-local. Hence, there is a natural transformation of functors $E_R \to (\quad)^\wedge_R$. If the group G is finitely generated, then the homomorphism $E_R G \to G^\wedge_R$ is surjective [4].

If $R = \mathbf{Z}/p$ and the group G is finitely generated, then $G^\wedge_R$ is isomorphic to the p-profinite completion [22] of G. If $R = \mathbf{Z}_P$, then [5]

$$G^\wedge_R \cong \lim_{\leftarrow} (G/\Gamma_k G)_P,$$

where $\Gamma_k G$ denotes the lower central series of the group G. In particular, if G is nilpotent, then the $\mathbf{Z}_P$-completion of G is isomorphic to G_P. However, since the R-completion functor is *not* idempotent on arbitrary groups, our previous considerations used to compare idempotent functors do not apply to it.

(Ho 1) P-localization of nilpotent spaces. In the pointed homotopy category of nilpotent CW-complexes, the local objects associated to the P-localization functor described by several authors (Adams [1], Bousfield-Kan [5], Hilton-Mislin-Roitberg [13], Sullivan [27]), are the nilpotent spaces whose homotopy (and integral homology) groups are P-local. Equivalently, P-local nilpotent spaces are those in which the nth power map $\Omega X \to \Omega X$, $\omega \mapsto \omega^n$, is a homotopy equivalence for every $n \in P'$, cf. [8, 21]. P-equivalences are maps $f\colon X \to Y$ inducing isomorphisms $f_*\colon H_k(X; \mathbf{Z}_P) \cong H_k(Y; \mathbf{Z}_P)$ for all k. For every nilpotent group G, the space $K(G,1)_P$ is a $K(G_P, 1)$.

(Ho 2) P-localization of spaces. As proved in [8, 9, 10], there exists an idempotent functor in the pointed homotopy category of CW-complexes whose local objects are those spaces in which the nth power map $\Omega X \to \Omega X$, $\omega \mapsto \omega^n$, is a homotopy equivalence for every $n \in P'$. We denote this functor by $(\quad)_P$ and call it *P-localization.* It has been shown [8] that it extends P-localization of nilpotent spaces. Furthermore, it is related to P-localization in the category of groups by

$$G_P \cong \pi_1 K(G,1)_P.$$

The associated class of P-equivalences is properly contained in the class of all $H_*(\ \ ;\mathbf{Z}_P)$-equivalences.

(Ho 3) h_*-localization of spaces. For each additive homology theory h_* in the pointed homotopy category of CW-complexes, Bousfield proved in [3] the existence of an *h_*-localization* functor, i.e. an idempotent functor whose equivalences are maps $f\colon X \to Y$ inducing isomorphisms $f_*\colon h_k(X) \cong h_k(Y)$ for all k. For any abelian group R, HR-localization E_R in the category of groups is related to the $H_*(\ \ ;R)$-localization functor (which we denote by E_R as well) by

$$E_R G \cong \pi_1 E_R K(G,1).$$

The $H_*(\ \ ;\mathbf{Z}_P)$-localization functor (written E_P to simplify the notation) extends P-localization of nilpotent spaces to all spaces, and it is not difficult to see that E_P is indeed *final* in the family of all such extensions [10]. In particular, if $(\ \ )_P$ is the P-localization functor, then there is a natural transformation of functors $(\ \ )_P \to E_P$. The map $X_P \to E_P X$ is a homotopy equivalence in some cases, e.g. when the space X_P is nilpotent.

(Ho 4) R-completion of spaces. Given a ring R with 1, a nilpotent space X is called *R-nilpotent* [5] if all its homotopy groups $\pi_k X$, $k \geq 1$, are R-nilpotent. For each space X, Bousfield and Kan constructed a functorial *R-completion* $\phi\colon X \to R_\infty X$ by taking the homotopy inverse limit of a cofinal diagram in the system of all targets of maps from X to R-nilpotent spaces. It has the property that, if $f\colon X \to Y$ is an $H_*(\ \ ;R)$-equivalence, then the induced map $f_*\colon R_\infty X \to R_\infty Y$ is a homotopy equivalence. A space X is called *R-good* if $\phi\colon X \to R_\infty X$ is an $H_*(\ \ ;R)$-equivalence. For every connected space X, the space $R_\infty X$ is $H_*(\ \ ;R)$-local [5, II, 2.8], and hence there is a natural map $E_R X \to R_\infty X$, which is a homotopy equivalence if and only if X is R-good.

Since nilpotent spaces are R-good for $R = \mathbf{Z}_P$ [5], the functor $(\mathbf{Z}_P)_\infty$ coincides with P-localization on nilpotent spaces. However, $(\mathbf{Z}_P)_\infty$ is not idempotent on arbitrary spaces, and therefore it cannot be compared to the previous functors using the methods of this section.

If X is a space such that $H_k(X; \mathbf{Z}/p)$ is finite for all k, then $(\mathbf{Z}/p)_\infty X$ has the homotopy type of the p-profinite completion $X_p^\wedge$ in the sense of Sullivan [27]. If G is a finite group and P a set of primes, then [5, VII, §4]

$$E_P K(G,1) \simeq (\mathbf{Z}_P)_\infty K(G,1) \simeq \prod_{p \in P} (\mathbf{Z}/p)_\infty K(G,1) \simeq \prod_{p \in P} K(G,1)_p^\wedge.$$

For every group G there is an epimorphism [4, p. 66]

$$\pi_1 R_\infty K(G,1) \twoheadrightarrow G_R^\wedge,$$

which is an isomorphism in many cases, e.g. for G free, G nilpotent or G finite.

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