Recent Advances in Unstable Localization

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Dedicated to Peter Hilton on the occasion of his 70th birthday

1. Introduction

The essentials of localization of 1-connected spaces —or, more generally, nilpotent spaces— at a set of primes P were solidly established between 1970 and 1975. Since then, the monograph by Hilton, Mislin and Roitberg [**HMR**] has been a fundamental piece of reference which has made the theory available to a broad public of potential users. Indeed, many of these became successful users and have been enriching the theory during two decades.

The insight of Bousfield and Kan [**BK**], [**Bo1**], [**Bo3**] broadened largely the domain of application of localization techniques. However, it became clear that dealing with nonnilpotent spaces carried considerable difficulties, and often would lead to finding mysterious effects of certain functors on the homotopy groups of spaces.

When, shortly afterwards, Bousfield transferred that machinery to stable homotopy theory [**Bo4**], the scope of localization increased again. Nowadays, most attempts to determine the structure of the homotopy groups of spheres require localization as a basic tool. One of its deepest forms is localization with respect to Morava K-theories, which plays a key role in the study of periodicity in homotopy theory [**Ra1**], [**DHS**], [**HS**].

I cannot attempt to present the status of all what is being currently investigated in connection with localization. My aim in this article is to collect instead a certain number of new ideas which have arisen more or less since 1988. Although the basic concepts behind these ideas are not new to specialists, several observations and applications have been recognized as truly original. Furthermore, they seem to be far from exhausted as sources of new results.

The first half of the paper contains a summary of our own contribution. The results in Section 4 are going to appear in more detailed form as a joint paper with George Peschke [**CP**]. Indeed, the influence of Peschke's philosophy is obvious in many parts of the present survey, as well as the deep input from Markus Pfenniger.

Before presenting that material, we explain in Section 3 why the naïve attempt of constructing a reflection onto the class of spaces with P-local homotopy groups cannot possibly work in the based homotopy category of CW-complexes $\mathcal{H}$ (not necessarily nilpotent). The key point in overcoming this difficulty is to take full advantage of the topology of (based) mapping spaces $\mathrm{map}_*(X,Y)$, instead of restricting attention to the "purely categorical" device $[X,Y]$, the set of morphisms in $\mathcal{H}$ from X to Y. Thus, in order to make invertible a prime q in the homotopy groups of spaces, one should not take as targets for the functor those spaces for which

$$q^*\colon \pi_n(X) \to \pi_n(X)$$

is bijective for all n, where q^* is induced by the degree q map of S^n; instead, one should consider spaces X for which

$$q^*\colon \Omega^n X \to \Omega^n X$$

is a homotopy equivalence for all n (in fact, there is no restriction in imposing this condition only for $n = 1$).

This idea was contained in embryonal form in [**Pe2**] and was pursued further in [**CP**], [**CPP**]; but it only took its full force when Dror Farjoun revived and developed an old observation of Bousfield [**Bo3**], by showing that it was possible to "invert" *any* map $f\colon A \to B$ in $\mathcal{H}$ in that broader sense, not only $q\colon S^n \to S^n$. More precisely, the class of spaces X for which

$$f^*\colon \mathrm{map}_*(B,X) \to \mathrm{map}_*(A,X)$$

is a weak homotopy equivalence turns out to be reflective in $\mathcal{H}$ for every map f, while the class of spaces X for which

$$f^*\colon [B,X] \to [A,X]$$

is a bijection often fails to be reflective. This is good news for people having once tried to find optimal conditions on certain families of maps in $\mathcal{H}$ enabling these maps to be rendered invertible in a universal way (not by passing to the category of fractions, but by means of a suitable functor within $\mathcal{H}$). This is very tricky, if attacked with classical category-theoretical tools; cf. [**Ad2**], [**DH**].

In the second half of the present article, a few aspects of that new approach to localization have been selected, by attempting to sketch the main ideas and including findings of Dror Farjoun [**Fa1**], [**Fa2**], [**Fa3**], [**Fa4**], [**Fa5**], Dror Farjoun–Smith [**FS**], Bousfield [**Bo5**], and Neisendorfer (unpublished). I am indebted to them for keeping me informed and for many tutorials. I also acknowledge several ideas from members of our topology seminar in Barcelona, which arose while reading the above preprints in early 1993, and have been included in my own presentation of the subject here.

Finally, I take the liberty to repeat under these acknowledgements certain feelings that I already expressed on the occasion of the meeting in Montréal. I belong to the youngest generation having entered into mathematics by reading the work of Peter Hilton, which is, as you know, a very pleasant experience. In addition, I am in big debt with him for teaching me further himself, not only localization, nor only mathematics. *This* is an even more pleasant experience. In the exposition

which follows, the successful parts reflect what I could learn from him. For the unsuccessful, I apologize to him and to the reader.

2. Reflective Subcategories

Homotopy theorists did not invent localization, but adapted it so as to be useful in homotopy theory. Therefore, most of the current terminology about localization originated in commutative algebra or category theory, and many results familiar to homotopy theorists admit in fact a more abstract formulation, which is often more enlightening.

Thus, we adopt in this paper the earlier point of view of Adams [**Ad2**], by emphasizing that localizations may be viewed as *idempotent monads*, about which there is an extensive literature available; see e.g. [**DFH**] or [**BW**] and their references. A *monad* or *triple* $\mathbf{E} = (E, \eta, \mu)$ in a category $\mathcal{C}$ consists of a functor $E\colon \mathcal{C} \to \mathcal{C}$ together with natural transformations $\eta\colon \mathrm{Id} \to E$ (called *unit*) and $\mu\colon E^2 \to E$ (called *multiplication*) such that

$$\mu \cdot \mu E = \mu \cdot E\mu \qquad \text{and} \qquad \mu \cdot \eta E = \mu \cdot E\eta = \mathrm{Id}.$$

A monad (E, η, μ) is *idempotent* if $E\eta = \eta E$; this condition is equivalent to μ being a natural equivalence of functors; see [**De**].

An idempotent monad $\mathbf{E} = (E, \eta, \mu)$ is characterized by either its class $\mathcal{S}(E)$ of *E-equivalences* (morphisms f such that Ef is invertible) or its class $\mathcal{D}(E)$ of *E-local objects* (objects in the image of E); see [**Ad2**] for further details. For every object X of $\mathcal{C}$, the morphism $\eta_X\colon X \to EX$ will be called the *E-localization* of X. It is an E-equivalence, which is in fact terminal among all E-equivalences going out of X, and at the same time initial among all morphisms from X to E-local objects. Thus, if we view E as a functor from $\mathcal{C}$ to $\mathcal{D}(E)$, then it is left adjoint to the inclusion of $\mathcal{D}(E)$ into $\mathcal{C}$ (we will not distinguish between a class of objects and the full subcategory with those objects). The term “localization” seems to be appropriate in this general setting, since the effect of E is precisely “inverting” a certain class of arrows, namely all E-equivalences.

As observed in [**CPP**], it turns out to be very convenient to distill one further abstract notion —the notion of *orthogonal pair*— from the properties of localizations. If $\mathcal{C}$ is any category, an object X and a morphism $f\colon A \to B$ are called *orthogonal* (this term has been borrowed from Freyd and Kelly [**FK**]) if

$$f^*\colon \mathcal{C}(B, X) \cong \mathcal{C}(A, X),$$

that is, if for any morphism $g\colon A \to X$ there is a unique $h\colon B \to X$ such that $hf = g$. For example, the abelianization $\varphi\colon G \to G/[G, G]$ of a group G is orthogonal to all abelian groups.

If $\mathcal{S}$ is a class of morphisms in $\mathcal{C}$, we denote by $\mathcal{S}^\perp$ the class of objects orthogonal to all morphisms in $\mathcal{S}$. If $\mathcal{S}$ consists of a single morphism f —which will often be the case in this article— we will abbreviate $\{f\}^\perp$ to $f^\perp$. For a class of objects $\mathcal{D}$, we use the notation $\mathcal{D}^\perp$ in the same way.

Definition 2.1. An *orthogonal pair* $(\mathcal{S}, \mathcal{D})$ in $\mathcal{C}$ consists of a class of morphisms $\mathcal{S}$ and a class of objects $\mathcal{D}$ such that $\mathcal{S}^\perp = \mathcal{D}$ and $\mathcal{D}^\perp = \mathcal{S}$.

If $(\mathcal{S}, \mathcal{D})$ is an orthogonal pair, then $\mathcal{S}$ and $\mathcal{D}$ are *saturated*, meaning that $\mathcal{D}^{\perp\perp} = \mathcal{D}$ and $\mathcal{S}^{\perp\perp} = \mathcal{S}$. Also, taking three times the orthogonal class is exactly the same as doing it once. Hence, for example, $(\mathcal{S}^{\perp\perp}, \mathcal{S}^{\perp})$ is an orthogonal pair for every class of morphisms $\mathcal{S}$, which we call the orthogonal pair *generated* by $\mathcal{S}$.

A major source of interesting orthogonal pairs is, of course, the theory of idempotent monads. Indeed, for every idempotent monad (E, η, μ), the classes $\mathcal{S}(E)$ and $\mathcal{D}(E)$ of E-equivalences and E-local objects form an orthogonal pair. At this point it might be asked if, given a monad $\mathbf{E} = (E, \eta, \mu)$, the fact that $\mathcal{S}(E)$, $\mathcal{D}(E)$ form an orthogonal pair implies that $\mathbf{E}$ is idempotent. This has been answered in the negative in [**CFT**].

It is not rare to find researchers looking for a proof of existence of certain localization functors (or, more generally, of left adjoints). Again, there is an extensive literature about that. We will be interested in those situations where a specific orthogonal pair $(\mathcal{S}, \mathcal{D})$ is given, and the question is to decide whether there is an idempotent monad (E, η, μ) such that $\mathcal{S} = \mathcal{S}(E)$ and $\mathcal{D} = \mathcal{D}(E)$. In fact, we will even be more restrictive, since all instances of this problem in the sequel will be of the following kind.

Problem 2.2. *Given a category $\mathcal{C}$ and a morphism f, is the orthogonal pair $(f^{\perp\perp}, f^{\perp})$ associated with an idempotent monad?*

Note that, if coproducts exist in the category $\mathcal{C}$, then asking the above question for a single morphism f is equivalent to asking it for a set of morphisms $\{f_i \mid i \in I\}$, since the following conditions for an object X are equivalent:

(1) X is orthogonal to each f_i.
(2) X is orthogonal to the coproduct f of the set $\{f_i\}$.

In those cases when the answer to Problem 2.2 is affirmative, it is said that the subcategory $f^{\perp}$ is *reflective*. If (E, η, μ) is a solution, then it is of course unique up to an isomorphism of monads. For an object X, the E-localization $\eta_X \colon X \to EX$ is sometimes called a *reflection* of X onto the subcategory $f^{\perp}$. For example, abelianization is a reflection in the category of groups onto the subcategory of abelian groups, which is indeed of the form $f^{\perp}$; see the last paragraph of this article.

The most general form of Problem 2.2 (involving a possibly proper class of morphisms $\mathcal{S}$ instead of a single f) is called the *orthogonal subcategory problem*. It admits a solution in complete or cocomplete categories under rather mild assumptions; see [**Bo3**], [**Ke**], [**Pf**]. However, the situation is more complicated in categories such as the homotopy category of CW-complexes. In [**CPP**] we discuss what can be done in this category, and more generally in categories where coproducts and weak colimits exist.

3. Spaces with P-Local Homotopy Groups

We next specialize to the realm of homotopy theory and give one fundamental example. Let us denote by $\mathcal{H}$ the based homotopy category of CW-complexes. Thus a space X and a (based) map $f \colon A \to B$ are orthogonal if $f^* \colon [B, X] \to [A, X]$ is a bijection of (based) homotopy classes of maps. We denote by $\mathcal{N}$ the full subcategory of $\mathcal{H}$ of nilpotent spaces.

From now on we fix a set of primes P (which may be empty) and denote, as usual, by P' its complement (which we normally assume not to be empty!). For every prime $q \in P'$, let

$$\rho_{q,n} \colon S^n \to S^n$$

be the standard map of degree q. Let f be the coproduct of the maps $\rho_{q,n}$ for $q \in P'$ and $n \geq 1$. We will be concerned with the orthogonal subcategory problem (Problem 2.2) for this particular map f, firstly in the category $\mathcal{N}$, and secondly in the whole category $\mathcal{H}$.

In any case, a space X belongs to $f^{\perp}$ if and only if

$$(\rho_{q,n})^* \colon \pi_n(X) \cong \pi_n(X)$$

for all $q \in P'$ and $n \geq 1$. But $(\rho_{q,n})^*$ is just multiplication by q if $n \geq 2$, and the qth power map for $n = 1$ (we will denote the fundamental group multiplicatively at all times). Hence, the class $f^{\perp}$ consists of spaces X whose higher homotopy groups admit a $\mathbb{Z}_P$-module structure (where $\mathbb{Z}_P$ denotes the ring of integers localized at P), and whose fundamental group is *uniquely* P'*-radicable*, i.e., the qth power map $x \mapsto x^q$ is bijective in $\pi_1(X)$ for all $q \in P'$. We call such groups P*-local*, as in [**HMR**] or [**Ri1**], both in the commutative and the noncommutative case. That is, spaces in $f^{\perp}$ are spaces with P-local homotopy groups.

It is well known that $f^{\perp}$ is reflective in $\mathcal{N}$. The associated idempotent monad is the P-localization described in the early seventies by Hilton–Mislin–Roitberg and Bousfield–Kan, after the first insight of Sullivan [**Su**].

However, it may be surprising to discover that

Theorem 3.1. *$f^{\perp}$ fails to be reflective in $\mathcal{H}$.*

The argument is related to the following observation of Mislin; cf. [**Fa1**]: The class of 1-connected spaces is not reflective in $\mathcal{H}$. For suppose it were; then there would be a map $\eta \colon \mathbb{R}P^2 \to Y$ universal among all maps from the real projective plane to 1-connected spaces. In particular, η would be orthogonal to a $K(\mathbb{Z}, 2)$, so that

$$H^2(Y) \cong H^2(\mathbb{R}P^2) \cong \mathbb{Z}/2.$$

But, since $H_1(Y) = 0$,

$$H^2(Y) \cong \mathrm{Hom}(H_2(Y), \mathbb{Z}),$$

and this group can never be isomorphic to $\mathbb{Z}/2$. In summary, it is not possible to "1-connectify" an arbitrary space X as if we were abelianizing a group (the universal covering map $\tilde{X} \to X$ goes in the opposite direction!).

Using a similar line of argument, in order to prove Theorem 3.1 we may use the fact that $H^2(G; A[G]) \neq 0$ for every noncyclic subgroup G of $\mathbb{Q}$ and every nonzero abelian group A (see [**Ca**] and the references therein). Here $A[G]$ denotes the abelian group of formal sums of elements of G with coefficients in A, which is a $\mathbb{Z}[G]$-module under the multiplication of G.

Suppose that there is a map $\eta \colon S^1 \to Y$ universal among all maps from the circle to spaces with P-local homotopy groups. Then η is orthogonal to the classifying space of any P-local discrete group G. But the bijection

$$\eta^* \colon [Y, K(G,1)] \cong [S^1, K(G,1)]$$

is equivalent to the bijection

$$\eta^* \colon \mathrm{Hom}(\pi_1(Y), G) \cong \mathrm{Hom}(\pi_1(S^1), G),$$

and this tells us that $\eta_*\colon \pi_1(S^1) \to \pi_1(Y)$ is orthogonal to all P-local groups G. Hence, $\pi_1(Y) \cong \mathbb{Z}_P$ (see the remarks about P-localization of arbitrary groups in Section 4).

Now let L be a "twisted Eilenberg–Mac Lane space" with

$$\pi_1(L) \cong \mathbb{Z}_P, \qquad \pi_2(L) \cong \mathbb{Z}_P[\mathbb{Z}_P], \qquad \pi_k(L) = 0 \quad \text{if } k \geq 3,$$

where the action of π_1 on π_2 is multiplication (this is a special instance of the action of a group G on $A[G]$). In our case, the single twisted k-invariant [**Hill**] of L is necessarily zero. Spaces of this kind classify cohomology with twisted coefficients, in a certain precise sense; see Theorem 7.18 in [**Gi**] or §3 of [**DH**].

Since L has P-local homotopy groups, there is a bijection

$$\eta^*\colon [Y, L] \cong [S^1, L].$$

By restricting this bijection to maps inducing the identity (resp. an inclusion) of fundamental groups, we infer that

$$H^2(Y; \mathbb{Z}_P[\mathbb{Z}_P]) \cong H^2(S^1; \mathbb{Z}_P[\mathbb{Z}_P]),$$

which is zero, while the Cartan–Leray spectral sequence for the universal cover $\widetilde{Y}$ shows that $H^2(Y; \mathbb{Z}_P[\mathbb{Z}_P])$ must contain a subgroup isomorphic to $H^2(\mathbb{Z}_P; \mathbb{Z}_P[\mathbb{Z}_P])$. But this group, as we said at the beginning of this discussion, is nonzero. This yields the desired contradiction.

Theorem 3.1 might deceive (or relieve) anyone having tried to extend the core of Hilton–Mislin–Roitberg to arbitrary CW-complexes, not necessarily nilpotent. It becomes clear that the most obvious approach cannot work.

But it is known, since Bousfield [**Bo1**], that there *are* idempotent monads in $\mathcal{H}$ extending P-localization of nilpotent spaces. In fact, as we shall discuss in Section 5, there are *many* such monads. Yet, the class of local spaces associated with any one of these monads must be more restricted than the class of spaces with P-local homotopy groups. For example, Bousfield gave in Theorem 5.5 of [**Bo1**] a purely algebraic description of $H_*(\quad; \mathbb{Z}_P)$-local spaces, which involves a complicated condition on the fundamental group and an analogous condition on its action on the higher homotopy groups. These conditions force the homotopy groups of $H_*(\quad; \mathbb{Z}_P)$-local spaces to be P-local, but much more than that. In the next section we describe another interesting class of spaces which includes all P-local nilpotent spaces and is reflective in $\mathcal{H}$.

Before closing this section, we ask a simple question which is related to our previous discussion.

QUESTION 3.2. *Is the class $\mathcal{D}$ of 1-connected rational spaces reflective in $\mathcal{H}$?*

The plausible answer is, of course, no; but we have been unable to prove it so far. The less ambitious reader may undertake the easier exercise of finding a map f such that $\mathcal{D} = f^{\perp}$.

4. Localizing with Respect to Self Maps of the Circle

It has been long known that a connected space X is nilpotent if and only if the groups $[W, \Omega X]$ are nilpotent for every finite CW-complex W; see [**Ro**] and

Corollary X.3.8 in [**Wh**]. If we choose as a special case $W = S^{k-1}_+$ (where the subindex denotes a disjoint basepoint) we obtain

$$[S^{k-1}_+, \Omega X] \cong \pi_k(X) \rtimes \pi_1(X), \quad k \geq 1 \tag{4.1}$$

(see [**Pe1**]), where the semidirect product is referred to the ordinary action of the fundamental group on the higher homotopy groups. Of course, the assertion that $\pi_k(X) \rtimes \pi_1(X)$ is nilpotent is equivalent to the assertion that $\pi_1(X)$ is nilpotent and acts nilpotently on $\pi_k(X)$; cf. §2 in [**Hi2**]. Hence, the spaces S^0 and S^{k-1}_+ for $k \geq 2$ suffice to "recognize" nilpotent spaces by means of the above criterion. In fact, we might have equally well written S^{k-1}_+ for $k \geq 1$, since a group is nilpotent if and only if the conjugation action on itself is nilpotent.

Furthermore, as explained in [**Ro**], the natural map

$$[W, \Omega X] \to [W, \Omega(X_P)]$$

is a P-localization of groups when X is nilpotent, for every finite CW-complex W. Hence, $[W, \Omega Y]$ is a P-local nilpotent group for every P-local nilpotent space Y.

This suggests another possible approach —less naïve than the one sketched in Section 3— to extend the main existence results of Hilton–Mislin–Roitberg. Recall that we are calling *P-local* those groups G (nilpotent or not) in which the qth power map $x \mapsto x^q$ is bijective for all $q \in P'$.

Definition 4.1. A CW-complex X will be called *P-local* if the group $[W, \Omega X]$ is P-local for every CW-complex W.

In fact, this definition turns out to be equivalent to imposing that $[W, \Omega X]$ be P-local for every *finite* CW-complex W. To understand this, suppose that the latter holds for a space X. Then, in particular, because of (4.1), the groups $\pi_k(X) \rtimes \pi_1(X)$ are P-local for $k \geq 1$. But these groups are the basepoint-free homotopy groups of ΩX. Thus, the qth power map will be bijective on all these groups if and only if the qth power map

$$\rho_q \colon \Omega X \to \Omega X,$$

sending every loop ω to ω^q, is a homotopy equivalence —here we are using, of course, the Whitehead theorem, together with the fact that the connected components of ΩX are simple. But this condition (for every $q \in P'$) implies that $[W, \Omega X]$ is indeed P-local for every CW-complex W, not necessarily finite.

In the course of this argument, we have proved the following, which we label for later reference.

Theorem 4.2. *A CW-complex X is P-local if and only if the qth power map on the loop space ΩX is a homotopy equivalence for every $q \in P'$.*

In particular, if G is a discrete group, then the classifying space BG is P-local if and only if G is a P-local group. The next facts also follow from our previous discussion.

Proposition 4.3. *A CW-complex X is P-local if and only if it is orthogonal to the degree q maps*

$$\rho_{q,k} \colon \Sigma(S^{k-1}_+) \to \Sigma(S^{k-1}_+), \quad k \geq 1, \ q \in P'.$$

Here we view $\Sigma(S^{k-1}_+)$ as $S^1 \wedge (S^{k-1}_+)$, and let $\rho_{q,k}$ be the product of the degree q map on the first factor and the identity on the second factor. Observe that

$$\Sigma(S^{k-1}_+) \simeq S^k \vee S^1,$$

although the obvious co-H-structures on these two homotopy types are different in general, for $[\Sigma(S^{k-1}_+), X]$ is isomorphic to the semidirect product $\pi_k(X) \rtimes \pi_1(X)$, while $[S^k \vee S^1, X]$ is isomorphic to the direct product $\pi_k(X) \times \pi_1(X)$.

Proposition 4.4. *A CW-complex X is P-local if and only if $\pi_1(X)$ is a P-local group and, for every $k \geq 2$, the action*

$$\omega\colon \mathbb{Z}[\pi_1(X)] \to \mathrm{End}(\pi_k(X))$$

has the property that

$$\omega(1 + x + x^2 + \cdots + x^{q-1})$$

is invertible for all $x \in \pi_1(X)$ and $q \in P'$.

To check this, develop $(a, x)^q$ for an arbitrary element (a, x) of $\pi_k(X) \rtimes \pi_1(X)$. This is an old trick which goes back at least to [**Ba2**].

It is now perfectly natural to call *P-local* a $\mathbb{Z}[G]$-module A (where G is any group) if $1 + x + x^2 + \cdots + x^{q-1}$ is an automorphism of A for every $x \in G$ and every $q \in P'$. Thus $A \rtimes G$ is P-local if and only if G is a P-local group and A is a P-local $\mathbb{Z}[G]$-module.

The concept of "P-local module" or "P-local action" has appeared —implicitly or explicitly— several times in the literature, always in connection with the study of roots in semidirect products of groups [**Ga**], [**Pe2**], [**Re**]. For an extensive account of algebraic properties of P-local modules, see [**CP**].

Let us denote by $\mathcal{D}_P$ the class of P-local spaces in the above sense. Proposition 4.3 tells us that $\mathcal{D}_P = f^{\perp}$ for a certain map f in $\mathcal{H}$; namely, the wedge of the maps $\rho_{q,k}$ for $k \geq 1$ and $q \in P'$, where we can replace, if we wish, $\rho_{q,1}$ by the degree q map $S^1 \to S^1$. Hence, $\mathcal{D}_P$ is saturated and we may consider the orthogonal pair $(\mathcal{S}_P, \mathcal{D}_P)$, where $\mathcal{S}_P = (\mathcal{D}_P)^{\perp}$. We call maps in $\mathcal{S}_P$ *P-equivalences* of spaces.

Theorem 4.5. *The subcategory $\mathcal{D}_P$ is reflective in $\mathcal{H}$.*

In other words, for every space X there is a map $l\colon X \to X_P$ which is initial among all maps from X to spaces in $\mathcal{D}_P$. Thus, we have thrown away from our class of P-local spaces a bunch of conflictive spaces with "bad" actions of the fundamental group on the higher homotopy groups, such as the twisted Eilenberg–Mac Lane space L used in Section 3 to prove the non-reflectivity of the class of all spaces with P-local homotopy groups. Note that $\mathcal{D}_P$ is in fact a subclass of that class (just take $x = 1$ in Proposition 4.4).

The proof of Theorem 4.5 can be found in [**CP**]. For every space X, the P-localization map $l\colon X \to X_P$ is constructed as the homotopy direct limit of a system of P-equivalences

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

where X_{i+1} is constructed by attaching cells to X_i so as to create P'-roots in the semidirect products $\pi_k(X_i) \rtimes \pi_1(X_i)$ and make these roots unique. It should be emphasized that the process stops at the first infinite ordinal, so that there is no need of resorting to transfinite direct limits.

The possibility of such a construction was suggested to us by Dror Farjoun; in fact, as we explain in Section 6, this is a special case of his own general construction. It is inspired by Bousfield's work [**Bo1**], [**Bo3**] and, ultimately, by the work of Sullivan [**Su**], Mimura–Nishida–Toda [**MNT**], and Adams [**Ad1**].

It is also clear that the above construction is closely related to the procedure of P-localizing an arbitrary group, not necessarily nilpotent, by successively adjoining P'-roots and making them unique. This is far from being a new idea. It was first carried out by Baumslag in [**Ba1**] for free groups, and later generalized by Ribenboim to arbitrary groups in [**Ri1**]. It is fascinating that Ribenboim's construction of the P-localization $l\colon G \to G_P$ of an arbitrary group turns out to be completely analogous to Bousfield's construction of the localization with respect to a generalized homology theory. We tried to make clear the common pattern in [**CPP**], where we pointed out that the construction used to prove Theorem 4.5 is simply another instance of that general abstract procedure.

P-localization of groups may be viewed as the idempotent monad associated with the orthogonal pair $(f^{\perp\perp}, f^{\perp})$ where f is the free product of the maps

$$\rho_q\colon \mathbb{Z} \to \mathbb{Z},$$

where $\rho_q(1) = q$, for all $q \in P'$ (so that a group G belongs to $f^{\perp}$ if and only if G is P-local). Homomorphisms in $f^{\perp\perp}$ will be called *P-equivalences* of groups.

Since, for every space X, the P-localization map $l\colon X \to X_P$ is orthogonal to all classifying spaces of P-local discrete groups, we immediately obtain

THEOREM 4.6. *For every CW-complex X, the homomorphism*

$$l_*\colon \pi_1(X) \to \pi_1(X_P)$$

is a P-localization of groups.

However, we warn the reader that, in general, the map

$$[W, \Omega X] \to [W, \Omega(X_P)]$$

is far from being a P-localization of groups, contrary to what happens in the nilpotent case. The reason should be clear after the following discussion.

It is nowadays well known that Ribenboim's P-localization, which is defined on all groups, coincides with the classical P-localization when restricted to nilpotent groups [**Ga**], [**Ri2**]. However, this was not obvious in its origins [**Ri1**]. Similarly, the functor provided by Theorem 4.5 does extend the classical P-localization of nilpotent spaces, i.e.,

$$\tilde{H}_k(X_P) \cong \mathbb{Z}_P \otimes \tilde{H}_k(X)$$

for all k and

$$\pi_k(X_P) \cong \mathbb{Z}_P \otimes \pi_k(X)$$

for $k \geq 2$ whenever X is nilpotent. Again, this is not obvious from the construction; for a detailed proof, see [**CP**].

Thus, it may be asked what $\tilde{H}_*(X_P)$ and $\pi_*(X_P)$ look like when X is nonnilpotent. Unfortunately, very little can be said in general. The right hint is given by the following result of [**CP**], which was obtained using arguments of obstruction theory with twisted coefficients.

THEOREM 4.7. *A map $f\colon X \to Y$ between connected CW-complexes is a P-equivalence if and only if $f_*\colon \pi_1(X) \to \pi_1(Y)$ is a P-equivalence of groups and*

(1) *$f_*\colon H_*(X;A) \to H_*(Y;A)$ is an isomorphism for all P-local $\mathbb{Z}[\pi_1(Y)_P]$-modules A, or, equivalently,*
(2) *$f^*\colon H^*(Y;A) \to H^*(X;A)$ is an isomorphism for all P-local $\mathbb{Z}[\pi_1(Y)_P]$-modules A.*

In particular, for every space X, the homomorphism

$$l_*\colon H_*(X;\mathbb{Z}_P) \to H_*(X_P;\mathbb{Z}_P)$$

is an isomorphism. However, the groups $\tilde{H}_*(X_P)$ (with integer coefficients) may fail to be P-local in general. An example is obtained by choosing X to be a wedge of at least two circles, for which $H_1(X_P)$ is the abelianized of the P-localization of a free group. The reason why this contains a huge P'-torsion summand is explained in Theorem 37.3 of Baumslag's thesis [**Ba1**].

We are indebted to Shen Wenhuai for pointing out the following interesting duality:

(1) $l_*\colon \tilde{H}_*(X) \to \tilde{H}_*(X_P)$ is a P-equivalence for all spaces X, but the groups $\tilde{H}_k(X_P)$ need not be P-local.
(2) $l_*\colon \pi_*(X) \to \pi_*(X_P)$ need not be a P-equivalence, but the groups $\pi_k(X_P)$ are P-local for all spaces X.

Examples in which X is a $K(G,1)$ but X_P has a lot of higher homotopy are common. As a matter of fact, according to Theorem 4.7, $l\colon X \to X_P$ may be seen as "homology localization with twisted coefficients". The precise formulation is given in [**CP**], where it is pointed out that the construction of $(\ \ )_P$ can be modified so as to obtain homology localizations with "different degrees of twisting" on the coefficients. Hence, it is not surprising that $l\colon X \to X_P$ is closely related to the $H_*(\ \ ;\mathbb{Z}_P)$-localization $\eta\colon X \to E_P X$, which is the "totally untwisted" case. Indeed, Theorem 4.7 also tells us that there is a natural transformation of functors $(\ \ )_P \to E_P$, which is a homotopy equivalence in some cases. For example, as shown in [**CP**],

THEOREM 4.8. *Let G be a finite group and p a single prime. Then*

$$(BG)_p \simeq E_p(BG) \simeq (BG)_p^{\wedge}.$$

Hence, if G is finite and perfect, then $(BG)_p \simeq (BG^+)_p$, where the superscript denotes Quillen's plus-construction. It must be mentioned that this is false for infinite groups in general. For example, if G is locally free and perfect (such groups exist; see Lemma 3.1 in [**BDH**]), then G_p, the fundamental group of $(BG)_p$, contains a copy of G, while BG^+ is 1-connected. Still, P-localization of spaces is in many cases another instance in homotopy theory of a generic procedure, namely "doing something to the fundamental group by preserving homology to a certain extent", which tends, of course, to change quite drastically the higher homotopy groups in general.

5. Extending Localization Functors

The functor $(\ \ )_P$ described in the previous section is idempotent, it extends the classical P-localization of nilpotent spaces to all spaces, and it is distinct from Bousfield's $H_*(\ \ ;\mathbb{Z}_P)$-localization functor, which we keep denoting by E_P. As another example illustrating this last assertion, consider $X = S^1 \vee S^1$; then $\pi_1(X_P)$ is countable (since the P-localization of any countable group is countable), while $\pi_1(E_P X)$ is uncountable, by Proposition 4.4 in [**Bo2**]. The functor $(\ \ )_P$ has been called the "careful P-localization" by Dror Farjoun, in the sense that its effect on homotopy types is somehow less drastic than the effect of $\mathbb{Z}_P$-completion or $H_*(\ \ ;\mathbb{Z}_P)$-localization, while it still creates spaces with P-local homotopy groups.

This terminology suggests the possibility of partially ordering idempotent monads in $\mathcal{H}$ according to "how drastically" they change homotopy types. There is a simple way to develop this idea, which is again borrowed from [**FK**].

Definition 5.1. Given two orthogonal pairs in any category $\mathcal{C}$, we write

$$(\mathcal{S}_1, \mathcal{D}_1) \leq (\mathcal{S}_2, \mathcal{D}_2)$$

if $\mathcal{D}_1 \subseteq \mathcal{D}_2$.

Thus, the bigger orthogonal pair is the one with the bigger class of objects. Of course, this is equivalent to the condition $\mathcal{S}_2 \subseteq \mathcal{S}_1$.

It is readily checked that, given two idempotent monads $\mathbf{E}_1 = (E_1, \eta_1, \mu_1)$ and $\mathbf{E}_2 = (E_2, \eta_2, \mu_2)$ in the same category $\mathcal{C}$, the following facts are equivalent [**CFT**].

(1) $(\mathcal{S}(E_2), \mathcal{D}(E_2)) \leq (\mathcal{S}(E_1), \mathcal{D}(E_1))$.
(2) There is a (unique) morphism of monads $\mathbf{E}_1 \to \mathbf{E}_2$; i.e., a natural transformation of functors $E_1 \to E_2$ rendering commutative the obvious diagrams.

Now let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$, $(\mathcal{S}', \mathcal{D}')$ an orthogonal pair in $\mathcal{C}'$ and $(\mathcal{S}, \mathcal{D})$ an orthogonal pair in $\mathcal{C}$. We say that $(\mathcal{S}, \mathcal{D})$ *extends* $(\mathcal{S}', \mathcal{D}')$ if both $\mathcal{S}' \subseteq \mathcal{S}$ and $\mathcal{D}' \subseteq \mathcal{D}$. Then the following holds; cf. [**CPP**].

Proposition 5.2. *In the above hypotheses,*

$$\left((\mathcal{D}')^{\perp}, (\mathcal{D}')^{\perp\perp}\right) \leq (\mathcal{S}, \mathcal{D}) \leq \left((\mathcal{S}')^{\perp\perp}, (\mathcal{S}')^{\perp}\right),$$

where orthogonality is meant in $\mathcal{C}$.

Accordingly, we call $\left((\mathcal{D}')^{\perp}, (\mathcal{D}')^{\perp\perp}\right)$ the *minimal extension* of $(\mathcal{S}', \mathcal{D}')$, and $\left((\mathcal{S}')^{\perp\perp}, (\mathcal{S}')^{\perp}\right)$ the *maximal extension*. If each of these three orthogonal pairs is associated with an idempotent monad, then the monad (E', η', μ') associated with $(\mathcal{S}', \mathcal{D}')$ admits an *initial* and a *terminal* extension over $\mathcal{C}$, namely the idempotent monads associated, respectively, with the maximal orthogonal pair and the minimal orthogonal pair. Of course, any of these orthogonal pairs might fail to be associated with an idempotent monad. In that case, the existence of a terminal or an initial extension of (E', η', μ') would not be guaranteed.

We will discuss two special cases of this situation, one in homotopy theory and the other one in group theory. Thus let us consider firstly the case $\mathcal{C} = \mathcal{H}$, the based homotopy category of CW-complexes, and $\mathcal{C}' = \mathcal{N}$, the full subcategory of nilpotent CW-complexes. Let $\mathbf{E}' = (E', \eta', \mu')$ correspond to P-localization in $\mathcal{N}$, and $(\mathcal{S}', \mathcal{D}')$ be the associated orthogonal pair. Then we have

Theorem 5.3. *$\big((\mathcal{D}')^{\perp}, (\mathcal{D}')^{\perp\perp}\big)$ is the orthogonal pair associated with homology localization with $\mathbb{Z}_P$ coefficients. That is, E_P is terminal among all extensions of E' over $\mathcal{H}$.*

Conjecture 5.4. *$\big((\mathcal{S}')^{\perp\perp}, (\mathcal{S}')^{\perp}\big)$ is the orthogonal pair associated with P-localization. That is, $(\ \ )_P$ is initial among all extensions of E' over $\mathcal{H}$.*

The proof of Theorem 5.3 is elementary. A map $f\colon X \to Y$ in $(\mathcal{D}')^{\perp}$ is orthogonal to all P-local nilpotent spaces. Since these include all spaces $K(\mathbb{Z}_P, n)$ for $n \geq 1$, f is an $H^*(\ \ ;\mathbb{Z}_P)$-equivalence, and hence also an $H_*(\ \ ;\mathbb{Z}_P)$-equivalence. That is, $(\mathcal{D}')^{\perp}$ is contained in $\mathcal{S}(E_P)$. But all P-local nilpotent spaces are $H_*(\ \ ;\mathbb{Z}_P)$-local, and this implies that all $H_*(\ \ ;\mathbb{Z}_P)$-equivalences are in $(\mathcal{D}')^{\perp}$, so that $\mathcal{S}(E_P)$ is contained in $(\mathcal{D}')^{\perp}$, which completes the argument. (In fact, the second part of this argument is redundant, in view of Proposition 5.2.)

We have not been able to prove Conjecture 5.4 so far, although there is some strong evidence in its favour. If it turned out to be true, then we would have "captured" all instances of idempotent monads in $\mathcal{H}$ extending P-localization of nilpotent spaces. For any such monad (T, η, μ) and every space X, there would be natural maps

$$X_P \to TX \to E_P X$$

commuting with the units of the respective monads. This leads to results as the following one (which can be proved directly, without resorting to Conjecture 5.4).

Theorem 5.5. *Let G be a finite group and (T, η, μ) any idempotent monad in $\mathcal{H}$ extending p-localization of nilpotent spaces, for a fixed single prime p. Then $T(BG) \simeq (BG)_p$.*

In other words, there is only one (idempotent) way to "p-localize" the classifying space of a finite group; this improves Theorem 4.8.

Let us consider now the analogs in group theory of Theorem 5.3 and Conjecture 5.4. While the abstract setting is completely analogous, the conclusion turns out to be different. Recall from [**Bo2**] that for every abelian group R there is an idempotent functor in the category of groups, called *HR-localization* and denoted by E_R, with the property that

$$E_R(\pi_1(X)) \cong \pi_1(E_R X)$$

for all spaces X, where $E_R X$ denotes the $H_*(\ \ ;R)$-localization of X. In the case $R = \mathbb{Z}_P$, we consistently abbreviate $E_{\mathbb{Z}_P}$ to E_P. The orthogonal pair associated with E_R in the category of groups is generated by the *HR-maps*, i.e., homomorphisms $\varphi\colon G \to K$ such that $H_1(\varphi)$ is iso and $H_2(\varphi)$ is epi. That is, HR-maps are precisely homomorphisms induced at the fundamental group by $H_*(\ \ ;R)$-equivalences of spaces; cf. [**Bo1**].

Take $\mathcal{C} = \mathcal{G}r$, the category of groups, and $\mathcal{C}'$ the full subcategory of nilpotent groups. Let $\mathbf{E}' = (E', \eta', \mu')$ correspond to P-localization of nilpotent groups, and let $(\mathcal{S}', \mathcal{D}')$ be the associated orthogonal pair. We have

Theorem 5.6. *$\big((\mathcal{D}')^{\perp}, (\mathcal{D}')^{\perp\perp}\big)$ is associated with a certain idempotent monad $\mathbf{L}_P = (L_P, \eta, \mu)$, which is not isomorphic to $H\mathbb{Z}_P$-localization.*

Theorem 5.7. *$\big((\mathcal{S}')^{\perp\perp}, (\mathcal{S}')^{\perp}\big)$ is the orthogonal pair associated with P-localization. That is, $(\ \ )_P$ is initial among all extensions of E' over $\mathcal{G}r$.*

Now it is Theorem 5.7 the one admitting an elementary proof. A group G in $(\mathcal{S}')^\perp$ is orthogonal to all P-equivalences of nilpotent groups. Since these include the maps $\rho_q\colon \mathbb{Z} \to \mathbb{Z}$, $q \in P'$, we infer that G is a P-local group. That is, $(\mathcal{S}')^\perp$ is contained in the class of P-local groups. The converse inclusion follows from Proposition 5.2.

Theorem 5.6 was proven, with different methods, in [**BT**] and [**CFT**]. The monad $\mathbf{L}_P$ turns out to be, in a certain precise sense, "the best idempotent approximation" to nilpotent $\mathbb{Z}_P$-completion. The nilpotent $\mathbb{Z}_P$-completion of a group G is defined as

$$\widehat{G}_P = \varprojlim (G/\Gamma^i G)_P,$$

where $\Gamma^i G$ denotes the lower central series of G. The group $\widehat{G}_P$ is not nilpotent in general, but it is "the best approximation of G by means of P-local nilpotent groups".

This functor $\widehat{(\ \ )}_P$ is part of a monad, which fails to be idempotent. For example, if F is a free group on a countably infinite set of free generators, then $(\widehat{\widehat{F}_P})_P$ is not isomorphic to $\widehat{F}_P$; see §13 of [**Bo2**] and Proposition IV.5.4 in [**BK**]. However, in the category of groups (as in any category which is complete and well-powered), for every given monad $\mathbf{T} = (T, \eta, \mu)$ there is an idempotent monad $\mathbf{T}' = (T', \eta', \mu')$ with the same class of equivalences as $\mathbf{T}$, i.e., such that for a map φ, the map $T'\varphi$ is invertible if and only if $T\varphi$ is invertible; see [**Fak**] and [**CFT**]. In this situation, we call $\mathbf{T}'$ the *idempotentification* of $\mathbf{T}$. With this terminology, the idempotent monad $\mathbf{L}_P$ in Theorem 5.6 is the idempotentification of nilpotent $\mathbb{Z}_P$-completion.

The orthogonal pair associated with $\mathbf{L}_P$ is strictly smaller than the one associated with $H\mathbb{Z}_P$-localization, since it follows from the construction of $\mathbf{L}_P$ that $L_P G = \widehat{G}_P$ if G is finitely generated, while for a free group F on two free generators the natural map $E_P F \to \widehat{F}_P$ is not iso; see Proposition 4.4 in [**Bo2**]. Thus there are, for every group G, natural maps

$$G_P \to E_P G \to L_P G \to \widehat{G}_P,$$

which need not be isomorphisms. (Yet, if G is nilpotent, they are.)

Of course, one could investigate the same idea in homotopy theory, where the analog of nilpotent $\mathbb{Z}_P$-completion is Bousfield–Kan $\mathbb{Z}_P$-completion $(\mathbb{Z}_P)_\infty$, which fails to be idempotent on arbitrary spaces. The surprise is that, contrary to what happens in the category of groups, the idempotentification of $(\mathbb{Z}_P)_\infty$ is nothing else than $H_*(\ \ ;\mathbb{Z}_P)$-localization —which we know it is indeed the terminal extension of P-localization of nilpotent spaces to all spaces (Theorem 5.3). In other words, there is no gap left between E_P and $(\mathbb{Z}_P)_\infty$ which could be occupied by some other idempotent functor.

In fact, it is true in general that, using the same notation as above, if the right Kan extension of the inclusion of $\mathcal{D}'$ in $\mathcal{C}$ along itself exists, then it is part of a monad in $\mathcal{C}$, and if the idempotentification of this monad exists, then it provides the terminal extension of $\mathbf{E}'$ over $\mathcal{C}$; see [**CFT**]. We are not aware of any analogous procedure to obtain the initial extension in general.

Note that the efforts to prove Theorem 5.6 would have been superfluous if the following question had a negative answer:

QUESTION 5.8. *Do there exist orthogonal pairs* $(\mathcal{S},\mathcal{D})$ *in the category of groups which are not associated with any idempotent monad?*

If $(\mathcal{S},\mathcal{D})$ is generated by either a set of groups or a set of homomorphisms —where distinction is made between a "set" and a "proper class"— then $(\mathcal{S},\mathcal{D})$ is associated with an idempotent monad; cf. [**Bo3**], [**Pf**], [**CPP**], [**CFT**]. Thus the following question is equally relevant.

QUESTION 5.9. *Do there exist orthogonal pairs* $(\mathcal{S},\mathcal{D})$ *in the category of groups which are neither generated by a set of groups nor by a set of homomorphisms?*

From Section 3 we know counterexamples to Question 5.8 in the based homotopy category of CW-complexes $\mathcal{H}$. However, as we shall point out later, Question 5.9 is still significant in $\mathcal{H}$.

6. Localizing with Respect to Any Map

We have seen in Section 3 that certain orthogonal pairs of the form $(f^{\perp\perp}, f^{\perp})$ in the based homotopy category of CW-complexes are not associated with any idempotent monad. However, this is only due to the fact that standard orthogonality is not the "best" concept to look at in homotopy theory. Indeed, given two spaces X, Y, the set $[X,Y]$ is only part of the richer structure of the set $\mathrm{map}_*(X,Y)$ of based maps from X to Y, endowed with the compact-open topology. Therefore, it is a good idea to consider the following notion.

DEFINITION 6.1. Let $f\colon A\to B$ be any map. A CW-complex X is *f-local* or *f-periodic* if the induced map

$$f^*\colon \mathrm{map}_*(B,X)\to \mathrm{map}_*(A,X)$$

is a weak homotopy equivalence.

Of course, if X is f-local, then X is orthogonal to f in $\mathcal{H}$, since the condition imposed in the definition implies in particular that $f^*\colon [B,X]\to[A,X]$ is bijective. However, being f-local is much more restrictive in general than being orthogonal to f. For example, if $f\colon S^1\to S^1$ is the degree q map for some prime q, then X is orthogonal to f if and only if $\pi_1(X)$ is uniquely q-radicable, while X is f-local if and only if the qth power map $\Omega X\to\Omega X$ is a weak homotopy equivalence (in this special case, we could delete "weak"), and, as we have explained in Section 4, this implies certain additional conditions on the higher homotopy groups of X.

THEOREM 6.2. *For every map $f\colon A\to B$, the class of f-local spaces is reflective in $\mathcal{H}$.*

This powerful result has been proved by Dror Farjoun in [**Fa1**]. An earlier version was sketched in §7 of [**Bo3**]; see also [**Bo5**]. We denote by $l\colon X\to L_fX$ the localization given by Theorem 6.2, and refer to it as *f-localization*. The corresponding equivalences will be called *f-equivalences*. The following result is implicit in the proof of Theorem 6.2.

THEOREM 6.3. *Let $f\colon A\to B$ be a map between n-connected spaces. Then the f-localization map $l\colon X\to L_fX$ induces isomorphisms $l_*\colon \pi_k(X)\to\pi_k(L_fX)$ for $k\le n$.*

Note that Theorem 4.5 is just a special case of Theorem 6.2. However, an important feature of Dror Farjoun's explicit construction is that it is functorial in the topological category. Thus, if we are given a commutative diagram of spaces and maps, the diagram obtained by applying L_f is again strictly commutative, not just up to homotopy. Other constructions, such as Sullivan's [**Su**], the one in Hilton–Mislin–Roitberg [**HMR**], or the one sketched in Section 4, are only functorial in the homotopy category. Since the class of local objects only determines an idempotent monad up to isomorphism, for every space X there is a wide variety of choices for the space X_P, all of which, however, are homotopy equivalent. In this context, Dror Farjoun's construction provides, as a special case, a functorial model for X_P.

It is interesting that the lack of functoriality of the earlier versions of P-localization motivated Anderson's paper [**An**], where a certain construction is described which turns out to be functorial in the topological category and coincides, up to homotopy, with Sullivan's P-localization of 1-connected spaces. Now we know that Anderson's construction is precisely g-localization, with respect to the following map g. Take the homotopy cofibre M of the map f giving rise to P-localization (i.e., the wedge of the degree q maps of S^1 for $q \in P'$) and consider the trivial map $g\colon M \to \mathrm{pt}$. Hence, Anderson's work is also a major precedent of Dror Farjoun's construction. In the same line of reasoning we find the work of Bendersky [**Be**], who constructed a functorial *semilocalization*, that is, a functor which preserves the fundamental group of an arbitrary space X and P-localizes its higher homotopy groups $\pi_k(X)$, $k \geq 2$. Again, Bendersky's functor is a special instance of localization with respect to a map, which turns out to be in this case Σf, the suspension of the map f inducing P-localization; see Example 3.6 in [**CPP**].

Dror Farjoun has asked the following question:

QUESTION 6.4. *Is it true that every idempotent monad in $\mathcal{H}$ (or in some subcategory) is f-localization for a certain map f?*

All examples which have been checked until now support a positive answer. However, a solution to this problem might require a deep input from set theory, for it is related to Question 5.9. First of all, it does not seem easy to decide under which hypotheses one can ensure that the class of equivalences associated with a given idempotent monad in $\mathcal{H}$ is generated by some set of maps, not even if "generated" is understood in the sense of Definition 6.1.

In order to construct the E_*-localization functor for a generalized homology theory E_*, Bousfield had to solve a difficulty of a similar kind. In that case, it is possible to consider the following map f.

THEOREM 6.5. *Given a generalized homology theory E_* (satisfying the limit axiom), let f be the wedge of a set of representatives of all isomorphism classes of E_*-equivalences $A \to B$ where the cardinality of the set of cells in A and B is not bigger than the cardinality of $E_*(\mathrm{pt})$. Then the following assertions are equivalent for a space X:*

(1) *X is E_*-local.*
(2) *X is f-local.*
(3) *X is orthogonal to f.*

The key ingredient in the proof of this result is Lemma 11.3 in [**Bo1**]. Therefore, homology localizations are also special cases of localization with respect to a map.

If E_* is ordinary homology with integer coefficients, and we consider the map $g\colon C \to \mathrm{pt}$, where C is the homotopy cofibre of the map f defined in Theorem 6.5, then the g-localization $l\colon X \to L_g X$ turns out to be precisely Quillen's plus-construction relative to the maximal perfect subgroup of $\pi_1(X)$. Observe that this fact suggests a way to define a "plus-construction" associated with any generalized homology theory E_*, by considering the homotopy cofibre C of the map f inducing E_*-localization, and localizing with respect to $g\colon C \to \mathrm{pt}$. The properties of this functor are being currently studied by José Luis Rodríguez.

Localizations with respect to maps of the form $f\colon W \to \mathrm{pt}$ occur so often and have such pleasant properties that deserve some comments. If $f\colon W \to \mathrm{pt}$, then a space X is f-local if and only if $\mathrm{map}_*(W, X)$ is weakly contractible, which is equivalent to the condition

$$\pi_k(\mathrm{map}_*(W, X)) = 0 \qquad \text{for all } k \geq 0.$$

But $[S^k, \mathrm{map}_*(W, X)] \cong [S^k \wedge W, X]$. Therefore, we have

Proposition 6.6. *If $f\colon W \to \mathrm{pt}$, then a space X is f-local if and only if $[\Sigma^k W, X] = 0$ for $k \geq 0$.*

That is, X is f-local if and only if X is orthogonal to the maps $\Sigma^k W \to \mathrm{pt}$ for $k \geq 0$. It might be asked if it is possible to write down a similar description of f-local spaces in the case of a general map $f\colon A \to B$, i.e., in terms of orthogonality in the usual sense. Note that, if A and B are suspensions, then X is f-local if and only if X is orthogonal to $f\colon A \to B$ and to

$$f \wedge \mathrm{id}\colon A \wedge S^k_+ \to B \wedge S^k_+ \qquad \text{for } k \geq 1;$$

(of course, Proposition 4.3 is a special case). Indeed, under the assumption that A and B are suspensions, the spaces $\mathrm{map}_*(A, X)$ and $\mathrm{map}_*(B, X)$ are H-spaces, and hence basepoint-free homotopy groups suffice to recognize a weak homotopy equivalence $\mathrm{map}_*(B, X) \to \mathrm{map}_*(A, X)$. The space $A \wedge S^k_+$ is sometimes denoted by $A \rtimes S^k$; cf. [**Fa1**].

If f is of the form $W \to \mathrm{pt}$, then it is appropriate to write P_W instead of L_f, where the letter P stands for "Postnikov" (!). This was suggested by Bousfield and Dror Farjoun after observing that a space X is local with respect to $f\colon S^n \to \mathrm{pt}$ if and only if $\pi_k(X) = 0$ for $k \geq n$; combining this fact with Theorem 6.3, one obtains

Theorem 6.7. *For any connected space X and $n \geq 1$, $P_{S^n} X$ has the homotopy type of the $(n-1)$th term in the Postnikov tower of X. Thus*

$$\pi_k(P_{S^n} X) \cong \begin{cases} \pi_k(X) & \text{if } k \leq n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the homotopy fibre of $l\colon X \to P_{S^n} X$ is the $(n-1)$-connected cover $X\langle n-1\rangle$, which can be built out of copies of S^{n-1} via cofibrations, i.e., starting from one point and attaching cells of dimension $\geq n$.

In general, we can regard $P_W X$ as the result of reducing X "modulo all of its W-information"; moreover, the above example suggests that the homotopy fibre of the map $l\colon X \to P_{\Sigma W} X$ might be "the best approximation to X built out of copies of W via cofibrations" —possibly under additional hypotheses. This idea can indeed be made rigorous and turns out to be a rich source of interesting results [**Fa3**], [**Fa4**], [**Fa5**]. Among other things, if W is E_*-acyclic for a generalized homology theory E_*, so are all spaces obtained from W via (pointed) homotopy colimits of any kind.

Hence, the study of P_W (and $P_{\Sigma^k W}$ for $k \geq 1$), together with the homotopy fibres of the maps $X \to P_{\Sigma^k W}X$, should lead to a better understanding of the class of E_*-acyclic spaces.

The use of the term *f-periodic*, which has been introduced in Definition 6.1 with the same meaning as f-local, is motivated by the following special situation. Let

$$v_n \colon \Sigma^d M \to M$$

be a v_n-self map of a finite p-local CW-complex M of type n (see chapter 1 of [**Ra2**]); that is, v_n induces an isomorphism in $K(n)_*$ and is zero in $K(m)_*$ for $m \neq n$. The existence of such maps for every n is ensured by [**HS**]. Then for every space X and every $k \geq 0$ we have maps

$$(v_n)^* \colon [\Sigma^k M, X] \to [\Sigma^{k+d} M, X], \tag{6.1}$$

which are isomorphisms if X is v_n-local. But $[\Sigma^k M, X]$ is usually denoted by $\pi_k(X; M)$. Hence, v_n-local —or, so to say, v_n-periodic— spaces satisfy

$$\pi_k(X; M) \cong \pi_{k+d}(X; M)$$

for all k, that is, they have indeed periodic homotopy groups. However, a v_n-periodic space may have a lot more of algebraic structure in its homotopy groups than mere periodicity, unless v_n has been chosen to be a suspension. The situation is completely analogous to the phenomenon which we described in Section 4 in the case of P-local spaces (which can actually be viewed as the case $n = 0$). This has been studied in [**Fa1**] for $n = 1$.

In fact, the analogy goes further. We may think of localization with respect to $p \colon S^2 \to S^2$, which is a "v_0-map", as a geometric construction inducing a certain algebraic construction on homotopy, namely

$$\pi_k(L_p X) = \mathbb{Z}[1/p] \otimes \pi_k(X) \qquad \text{for } k \geq 2$$

(while $\pi_1(X)$ remains unchanged), where p "acts" on $\pi_k(X)$ via the appropriate suspension $p \colon S^k \to S^k$. More generally, for any v_n-map

$$v_n \colon \Sigma^d M \to M,$$

we may view v_n as an operator of degree d on the homotopy groups $\pi_*(X; M)$, by (6.1). In this situation, it is common practice to invert algebraically the operator v_n by considering the groups

$$v_n^{-1}\pi_k(X; M) = \mathbb{Z}\left[v_n, v_n^{-1}\right] \otimes_{\mathbb{Z}[v_n]} \pi_k(X; M).$$

Now one could look for a geometric construction L_{v_n} inducing this algebraic construction on homotopy, that is, such that

$$\pi_k(L_{v_n} X; M) \cong v_n^{-1}\pi_k(X; M) \qquad \text{for } k \geq 2.$$

This has been done by Bousfield in [**Bo5**]. The functor L_{v_n} is essentially localization with respect to a v_n-map, desuspended as many times as possible (but one!). However, for technical reasons, Bousfield considered instead localization with respect to a certain map of the form $f \colon \Sigma W \to \mathrm{pt}$, since these tend to preserve fibrations to a larger extent. Indeed, for any space W, if $E \to X$ is a fibration with fibre F, then the homotopy fibre of the natural map from $P_{\Sigma W} F$ to the homotopy fibre of

$$P_{\Sigma W} E \to P_{\Sigma W} X$$

is a product of Eilenberg–Mac Lane spaces [**FS**]. Furthermore, under suitable assumptions on W, this "error term" can be reduced to a single Eilenberg–Mac Lane space [**Bo5**].

If suitably defined, the functor L_{v_n} does not depend on the choice of a v_n-map. Hence, this construction allows to define for every space X a natural tower [**Bo5**]

$$L_{v_0} X \leftarrow L_{v_1} X \leftarrow L_{v_2} X \leftarrow \cdots$$

which deserves to be called the *unstable chromatic tower* of X, in view of the fact that it provides successive approximations to X by spaces showing higher and higher sorts of periodicity.

We conclude this section with a few additional remarks on the effect of L_f on fibrations. There are plenty of examples showing how far is L_f from preserving fibrations in general. However, the following holds.

THEOREM 6.8. *Let $f\colon A \to B$ be any map, and $p\colon E \to X$ a fibration with fibre F. If $L_f F$ is contractible, then $L_f(p)$ is a homotopy equivalence.*

The idea here is to use fibrewise localization [**Fa3**], [**Br**]. This is a map $E \to \bar{E}$ over X which is an f-equivalence and such that $\bar{E} \to X$ is a fibration with fibre $L_f F$. After ensuring the existence of such a functor, Theorem 6.8 follows immediately.

In the special case when f is of the form $W \to \mathrm{pt}$, more can be said. In this case, we have, among other results [**Fa3**], [**Bo5**]:

THEOREM 6.9. *Let $f\colon W \to \mathrm{pt}$ be given, and $F \to E \to X$ be a fibration with X connected.*

1. *If F is f-local, then $L_{\Sigma f}$ preserves the fibration.*
2. *If X is f-local (or, more generally, if $L_{\Sigma f} X \simeq L_f X$), then L_f preserves the fibration.*
3. *There is a fibration of the form*

$$L_f F \to \bar{E} \to L_{\Sigma f} X,$$

together with an f-equivalence $E \to \bar{E}$. Moreover, the space $\bar{E}$ is Σf-local. If E is f-local, F is a group and the fibration $F \to E \to X$ is principal, then $\bar{E} \simeq E$.

The machinery presented in this section has been the starting point of a fruitful research by Dror Farjoun, Bousfield, Nofech [**No**], Blanc–Thompson [**BTh**], and others. We will be content to close this article with one illuminating application. Although the main result in the next section —Neisendorfer's theorem— dates from 1991 and has already become folklore, it has not been published by its author so far. The only written references are contained in [**Fa3**] and [**ABN**].

7. Neisendorfer Localization

One of the most appealing possibilities offered by Theorem 6.2 is to play the following game. Choose a nicely looking map $f\colon A \to B$ and study the properties of the functor L_f, in the hope that they are relevant.

An interesting choice is $f\colon B(\mathbb{Z}/p) \to \mathrm{pt}$ for a fixed prime p. In this section we discuss the properties of the corresponding functor L_f, which we will abbreviate to L for simplicity.

Since f is an f-equivalence, $LB(\mathbb{Z}/p)$ is contractible. By recalling that finite p-groups are solvable and repeatedly using Theorem 6.8, we find that LBG is contractible for every finite p-group G.

Also, if $X = \operatorname{hocolim} X_i$, then

$$\mathrm{map}_*(X,Y) \simeq \operatorname{holim} \mathrm{map}_*(X_i,Y)$$

for every space Y. It follows that LBG is actually contractible for every locally finite p-torsion group G, not necessarily finite. But much more is true. Let G be any (discrete) abelian group such that $LBG \simeq \mathrm{pt}$. Then $LK(G,n) \simeq \mathrm{pt}$ for $n \geq 1$. To prove this, use the universal fibrations

$$K(G,n-1) \to \mathrm{pt} \to K(G,n)$$

and an obvious induction, again relying on Theorem 6.8.

Hence, the functor L is quite destructive, for it annihilates all Eilenberg–Mac Lane spaces whose single homotopy group is p-torsion. However, as we next show, L is not harmful at all to p'-torsion. Note that BG is f-local if and only if $\mathrm{map}_*(B(\mathbb{Z}/p),BG)$ is weakly contractible, which is equivalent to $[B(\mathbb{Z}/p),BG]=0$, or $\mathrm{Hom}(\mathbb{Z}/p,G)=0$. Hence, BG is f-local if and only if G is p-torsionfree. Similarly, observe that the map $f\colon B(\mathbb{Z}/p) \to \mathrm{pt}$ is a p'-equivalence, and therefore

Proposition 7.1. *Every p'-local space is f-local.*

Next, the group extension

$$0 \to \mathbb{Z} \to \mathbb{Z}[1/p] \to \mathbb{Z}/p^\infty \to 0,$$

which gives rise to a fibration

$$K(\mathbb{Z}/p^\infty, n-1) \to K(\mathbb{Z},n) \to K(\mathbb{Z}[1/p],n),$$

tells us that

$$LK(\mathbb{Z},n) \simeq K(\mathbb{Z}[1/p],n) \qquad \text{if } n \geq 2.$$

Hence, L does not easily "digest" free abelian groups. But p-completion helps digestion, for if a simple space X is p'-local, then $\tilde{H}_*(X;\mathbb{Z}/p)=0$, and hence $\widehat{X}_p \simeq \mathrm{pt}$. Therefore, if G is *any* abelian group, then

$$\left(LK(G,n)\right)_p \simeq \mathrm{pt} \qquad \text{for } n \geq 2.$$

This might seem implausible, but note that we have omitted so far the study of the case $n=1$, where the deepest input actually occurs. Indeed, $K(\mathbb{Z},1)=S^1$ is a finite CW-complex, and Miller's theorem [**Mill**], stating that $\mathrm{map}_*(BG,X)$ is weakly contractible for every locally finite group G and every finite-dimensional CW-complex X, tells us that all finite-dimensional CW-complexes are f-local.

In particular, $LK(\mathbb{Z},1) = K(\mathbb{Z},1)$. Hence, there *is* something escaping from the devastating effect of the composition of L followed by p-completion. This leads to the following result of Neisendorfer. It is interesting to observe the analogy with work of Mislin [**Mis**].

THEOREM 7.2. *Let X be a finite-dimensional 1-connected CW-complex. Suppose further that $\pi_2(X)$ is torsion. Then, for $n \geq 2$,*

$$(LX\langle n\rangle)_p \simeq \widehat{X}_p,$$

where $X\langle n\rangle$ denotes the n-connected cover of X, and L is localization with respect to $f\colon B(\mathbb{Z}/p) \to \mathrm{pt}$.

The proof uses our previous remarks, together with part (2) of Theorem 6.9 applied to the fibration

$$F \to X\langle n\rangle \to X.$$

Since X is f-local by assumption, we have a fibration

$$LF \to LX\langle n\rangle \to X$$

of simple spaces. Thus

$$(LF)_p \to (LX\langle n\rangle)_p \to \widehat{X}_p$$

is again a fibration. Now it suffices to check that $(LF)_p$ is contractible. Let Y be the homotopy fibre of the p'-localization map $F \to F_{p'}$. The assumption that $\pi_2(X)$ is torsion ensures that Y will be connected, for $\pi_1(F) \cong \pi_2(X)$. Since Y has finitely many homotopy groups, and these are p-torsion groups, LY is contractible. Hence, $LF \simeq L(F_{p'}) \simeq F_{p'}$, by Proposition 7.1, and the result follows.

As a consequence of Theorem 7.2, one obtains an easy proof of the following well known fact (cf. [**Se**], [**MN**]). All finite-dimensional 1-connected CW-complexes which are not contractible must have infinitely many nontrivial homotopy groups.

Theorem 7.2 has found another significant application in [**ABN**]. It also provides a counterexample to the following question. Given a map $f\colon A \to B$, one may consider the induced homomorphism $f_* : \pi_1(A) \to \pi_1(B)$. The orthogonal pair generated by f_* is reflective in the category of groups. In fact, the corresponding localization functor, which we denote by L_{f_*}, can be constructed exactly in the same way as Theorem 6.2 was proved. Then one could ask

QUESTION 7.3. *For which maps f is it true that the groups $\pi_1(L_f X)$ and $L_{f_*}\pi_1(X)$ are isomorphic for all spaces X?*

This turns out to be the case for most examples of f-localization which have been mentioned in this article, but not for $f\colon B(\mathbb{Z}/p) \to \mathrm{pt}$. Indeed, there exist finite CW-complexes X whose fundamental group contains nontrivial p-torsion, so that $\pi_1(L_f X)$ is not f_*-local in general.

Many results about f-localization can be transferred to the category of groups, where they have different descriptions and sometimes much more elementary proofs. For example, the group-theoretical analog of Theorem 6.8 is a triviality. Fibrewise P-localization of group extensions has been developed in [**CC**], after the first approach by Hilton [**Hi1**] in the nilpotent case. Its generalization to fibrewise φ-localization of group extensions for an arbitrary homomorphism $\varphi\colon \pi \to \nu$ does not offer major difficulties.

If φ is of the form $\varphi : \pi \to \{1\}$, then the effect of the functor L_φ can be easily described: For a given group G, the localization $L_\varphi G$ is the largest quotient of G on which all homomorphic images of π are trivial. For example, in the case when $\pi = \mathbb{Z}/p$, this amounts to killing the p-torsion of G. In general, if $\varphi\colon \pi \to \{1\}$, the kernel of $l\colon G \to L_\varphi G$ can be constructed as a direct limit, in a manner analogous to the construction of the *p-isolator* subgroup [**Ri1**] of an arbitrary group G (also called *p-radical*). However, for a more general homomorphism $\varphi\colon \pi \to \nu$, the kernel of $l\colon G \to L_\varphi G$ seems to be extremely difficult to characterize; see [**Ga**], [**BC**].

It might also be asked, as in Question 6.4, if every idempotent monad in the category of groups is φ-localization for some homomorphism φ. We do not know the answer, which is again related to the questions posed at the end of Section 5. One of the first idempotent monads which comes to mind in the category of groups is abelianization. We leave it to the reader to find a homomorphism φ for which the φ-local groups are precisely the abelian groups (the answer should make it clear how φ is to be chosen in the case of the reflection onto any variety of groups).

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