Depth and simplicity of Ohkawa's argument

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Abstract This is an expository article about Ohkawa's theorem stating that acyclic classes of representable homology theories form a set. We provide background in stable homotopy theory and an overview of subsequent advances in the study of Bousfield lattices. As a new result, we prove that there is a proper class of acyclic classes of nonrepresentable homology theories.

1 Introduction

The main purpose of this article is to present the statement and proof of Ohkawa's theorem [25, Theorem 2] without assuming expertise on the reader's part in homotopy theory. Thus in Section 2 and Section 3 we collect basic facts about homology theories, spectra, Spanier–Whitehead duality, and Adams representability.

Most of Ohkawa's article [25] was devoted to a discussion of injective hulls of spaces and spectra with respect to homology theories. After the publication of that article, it remained generally unnoticed that the proof of the fact that Bousfield classes of spectra form a set instead of a proper class did not depend on injective hulls —although it had likely been inspired by the study of those.

In fact, Ohkawa's theorem did not become widespread until Dwyer and Palmieri published in [10] another proof of the same result, motivated by earlier thoughts of Strickland [32], who studied jointly with Hovey and Palmieri [15, 17] the complete lattice resulting from the fact that Bousfield classes of spectra form a set. Their work triggered further progress in the understanding of chromatic homotopy theory [4, 37] and, more generally, tensor triangulated categories [11, 18, 36], including derived categories of commutative rings.

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We present Ohkawa's proof of [25, Theorem 2] without changing anything substantial from the original argument, in order to illustrate both its simplicity and the far-reaching depth of the idea behind it. Recent generalizations of Ohkawa's theorem in the context of triangulated categories by Iyengar–Krause [18] and Stevenson [31] used different methods, but the general form of the same result described in [8] for non necessarily stable combinatorial model categories was proved using precisely a version of Ohkawa's argument.

All the variants of Ohkawa's theorem published so far include representability as a crucial ingredient. In its original formulation, it was indeed a statement about representable homology theories, whose featuring property is that they preserve coproducts and filtered colimits. This property is essential in the proof of Ohkawa's theorem given by Dwyer and Palmieri in [10], which is based on the fact that every CW-spectrum is a filtered union of its finite subspectra. Additivity and exactness are also fundamental hypotheses in [18, Theorem 2.3] for the validity of Ohkawa's theorem in well generated tensor triangulated categories.

The proof of the version of Ohkawa's theorem presented in [8] no longer requires additivity nor exactness —not even homotopy invariance— but it is a result about endofunctors in combinatorial model categories preserving λ-filtered colimits for some regular cardinal λ; see [9] in this volume for details.

One could ask if this assumption can be weakened further. In Section 7 we show that if one considers non necessarily representable homology theories without any extra assumption, then there is a proper class of distinct Bousfield classes of those.

Acknowledgements The author wishes to acknowledge the kind hospitality of the University of Nagoya during the memorial conference for Professor Ohkawa held in August 2015. The content of Section 7 is based on joint work of the author with Pau Casassas and Fernando Muro. The author was supported by the Agency for Management of University and Research Grants of Catalonia with references 2014 SGR 114 and 2017 SGR 585, and the Spanish Ministry of Economy and Competitiveness under AEI/FEDER research grants MTM2013-42178-P and MTM2016-76453-C2-2-P, as well as grant MDM-2014-0445 awarded to the Barcelona Graduate School of Mathematics.

2 Homology theories

Generalized homology theories were studied by G. W. Whitehead in [34] after the discovery of K-theory and other functorial constructions on spaces that satisfied the Eilenberg–Steenrod axioms [12] except the dimension axiom. In order to state these axioms in a simple way, we will only consider reduced homology theories and restrict their scope to CW-complexes, that is, topological spaces constructed by successively attaching cells of increasing dimensions [35, Section 5].

For $n \geq 0$, the *n-skeleton* $X^{(n)}$ of a CW-complex X is the union of its cells of dimension lower than or equal to n. A *pointed* CW-complex is a pair consisting of a CW-complex X and a distinguished 0-cell x_0. Pointed CW-complexes form a category whose morphisms are continuous maps $f\colon X \to Y$ with $f(X^{(n)}) \subseteq f(Y^{(n)})$ for all n and $f(x_0) = y_0$.

A *reduced homology theory* is a collection of functors $\{h_n\}_{n\in\mathbb{Z}}$ from pointed CW-complexes to abelian groups with the following properties:

- *Homotopy invariance:* If two maps $f,g\colon X\to Y$ are homotopic, then the induced homomorphisms $h_n(f)$ and $h_n(g)$ coincide for all n.
- *Exactness:* Every inclusion $i\colon A\hookrightarrow X$ of a subcomplex induces, for all n, an exact sequence of abelian groups

$$h_n(A)\longrightarrow h_n(X)\longrightarrow h_n(X/A).$$

- *Suspension isomorphism:* There is a natural isomorphism $h_n(X)\cong h_{n+1}(\Sigma X)$ for all n and all X, where $\Sigma X = S^1\wedge X$.

Here and throughout we denote by S^n the n-sphere and by $\wedge$ the smash product, i.e., the quotient of the cartesian product by the one-point union of pointed spaces. The space ΣX is called the *suspension* of X. The exactness axiom and the suspension isomorphism axiom are usually replaced by long exact sequences for pairs of spaces and the excision axiom in the case of nonreduced homology theories. Passage from reduced to nonreduced and conversely can be done as explained in [33, Sections 7.34 and 7.35] or in [34, Section 5]. Generalized cohomology theories are defined in the same way, but contravariantly.

The graded abelian group $h_*(S^0)$ is called the *coefficients* of h_*. A reduced homology theory $\{h_n\}_{n\in\mathbb{Z}}$ is *ordinary* if $h_n(S^0)=0$ for $n\neq 0$. Otherwise it is called *extraordinary* or *generalized*. Examples include the following, among many others:

- Complex K-theory, for which $\tilde{K}_*(S^0)=\mathbb{Z}[t,t^{-1}]$ with t in degree 2.
- Complex cobordism, such that $\widetilde{MU}_*(S^0)=\mathbb{Z}[x_1,x_2,\dots]$ with x_i in degree $2i$.
- Morava K-theories, with $\widetilde{K(n)}_*(S^0)=\mathbb{F}_p[v_n,v_n^{-1}]$ and v_n in degree $2(p^n-1)$.

It follows from results in [12] that, if a homology theory is ordinary, then there is an abelian group G such that $h_n(X)\cong\tilde{H}_n(X;G)$ for all finite CW-complexes X and all n, where $\tilde{H}_n$ denotes reduced singular homology. This result was extended by Milnor in [21] to arbitrary CW-complexes, not necessarily finite, under the following additional assumption. A reduced homology theory $\{h_n\}_{n\in\mathbb{Z}}$ is called *additive* if it satisfies the *Milnor axiom* about preservation of coproducts:

$$h_n\left(\bigvee_{i\in I} X_i\right)\cong\bigoplus_{i\in I} h_n(X_i)$$

for every set of indices I and all n. This property is a consequence of the previous axioms if the set of indices I is finite, but it is not if I is infinite. If h_* is additive and ordinary, then the natural isomorphism $h_*\cong\tilde{H}_*(-;h_0(S^0))$ can be proved by comparing the respective cellular chain complexes, as in [13, Theorem 4.5.9]. A similar argument yields the following more general result, whose proof is given in [29, Proposition II.3.19] and [33, Theorem 7.55].

Proposition 2.1 *If a natural transformation $h'_*\to h_*$ of additive homology theories induces an isomorphism $h'_*(S^0)\cong h_*(S^0)$, then it also induces an isomorphism $h'_*(X)\cong h_*(X)$ for every CW-complex X.*

3 Spectra and representability

There are several different models for the homotopy category of spectra. Here we consider CW-spectra for consistency with the rest of the article. A *CW-spectrum* is a sequence of pointed CW-complexes $E = \{E_n\}_{n\in\mathbb{Z}}$ together with subcomplex inclusions $\Sigma E_n \hookrightarrow E_{n+1}$ for all n. Each CW-complex X yields a CW-spectrum with $X_n = \Sigma^n X$ if $n \geq 0$ and $X_n = *$ (a single point) for $n < 0$. We will not distinguish notationally a CW-complex from the corresponding CW-spectrum, and will omit "CW" from now on for shortness.

Spectra can be suspended and desuspended:

$$(\Sigma^k E)_n = E_{n+k} \quad \text{for } k \in \mathbb{Z}.$$

A *stable cell* of a spectrum E is a cell $c \subset E_n$ for some n, which is identified with $\Sigma^k c \subset E_{n+k}$ for $k \geq 1$. If c is a d-cell in E_n then it represents a $(d-n)$-cell of E. A spectrum with only a finite number of distinct stable cells is called *finite*. More generally, the *cardinality* of a spectrum is the cardinality of its set of stable cells.

Maps between spectra are defined up to cofinality [3, 33], and homotopies between maps of spectra are defined similarly as for topological spaces. We denote by $[X,Y]$ the set of homotopy classes of maps $X \to Y$. Suspension induces bijections

$$[X,Y] \cong [\Sigma^k X, \Sigma^k Y] \tag{1}$$

for all k and all spectra X and Y. Moreover there is a natural homotopy equivalence $\Sigma E \simeq S^1 \wedge E$ for every spectrum E. Consequently, the homotopy category of spectra is additive, since $[X,Y] \cong [\Sigma^2 X, \Sigma^2 Y]$ and the latter has a natural abelian group structure for all X and Y, resulting from the pinch map $S^2 \to S^2 \vee S^2$ on the domain.

Moreover, the homotopy category of spectra is *triangulated*. This means that each map $f\colon X \to Y$ fits into a *cofibre sequence* $X \to Y \to C$ that expands into

$$\cdots \longrightarrow X \xrightarrow{\ f\ } Y \longrightarrow C \longrightarrow \Sigma X \xrightarrow{\ \Sigma f\ } \Sigma Y \longrightarrow \cdots \tag{2}$$

in such a way that certain axioms are satisfied [16, 20, 24]. Most notably, (2) yields long exact sequences of abelian groups by applying $[E,-]$ or $[-,E]$ to it, where E is any spectrum. Indeed, it is a feature of spectra that there is no distinction between fibre sequences and cofibre sequences, in contrast with spaces.

The *homotopy groups* of a spectrum $E = \{E_n\}_{n\in\mathbb{Z}}$ are defined as

$$\pi_k(E) = [\Sigma^k S^0, E] \cong \operatorname{colim}_n \pi_{k+n}(E_n) \quad \text{for } k \in \mathbb{Z}.$$

A map of spectra $X \to Y$ inducing isomorphisms $\pi_k(X) \cong \pi_k(Y)$ for all k is a homotopy equivalence [3, Corollary III.3.5]. It is also remarkable that

$$\pi_k(X \vee Y) \cong \pi_k(X) \oplus \pi_k(Y) \tag{3}$$

for all k, since $X \to X \vee Y \to Y$ is a split cofibre sequence.

The stable homotopy groups of the sphere spectrum are of utmost importance. If $k>0$ then $\pi_k(S^0)$ is finite [27], while $\pi_0(S^0)$ is infinite cyclic, and if $k<0$ then $\pi_k(S^0)=0$. We state the following consequence for its use in Section 5.

Lemma 3.1 *The set of homotopy types of finite spectra is countable, and given any two finite spectra A and B the abelian group $[A,B]$ is finitely generated.*

Proof. For every finite spectrum A there is a finite CW-complex X and an integer k such that $A \simeq \Sigma^k X$, and if two CW-complexes are homotopy equivalent then their suspension spectra are also homotopy equivalent. Hence our first claim follows from the fact that every finite CW-complex is homotopy equivalent to a finite polyhedron; cf. [29, Lemma II.3.16].

To prove the second claim, observe first that, for every finite spectrum B, each of its homotopy groups $\pi_k(B)$ is finitely generated since it is obtained by means of finitely many group extensions starting from homotopy groups of spheres and using cofibre sequences as in (2) corresponding to the cells of B. Arguing in the same way, if A is another finite spectrum then the abelian group $[A,B]$ is finitely generated since it is obtained in finitely many steps starting from homotopy groups of B and using cofibre sequences determined by the cells of A. □

As shown in [34, Theorem 5.2], every spectrum E defines a homology theory as

$$E_n(X) = \pi_n(E \wedge X) \tag{4}$$

and similarly E defines a cohomology theory as

$$E^n(X) = [\Sigma^{-n}X, E], \tag{5}$$

where X is any pointed CW-complex. In fact (4) and (5) make perfectly sense if X is a spectrum, with any version of a smash product for spectra [2]; for instance, $(E\wedge X)_{2n} = E_n \wedge X_n$ and $(E\wedge X)_{2n+1} = E_{n+1}\wedge X_n$.

Thus E_* defines a homology theory on spectra, meaning that it is a functor from spectra to graded abelian groups which is homotopy invariant and exact in the sense that every cofibre sequence $X \to Y \to C$ of spectra yields an exact sequence

$$E_n(X) \longrightarrow E_n(Y) \longrightarrow E_n(C)$$

for every n, and there is a natural isomorphism $E_n(X) \cong E_{n+1}(\Sigma X)$ for all X.

Similarly, E^* is a cohomology theory on spectra. It is clear from (5) that E^* sends coproducts to products, and it is also true that E_* preserves coproducts, by the following argument. Recall that a partially ordered set I is *filtered* if for every two elements i and j there is another element k such that $i \le k$ and $j \le k$.

Lemma 3.2 *For every spectrum E, the homology theory E_* preserves coproducts and sends filtered unions of subspectra to filtered colimits.*

Proof. As shown, for instance, in [33, Lemma 8.34], if a spectrum X is a filtered union of subspectra X_i then the inclusions $X_i \hookrightarrow X$ induce an isomorphism

$\mathrm{colim}_i\,[F,X_i] \cong [F,X]$ for every finite spectrum F. Moreover, $E\wedge X$ is also a filtered union of its subspectra $E\wedge X_i$. Therefore, since $\Sigma^n S^0$ is a finite spectrum,

$$E_n(X) = \pi_n(E\wedge X) = [\Sigma^n S^0, E\wedge X] \cong \mathrm{colim}_i\,[\Sigma^n S^0, E\wedge X_i] = \mathrm{colim}_i\, E_n(X_i).$$

As a special case, E_n preserves coproducts because every coproduct of spectra is a filtered union of finite coproducts and E_n preserves these by (3). □

The homology theory E_* and the cohomology theory E^* given by (4) and (5) are said to be *represented* by the spectrum E. Singular (co)homology with coefficients in G is represented by the Eilenberg–Mac Lane spectrum $\{K(G,n) \mid n \geq 0\}$, and complex K-theory is represented by the spectrum consisting of the unitary group U in odd dimensions and $\mathbb{Z}\times BU$ in even dimensions (where BU is the classifying space of U), with structure maps given by Bott periodicity $\Omega^2 BU \simeq \mathbb{Z}\times BU$.

Brown's representability theorem [7, Theorem II] for cohomology theories with countable coefficients was extended by Adams in [2, Theorem 1.6] by showing that every cohomology theory defined on finite CW-complexes is represented by some spectrum (not necessarily finite). This leads to the following central result.

Theorem 3.3 (Adams) *Every additive homology theory on CW-complexes is represented by some spectrum.*

Proof. As a consequence of Alexander duality, if X is a finite nonempty proper subcomplex of S^n then there is a finite subcomplex $D_n X$ of $S^n \smallsetminus X$ such that

$$E^k(X) \cong E_{n-k-1}(D_n X) \tag{6}$$

for all k and every spectrum E; see [30, p. 199]. Hence each homology theory h_* defines by means of such duality a cohomology theory on finite CW-complexes, as shown in [34, Corollary 7.10], which is representable by Adams' extension of Brown's theorem. Then the representing spectrum E defines an additive homology theory E_* whose restriction to finite CW-complexes is naturally isomorphic to the restriction of h_*. Moreover, for every CW-complex X and every n the group $E_n(X)$ is the colimit of $E_n(X_i)$ where $\{X_i\}_{i\in I_X}$ is the filtered set of all finite subcomplexes of X; see [33, Corollary 8.35]. Hence there is a natural transformation $E_* \to h_*$ inducing an isomorphism $E_*(S^0) \cong h_*(S^0)$. If h_* is also additive, this implies that $E_*(X) \cong h_*(X)$ for all X, by Proposition 2.1. □

The stable analogue of (6) is as follows; cf. [3, Part III, § 5]. Each finite spectrum A admits a homotopy unique *Spanier–Whitehead dual* DA, which is also finite and is equipped with a map

$$DA \wedge A \longrightarrow S^0$$

inducing isomorphisms $[X, Y\wedge DA] \cong [X\wedge A, Y]$ and $[X, A\wedge Y] \cong [DA\wedge X, Y]$ for all spectra X and Y; cf. [33, Theorem 14.34]. Therefore $DDA \simeq A$ and

$$E^{-n}(A) \cong E_n(DA) \tag{7}$$

for all spectra E and all n. Using Spanier–Whitehead duality it follows with the same argument as in the proof of Theorem 3.3 that every additive homology theory on spectra is represented by some spectrum [20, Chapter 4, Theorem 16].

4 Bousfield equivalence classes of spectra

Given two spectra E and X, the spectrum X is called *E_*-acyclic* if $E_*(X)=0$, where E_* denotes the homology theory represented by E as in (4). Two spectra E and F are called *Bousfield equivalent* if the classes of E_*-acyclic spectra and F_*-acyclic spectra coincide. Since the statement that $E_n(X)=0$ for all n is equivalent to the statement that $E\wedge X\simeq 0$, where 0 denotes here the one-point spectrum, two spectra E and F are Bousfield equivalent if and only if

$$\{X \mid E\wedge X\simeq 0\}=\{X \mid F\wedge X\simeq 0\}. \tag{8}$$

It is also true that E and F are Bousfield equivalent if and only if E_*-localization and F_*-localization are naturally isomorphic. Here *E_*-localization* is meant in the sense of [5], where it was proved that for every spectrum X and every representable homology theory E_* there is a map $l\colon X\to L_EX$ such that $E_n(l)$ is an isomorphism for all n and L_EX is *E_*-local*, that is, for every map $f\colon A\to B$ such that $E_n(f)$ is an isomorphism for all n, the function $[B,L_EX]\to[A,L_EX]$ is bijective. Then L_E defines an exact endofunctor in the homotopy category of spectra such that a map $X\to Y$ induces a homotopy equivalence $L_EX\simeq L_EY$ if and only if it induces isomorphisms $E_n(X)\cong E_n(Y)$ for all n. Hence $L_EX\simeq 0$ if and only if X is E_*-acyclic. Therefore, the collection of E_*-acyclic spectra determines L_E up to a natural isomorphism.

Bousfield equivalence classes have been studied since the decade of 1980 in connection with homological localizations [5, 6, 26]. The Bousfield equivalence class of a spectrum E is usually denoted by $\langle E\rangle$, and it is also common to view $\langle E\rangle$ as the collection of all E_*-acyclic spectra. There is a partial order on Bousfield classes, namely $\langle E\rangle\le\langle F\rangle$ if and only if the class of F_*-acyclics is contained in the class of E_*-acyclics, or, equivalently, if there is a natural transformation $L_F\to L_E$ of coaugmented functors.

Thanks to Ohkawa's theorem, the collection of Bousfield classes becomes in fact a complete lattice with least upper bounds (*joins*) given by the wedge sum, and greatest lower bounds (*meets*) obtained as wedges of all lower bounds, which exist since there is only a set of those. The smash product provides lower bounds, but not greatest lower bounds in general. This lattice and other related lattices have been studied by a number of authors [4, 11, 14, 15, 17, 18, 23, 36, 37].

Ohkawa's injective hulls [25] are closely related to homological localizations. For a homology theory E_* on spectra, a spectrum Y is *E_*-injective* if, for every map $f\colon A\to B$ such that $E_n(f)$ is a monomorphism for all n, the function $[B,Y]\to[A,Y]$ is surjective. A map $h\colon X\to Y$ is an *E_*-injective enveloping map* if Y is E_*-injective and $E_n(h)$ is a monomorphism for all n, and, moreover, for all $g\colon Y\to Z$ and every n

the homomorphism $E_n(g)$ is monic if $E_n(g \circ h)$ is monic. In [25, Theorem 1] it was shown that if a homology theory E_* is representable then every spectrum X admits an E_*-injective enveloping map $h\colon X \to Y$, which is unique up to homotopy. Then Y is called an *injective hull* of X.

5 Okhawa's argument

Choose a set $\mathscr{F}$ of representatives of all homotopy types of finite spectra and a set $\mathscr{M}$ of representatives with domains and codomains in $\mathscr{F}$ of all isomorphism classes of maps between finite spectra in the stable homotopy category. Thus for each map $f\colon A \to B$ between finite spectra the set $\mathscr{M}$ contains a unique map $f_0\colon A_0 \to B_0$ where A_0 and B_0 are in $\mathscr{F}$ and there exist two homotopy equivalences $h_A\colon A_0 \to A$ and $h_B\colon B_0 \to B$ such that $f \circ h_A \simeq h_B \circ f_0$. By Lemma 3.1, $\mathscr{F}$ has cardinality $\aleph_0$ and $\mathscr{M}$ also has cardinality $\aleph_0$ since for every two finite spectra A and B the abelian group $[A,B]$ of homotopy classes of maps $A \to B$ is finitely generated.

Given two maps of spectra $g\colon X \to Y$ and $f\colon X \to E$, we say that f *extends to* Y if there exists a map $\tilde{f}\colon Y \to E$ such that $\tilde{f} \circ g \simeq f$. For a map $f\colon X \to E$ of spectra with $X \in \mathscr{F}$, we denote, as in [25],

$$t(f) = \{g\colon X \to Y \mid g \in \mathscr{M} \text{ and } f \text{ extends to } Y\}. \tag{9}$$

Hence $t(f) \in \mathscr{P}(\mathscr{M})$, where the latter denotes the set of subsets of $\mathscr{M}$. Next, for a spectrum E, let $t_E\colon \mathscr{F} \to \mathscr{P}(\mathscr{P}(\mathscr{M}))$ be the function defined as

$$t_E(X) = \{t(f) \mid f\colon X \to E\} \tag{10}$$

for each $X \in \mathscr{F}$, and call two spectra E and F *elementarily equivalent* if $t_E = t_F$, that is, if $t_E(X) = t_F(X)$ for every $X \in \mathscr{F}$.

For a spectrum E, we consider the homology theory E_* on spectra represented by E, namely $E_n(X) = \pi_n(E \wedge X)$ for $n \in \mathbb{Z}$ and every spectrum X. If $\{X_i\}_{i \in I_X}$ is the collection of all finite subspectra of X, then the inclusions $X_i \hookrightarrow X$ induce an isomorphism

$$\operatorname{colim}_{i \in I_X} E_n(X_i) \cong E_n(X) \tag{11}$$

for every n by Lemma 3.2, since I_X is filtered.

Theorem 5.1 (Ohkawa) *Suppose that two spectra E and F are elementarily equivalent, and let $f\colon X \to Y$ be any map of spectra. For each $n \in \mathbb{Z}$, the homomorphism $E_n(f)\colon E_n(X) \to E_n(Y)$ is monic if and only if $F_n(f)\colon F_n(X) \to F_n(Y)$ is monic.*

Proof. Suppose that $E_n(f)$ is a monomorphism, and let $\phi \in \operatorname{Ker} F_n(f)$. Our aim is to prove that $\phi = 0$.

Since F_n satisfies (11), there is a finite subspectrum $A \subseteq X$ and a class $\alpha \in F_n(A)$ such that $F_n(i_{A,X})(\alpha) = \phi$, where $i_{A,X}\colon A \to X$ denotes the inclusion. Therefore $F_n(f \circ i_{A,X})(\alpha) = 0$ and, using again the fact that F_n commutes with filtered colimits, we infer that there is a finite subspectrum $B \subseteq Y$ that contains $f(A)$ and such that $F_n(f')(\alpha) = 0$ if $f'\colon A \to B$ denotes the restriction of f:

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle i_{A,X}}\uparrow & & \uparrow{\scriptstyle i_{B,Y}} \\ A & \xrightarrow[\;f'\;]{} & B \end{array}$$

Let DA denote a Spanier–Whitehead dual of A. Then, since $F_n(A) \cong F^{-n}(DA)$, the class α is represented by a map $a\colon \Sigma^n DA \to F$, and the fact that $F_n(f')(\alpha) = 0$ implies that $a \circ \Sigma^n Df' \simeq 0$, where $Df'\colon DB \to DA$ is dual to f'.

Now replace $\Sigma^n DA$ by a homotopy equivalent finite spectrum belonging to $\mathscr{F}$ and choose a map $p\colon \Sigma^n DA \to P$ in $\mathscr{M}$ such that the following is a cofibre sequence:

$$\Sigma^n DB \xrightarrow{\Sigma^n Df'} \Sigma^n DA \xrightarrow{\;p\;} P \longrightarrow \Sigma^{n+1} DB.$$

Here the map $a\colon \Sigma^n DA \to F$ extends to P since $a \circ \Sigma^n Df' \simeq 0$, and this means precisely that $p \in t(a)$ as defined in (9).

Now $t(a) \in t_F(\Sigma^n DA)$ and, since we are assuming that $t_E = t_F$, we infer that $t(a) \in t_E(\Sigma^n DA)$. Therefore there is a map $b\colon \Sigma^n DA \to E$ with $t(a) = t(b)$. Thus $p \in t(b)$ and this implies that b extends to P.

Let $\beta \in E_n(A)$ be the class represented by b. Since b extends to P, we have that $b \circ \Sigma^n Df' \simeq 0$ and consequently $E_n(f')(\beta) = 0$. Since $E_n(f)$ is injective and $f \circ i_{A,X} = i_{B,Y} \circ f'$, it follows that $E_n(i_{A,X})(\beta) = 0$. Since E_n commutes with filtered colimits, there is a finite subspectrum $C \subseteq X$ containing A such that $E_n(i_{A,C})(\beta) = 0$.

Hence $b \circ \Sigma^n Di_{A,C} \simeq 0$ and therefore b extends to a homotopy cofibre Q of $\Sigma^n Di_{A,C}$, which we may choose so that the map $q\colon \Sigma^n DA \to Q$ is in $\mathscr{M}$:

$$\Sigma^n DC \xrightarrow{\Sigma^n Di_{A,C}} \Sigma^n DA \xrightarrow{\;q\;} Q \longrightarrow \Sigma^{n+1} DC.$$

Thus $q \in t(b)$, and using again that $t(a) = t(b)$, we find that $q \in t(a)$, and this means that $F_n(i_{A,C})(\alpha) = 0$. Hence $\phi = F_n(i_{A,X})(\alpha) = F_n(i_{C,X})F_n(i_{A,C})(\alpha) = 0$, from which it follows that $F_n(f)$ is indeed a monomorphism. Exchanging the roles of E and F completes the proof. □

Corollary 5.2 *If two spectra E and F are elementarily equivalent, then E and F are Bousfield equivalent.*

Proof. Suppose that E and F are elementarily equivalent, and suppose that a given spectrum X is E_*-acyclic. Then the map from X to the zero spectrum induces a monomorphism $E_n(X) \to 0$ for all n. According to Theorem 5.1, the homomorphism $F_n(X) \to 0$ is also a monomoprhism for all n, which means that X is F_*-acyclic. □

Hence Bousfield equivalence classes of spectra form a set of cardinality smaller than or equal to the cardinality of the set of elementary equivalence classes, which is at most $2^{2^{\aleph_0}}$.

6 Other proofs and extensions of Ohkawa's theorem

The argument given in Section 5 uses Spanier–Whitehead duality and the fact that representable homology theories commute with filtered colimits. An alternative proof not requiring the use of duality was published by Dwyer and Palmieri in [10]. Their argument works in every algebraic stable homotopy category, as shown in [19]. It can be summarized as follows in the case of spectra.

For a spectrum E and a homology class $c \in E_*(A)$, where A is a finite spectrum, define the *annihilator* of c as

$$\mathrm{ann}_A^E(c) = \{f\colon A \to B \mid B \text{ is finite and } E_*(f)(c) = 0\},$$

and let the *Ohkawa class* of E consist of all annihilators of all classes $c \in E_*(A)$ where A is a finite spectrum.

Theorem 6.1 (Dywer–Palmieri) *If two spectra E and F give rise to the same Ohkawa class, then they are in the same Bousfield equivalence class.*

Proof. If the Ohkawa class of F is contained into the Ohkawa class of E then the class of E_*-acyclics is contained in the class of F_*-acyclics. To prove this fact, suppose that $E_*(X) = 0$ and write X as a union of its finite subspectra. Given any class $c \in F_n(X)$, there is a finite subspectrum A of X and a class $a \in F_n(A)$ such that $F_n(i)(a) = c$ where $i\colon A \to X$ is the inclusion. By assumption there is a class $a' \in E_n(A)$ such that $\mathrm{ann}_A^F(a) = \mathrm{ann}_A^E(a')$. Since $E_*(X) = 0$, there is a finite subspectrum B of X containing A such that the homomorphism induced by the inclusion $j\colon A \to B$ satisfies $E_n(j)(a') = 0$; therefore $j \in \mathrm{ann}_A^E(a')$. Hence j also belongs to $\mathrm{ann}_A^F(a)$ and this implies that $F_n(j)(a) = 0$, so $c = 0$ in $F_n(X)$, as claimed. □

If the definition of Ohkawa classes is restricted after a choice of a set $\mathscr{F}$ of representatives of all homotopy types of finite spectra and a set $\mathscr{M}$ of representatives with domains and codomains in $\mathscr{F}$ of all homotopy classes of maps between finite spectra, as in Section 5 and as in [10], then the cardinality of the set of Ohkawa classes is bounded by $2^{2^{\aleph_0}}$, and hence so is the cardinality of the set of Bousfield equivalence classes of spectra. It is still unknown if this bound can be lowered.

Upper and lower bounds for the cardinality of the set of Bousfield classes in an arbitrary algebraic stable homotopy category have been given in [19] in terms of a generating set of small objects.

Dwyer and Palmieri proved in [11] that in the derived category of a truncated polynomial ring on countably many generators there is also only a set of Bousfield equivalence classes, and asked whether this was in fact the case in the derived category of every commutative ring —Bousfield equivalence of chain complexes of modules over a ring is defined as in (8) with the smash product replaced by the derived tensor product of chain complexes. This was known to be true for countable rings as shown in [10] and also for Noetherian rings due to a result of Neeman [22]: for a Noetherian commutative ring R there is a bijection between the Bousfield lattice in the derived category $\mathscr{D}(R)$ and the lattice of subsets of the spectrum of R.

However, it had already been observed in [23] that something very different happens for rings that fail to be Noetherian. Further results in this research direction have been obtained by Wolcott in [36].

Around 2010 Stevenson extended in an unpublished article the Dwyer–Palmieri argument to compactly generated triangulated categories equipped with a biexact and coproduct-preserving tensor product, hence proving that, indeed, the Bousfield lattice of the derived category $\mathscr{D}(R)$ is a set for every commutative ring R. Shorty after, Iyengar and Krause proved in [18] that the same result holds in any well generated tensor triangulated category. Their argument was based on a restricted Yoneda embedding of a triangulated category $\mathscr{T}$ with a set of α-compact generators for some cardinal α into the category of abelian presheaves over that set.

Still Okhawa's theorem remained a result about additive categories. Another step was made in [8] by showing that it holds in fact in the homotopy category of every combinatorial model category, not necessarily stable. A proof of this fact is presented in [9] using Rosický's result [28, Proposition 5.1] that, for a combinatorial model category $\mathscr{K}$, the composite

$$\mathscr{K} \longrightarrow \mathrm{Ho}\,\mathscr{K} \longrightarrow \mathbf{Set}^{(\mathrm{Ho}\,\mathscr{K}_\lambda)^{\mathrm{op}}}$$

of the canonical functor from $\mathscr{K}$ to its homotopy category followed by a restricted Yoneda embedding preserves λ-filtered colimits for a sufficiently large regular cardinal λ. Here $\mathscr{K}_\lambda$ is a set of representatives of isomorphism classes of λ-presentable objects in $\mathscr{K}$.

7 Nonrepresentable homology theories

If a homology theory is not representable, then it need not preserve colimits of any kind. Therefore its value on a spectrum need not be determined by its values on finite subspectra. For this reason, there is no hope that the argument used in the proof of Theorem 5.1 can be extended to non necessarily representable homology theories. In this section we show that, indeed, Ohkawa's theorem does not hold for nonrepresentable homology theories.

For an abelian group A and a cardinal α, we denote by A^α the cartesian product of α copies of A, that is, the abelian group of functions $\alpha \to A$. Moreover, we denote by SA^α the subgroup of A^α consisting of *shrinking* functions, that is, functions $\alpha \to A$ whose image has cardinality smaller than α.

Note that $F_\alpha A = A^\alpha / SA^\alpha$ defines an exact functor from the category of abelian groups to itself. This fact has the following consequence.

Theorem 7.1 *For every uncountable cardinal α there is a reduced homology theory h_*^α on pointed CW-complexes such that if X has less than α cells then $h_*^\alpha(X) = 0$ but there exists a CW-complex with α cells which is not h_*^α-acyclic.*

Proof. Consider the exact endofunctor $F_\alpha A = A^\alpha / SA^\alpha$ on the category of abelian groups. If the cardinality of A is less than α then the image of every function $\alpha \to A$

has cardinality smaller than α. Hence $SA^\alpha = A^\alpha$ and $F_\alpha A = 0$. On the other hand, there is an injective function $\alpha \to \oplus_{i<\alpha}\mathbb{Z}$ and hence $F_\alpha(\oplus_{i<\alpha}\mathbb{Z}) \neq 0$.

Next, define $h_n^\alpha = F_\alpha \circ \tilde{H}_n$ for all n, where $\tilde{H}_*$ denotes reduced singular homology. Since F_α is exact, h_*^α is a reduced homology theory. If X has less than α cells then the cardinality of $\tilde{H}_*(X)$ is smaller than α and therefore $h_*^\alpha(X) = 0$. However, for a wedge of α circles we have $H_1(\bigvee_{i<\alpha} S^1) \cong \oplus_{i<\alpha}\mathbb{Z}$ and this implies that $h_1^\alpha(\bigvee_{i<\alpha} S^1)$ is nonzero. □

We say that two homology theories h_* and h'_* (defined on spaces or spectra) are *Bousfield equivalent* if they have the same acyclics.

Corollary 7.2 *There is a proper class of distinct Bousfield equivalence classes of nonrepresentable homology theories of spaces or spectra.*

Proof. In the case of spaces, consider the collection $\{h_*^\alpha\}$ given by Theorem 7.1 where α runs through all uncountable cardinals. Then any two of them belong to distinct Bousfield equivalence classes since if $\beta > \alpha$ then there is a space X which is h_*^β-acyclic but not h_*^α-acyclic. The same argument is valid for spectra by defining similarly $h_*^\alpha = F_\alpha \circ H_*$ where H_* is ordinary homology with $\mathbb{Z}$ coefficients. □

However, if we fix an arbitrary regular cardinal λ then there is only a set of Bousfield equivalence classes of homology theories that preserve λ-filtered colimits. For a proof of this claim, see [8, Corollary 3.8].

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