Cohomological localizations and set-theoretical reflection

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Abstract Homological localizations of spaces and spectra have been a fundamental tool in algebraic topology since the decade of 1970, especially in the setting of chromatic homotopy. However, it is unknown whether the existence of *cohomological* localizations can be proved in ZFC or not. Although this is apparently a homotopy-theoretical problem, it turned out to be closely related with set-theoretical reflection principles and therefore with the existence of large cardinals. In this note we present the state of the art with enough background so that proofs of results are readable by both topologists and set theorists.

Introduction

The technique of computing homotopy groups of spaces one prime at a time was pioneered by Serre [30]. A remarkable result derived from Serre's work states that for every prime p the homotopy groups $\pi_k(S^n)$ of the n-sphere with $n \geq 2$ contain nonzero p-torsion elements for infinitely many values of k.

In 1961 Adams discovered that spheres can be embedded into CW-complexes with countably many cells in such a way that homology groups and homotopy groups are transformed into their p-local versions, and furthermore he proved that odd-dimensional spheres localized in this sense at primes $p \geq 5$ become homotopy associative H-spaces [2]. Subsequently, localization of 1-connected spaces at primes was thoroughly developed by several authors, including Bousfield–Kan [10], Hilton–Mislin–Roitberg [17], Mimura–Nishida–Toda [24], Sullivan [32], etc. Among many achievements, this technique opened the way into rational homotopy theory [28, 33].

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Supported by MCIN/AEI/10.13039/501100011033 under grant PID2020-117971GB-C22

From a category-theoretical point of view, localizing spaces at a prime p is equivalent to inverting up to homotopy the collection of all maps $X \longrightarrow Y$ that induce isomorphisms $H_n(X;\mathbb{Z}_{(p)}) \cong H_n(Y;\mathbb{Z}_{(p)})$ for all n, where H_n denotes singular homology and $\mathbb{Z}_{(p)}$ are the integers localized at p. Adams designed a convenient machinery for this purpose, involving idempotent monads, categories of fractions, and Brown representability, and showed that his construction of homological localizations was feasible for arbitrary representable homology theories. However, his presentation of results in [3, 4] contained a set-theoretical inaccuracy which was later repaired by Bousfield in [7]. We explain this in more detail in Section 1.

Bousfield extended his approach to spectra [8], and it was in the realm of stable homotopy where homological localizations were best understood and most useful, especially towards the study of chromatic phenomena [29]. Every finite p-local spectrum X is the homotopy inverse limit of its *chromatic tower* of localizations $L_{E(n)}X$, where $L_{E(0)}$ is rationalization and $L_{E(1)}$ is localization with respect to p-local complex K-theory. The homology theories $E(n)_*$ were defined by Johnson and Wilson [20] after earlier work of Brown–Peterson [12] and Morava. This result, known as the *chromatic convergence theorem*, opened the way to impressive advances in the calculation of homotopy groups of spheres.

Bousfield also showed that the Kan–Quillen model structure on the category of simplicial sets [27] can be modified by incorporating E_*-equivalences into the collection of weak equivalences for some spectrum E, and the fibrant spaces in the resulting model structure are the E_*-local Kan complexes. This idea was broadly generalized in what is nowadays called *Bousfield localizations* of model categories, and has found applications in various mathematical disciplines. By an E_*-equivalence we mean a map $X \longrightarrow Y$ that induces isomorphisms $E_n(X) \cong E_n(Y)$ for all n.

In an unpublished paper [9], Bousfield considered localizations with respect to *cohomology* theories, where one seeks to invert up to homotopy the E^*-equivalences for a spectrum E, i.e., the maps $X \longrightarrow Y$ inducing isomorphisms $E^n(Y) \cong E^n(X)$ for all n. He never supplied a proof of the existence of arbitrary cohomological localizations, although he showed that in many examples the class of E^*-equivalences coincides with the class of F_*-equivalences for some homology theory F_*. This was worked out further by Hovey in [18], where he conjectured that every cohomological Bousfield class is indeed a homological Bousfield class. While this is still an open problem in the category of spectra, a counterexample was found by Stevenson in the derived category of a non-Noetherian ring [31].

Alhough the lack of examples of cohomological Bousfield classes that are not homological Bousfield classes has diminished the practical interest of constructing cohomological localizations in homotopy theory, the problem of whether the existence of cohomological localizations can be proved or not using the ZFC axioms of set theory has remained as a challenging logical problem.

A first step was made in [13] by showing that *Vopěnka's principle* from set theory implies the existence of localizations with respect to arbitrary cohomology theories on simplicial sets. This result was based on previous knowledge of locally presentable and accessible categories, where it had been shown that Vopěnka's principle implies the existence of localizations onto limit-closed subcategories [1].

Using other methods, Przeździezcki proved in [26] that an E^*-localization can be constructed in ZFC if each of the spaces constituting E is a homotopy retract of a compact space. Another step came in [6] by showing that if arbitrarily large supercompact cardinals exist, then E^*-localization exists for all spectra E.

The existence of supercompact cardinals cannot be proved in ZFC, since they are inaccessible. They have an important place in the large-cardinal hierarchy [21], where Vopěnka's principle also belongs (much higher up). Large cardinals appear naturally in several areas of mathematics. For example, the existence of Grothendieck universes —an assumption very often made to justify the use of "small" sets— is equivalent to the existence of inaccessible cardinals.

The general form of the *reflection principle* in set theory says informally that every property of the universe of all sets is shared by some set. This principle can be formalized in different ways, some of which are related with large-cardinal axioms.

The concept of *structural reflection* was introduced by Bagaria and discussed in detail in [5]. It states that for a class C of structures of the same type there is a cardinal κ such that every $X \in C$ has a logically equivalent substructure of cardinality smaller than κ and isomorphic to some $Y \in C$. This assertion is implied by the Löwenheim–Skolem theorem if the class C is defined by an upward absolute formula —that is, a formula whose truth in a transitive model implies its truth in every larger model. For classes defined by formulas of higher complexity, structural reflection requires the existence of large cardinals.

In our case, the class of E_*-equivalences for a spectrum E can be defined by an upward absolute formula with E as a parameter. Consequently, the existence of arbitrary E_*-localizations can be proved in ZFC. Although this was of course known since [7], we emphasize that a proof can be given by means of a basic set-theoretical argument; see Theorem 2 below.

However, the complexity of defining E^*-equivalences seems to be higher in the Lévy hierarchy, since no upward absolute formula has been found for this purpose. The difficulty is that, in order to formalize the statement that a space X is E^*-acyclic, the collection of all functions from X to E has to be considered in some way, and sets of functions (for example, $2^{\mathbb{N}}$) are not upward absolute in general.

While it is conceivable that a proof of the existence of cohomological localizations can be given in ZFC, it is unreasonable to expect an explicit counterexample in ZFC, since such a counterexample would invalidate most of the large-cardinal hierarchy. In fact, the statement that arbitrary cohomological localizations exist is probably equivalent to some large-cardinal principle.

1 Homology theories and cohomology theories

By a *homology theory* we mean a generalized homology in the sense of the Eilenberg–Steenrod axioms [15], and similarly for a *cohomology theory*. We only consider homology and cohomology theories that are *reduced* (i.e., vanishing on a point) and *representable*, that is, defined by a spectrum as follows. A *spectrum E* is a collection of pointed spaces E_k for $k \geq 0$ together with structure maps $S^1 \wedge E_k \longrightarrow E_{k+1}$, where S^n denotes the n-sphere and $\wedge$ is the smash product, i.e., $A \wedge B$ is obtained by collapsing the one-point union $A \vee B$ within the product $A \times B$ for pointed spaces A and B. Every spectrum E yields a homology theory E_* by defining

$$E_n(X) = \mathrm{colim}_k \, [S^{n+k}, E_k \wedge X]$$

for $n \in \mathbb{Z}$ and every pointed space X, where $[-,-]$ denotes pointed homotopy classes of maps, and a cohomology theory E^* as

$$E^n(X) = \mathrm{colim}_k \, [S^k \wedge X, E_{n+k}]$$

for $n \in \mathbb{Z}$ as well. When $n \geq 0$, this simplifies to $E^n(X) \cong [X, E_n]$. For convenience we assume that E is an Ω-*spectrum*, that is, the adjoint maps $E_k \longrightarrow \Omega E_{k+1}$ of the structure maps of E are weak homotopy equivalences, where Ω denotes loops.

Note that X appears in the target of maps $S^{n+k} \longrightarrow E_k \wedge X$ for homology while it appears in the source of maps $S^k \wedge X \longrightarrow E_{n+k}$ for cohomology. This fact implies covariance of E_* but contravariance of E^*, a fundamental difference.

A map $X \longrightarrow Y$ of spaces is called an E_*-*equivalence* if it induces isomorphisms $E_n(X) \cong E_n(Y)$ for all n, and it is called an E^*-*equivalence* if it induces isomorphisms $E^n(Y) \cong E^n(X)$ for all n. An E_*-*localization* of a space X is a terminal E_*-equivalence going out of X, that is, an E_*-equivalence $\eta_X \colon X \longrightarrow L_E X$ such that for every E_*-equivalence $f \colon X \longrightarrow Y$ there is a map $g \colon Y \longrightarrow L_E X$ such that the composite $g \circ f$ is homotopic to η_X, and g is unique up to homotopy with this property. An E^*-*localization* is defined analogously.

1.1 Categories of fractions

The approach undertaken by Adams in [3, 4] to construct E_*-localizations for every homology theory E_* is summarized in this section. He worked in the category $\mathcal{H}$ whose objects are CW-complexes with basepoint and whose morphisms are pointed homotopy classes of maps. We write $f \simeq g$ to denote that f and g are homotopic.

If $\mathcal{S}$ denotes either the class of E_*-equivalences or the class of E^*-equivalences for a spectrum E, then $\mathcal{S}$ admits a *calculus of left fractions* as defined by Gabriel–Zisman in [16] as follows:

(i) $\mathcal{S}$ is closed under compositions.
(ii) For every pair of maps $s \colon W \longrightarrow X$ and $f \colon W \longrightarrow Y$ where $s \in \mathcal{S}$ there are maps $g \colon X \longrightarrow Z$ and $t \colon Y \longrightarrow Z$ with $t \in \mathcal{S}$ such that $g \circ s \simeq t \circ f$.
(iii) For every map $s \colon W \longrightarrow X$ in $\mathcal{S}$ and every pair of maps $f, g \colon X \longrightarrow Y$ with $f \circ s \simeq g \circ s$ there is a map $t \colon Y \longrightarrow Z$ in $\mathcal{S}$ such that $t \circ f \simeq t \circ g$.

Condition (ii) is satisfied by choosing a homotopy pushout of f and s, and condition (iii) holds using a homotopy coequalizer of f and g, with a Mayer–Vietoris argument in both cases; cf. [7, Lemma 3.6]. In addition to (i), for composable maps s and t, if two of s, t and $t \circ s$ are in $\mathcal{S}$ then the third is also in $\mathcal{S}$.

The *category of fractions* $\mathcal{S}^{-1}\mathcal{H}$ has the same objects as $\mathcal{H}$ and morphisms from X to Y are equivalence classes of zig-zags

$$X \xrightarrow{f} Z \xleftarrow{s} Y$$

where $s \in \mathcal{S}$, and two such zig-zags $(f, s)\colon X \to Z \leftarrow Y$ and $(f', s')\colon X \to Z' \leftarrow Y$ are defined to be equivalent if there is a space Z'' equipped with maps $g\colon Z \to Z''$ and $g'\colon Z' \to Z''$ in $\mathcal{S}$ such that $g' \circ f' \simeq g \circ f$ and $g \circ s \simeq g' \circ s'$. Composition is defined using (ii), and it is well defined thanks to (iii).

In the category $\mathcal{S}^{-1}\mathcal{H}$ each map $s\colon X \to Y$ in $\mathcal{S}$ has an inverse, namely the zig-zag $(\mathrm{id}, s)\colon Y \to Y \leftarrow X$. Moreover, there is a canonical functor $Q\colon \mathcal{H} \to \mathcal{S}^{-1}\mathcal{H}$ sending every map $f\colon X \to Y$ to $(f, \mathrm{id})\colon X \to Y \leftarrow Y$, and Q is universal among functors from $\mathcal{H}$ sending all maps in $\mathcal{S}$ to isomorphisms; see [16, Proposition 2.4].

However, there is a famous difficulty with the category $\mathcal{S}^{-1}\mathcal{H}$, namely we need to prove that it is *locally small*, i.e., it has only a set of morphisms between any two objects (not a proper class). As explained in the next subsection, this is feasible if $\mathcal{S}$ is the class of E_*-equivalences for a spectrum E, yet it is still an open problem (in ZFC) when $\mathcal{S}$ is the class of E^*-equivalences.

Once this difficulty is solved, Brown's representability theorem [11] ensures the existence of a right adjoint $R\colon \mathcal{S}^{-1}\mathcal{H} \to \mathcal{H}$ to $Q\colon \mathcal{H} \to \mathcal{S}^{-1}\mathcal{H}$, i.e., for all spaces X and Y there is a natural bijective correspondence

$$\mathcal{S}^{-1}\mathcal{H}(QX, Y) \cong \mathcal{H}(X, RY).$$

In order to use Brown representability, it is necessary that, for a fixed space Y, the functor sending each space X to $\mathcal{S}^{-1}\mathcal{H}(QX, Y)$ be set-valued rather than class-valued. This is the reason why we need that the category $\mathcal{S}^{-1}\mathcal{H}$ be locally small.

The adjoint pair Q, R yields an idempotent functor $L_E\colon \mathcal{H} \to \mathcal{H}$, namely the composite RQ equipped with the unit η of the adjunction. This idempotent functor is an E_*-localization on $\mathcal{H}$. Indeed, for every X the map $\eta_X\colon X \to L_E X$ is in $\mathcal{S}$, and it is a terminal map in $\mathcal{S}$ going out of X, as desired.

The properties of E_*-localization of spaces are analogous to those of the passage from abelian groups to $\mathbb{Q}$-vector spaces by formally inverting nonzero integers. In fact, if we choose $E_* = H_*(-;\mathbb{Q})$ as our homology theory (hence $\mathcal{S}$ is the class of singular homology equivalences with rational coefficients), then the resulting idempotent functor on the homotopy category $\mathcal{H}$ of pointed CW-complexes extends Sullivan's rationalization of 1-connected spaces [33]. This was one of the motivations of Adams' work, although the behaviour of $H_*(-;\mathbb{Q})$-localization on arbitrary spaces is much more difficult to describe than in the case of 1-connected spaces.

We next address the problem of proving that $\mathcal{S}^{-1}\mathcal{H}$ is locally small.

1.2 Solution-set conditions

A standard way to prove that a category of fractions is locally small is to impose the existence of a cofinal subset of the class $\{s\colon Y \to Z \mid s \in \mathcal{S}\}$ for every fixed Y. This cofinality condition was stated as Axiom 3.4 in [4] and also considered by Deleanu in [14], and reads as follows:

(A) For every space Y there is a subset $A = \{s_\alpha\colon Y \to Z_\alpha\}$ of $\mathcal{S}$ such that for every map $s\colon Y \to Z$ in $\mathcal{S}$ there is a map $s_\alpha\colon Y \to Z_\alpha$ in A and a map $g\colon Z \to Z_\alpha$ such that $g \circ s \simeq s_\alpha$.

This condition ensures that each zig-zag $(f, s)\colon X \to Z \leftarrow Y$ represents the same morphism in $\mathcal{S}^{-1}\mathcal{H}$ as $(g \circ f, s_\alpha)\colon X \to Z_\alpha \leftarrow Y$ for some α, and there is only a set of those. Consequently, $\mathcal{S}^{-1}\mathcal{H}(X, Y)$ is a set for all X and Y, as wanted.

Unfortunately, there seems to be no way to check a priori that condition (A) holds for E_*-equivalences nor for E^*-equivalences. This is the reason why Adams' approach was not considered to be conclusive at that moment.

However, as observed by Fiedorowicz in [4, § 8], the fact that $\mathcal{S}^{-1}\mathcal{H}$ is locally small can also be inferred from the following solution-set condition, which is much more useful than (A):

(B) For all spaces X and Y there is a set of zig-zags $B = \{(f_\alpha, s_\alpha)\colon X \to Z_\alpha \leftarrow Y\}$ with $s_\alpha \in \mathcal{S}$ such that for every $(f, s)\colon X \to Z \leftarrow Y$ with $s \in \mathcal{S}$ there exists $(f_\alpha, s_\alpha) \in B$ and a map $g\colon Z_\alpha \to Z$ such that $g \circ f_\alpha \simeq f$ and $g \circ s_\alpha \simeq s$.

In other words, condition (B) imposes that the category of zig-zags from X to Y where the backward arrow is in $\mathcal{S}$ has a weakly initial small subcategory. This ensures that each (f, s) represents the same morphism as (f_α, s_α) for some α, and hence it follows again that $\mathcal{S}^{-1}\mathcal{H}(X, Y)$ is a set for all X and Y.

Condition (B) holds for the class $\mathcal{S}$ of E_*-equivalences if E is any spectrum. The following argument is a rewriting of [4, Lemma 8.3] or [7, Lemma 11.3].

Theorem 1 (Existence of homological localizations: topological proof)
E_*-localization exists for every spectrum E.

Proof In order to prove that E_*-localization of CW-complexes exists for every spectrum E it suffices to prove that condition (B) holds for the class of E_*-equivalences.

Given X and Y, let κ be an infinite cardinal bigger than the cardinality of the sets of cells of X and Y and bigger than the cardinality of the abelian group $E_*(S^0)$. It then follows by means of the Atiyah–Hirzebruch spectral sequence that the cardinality of $E_*(X)$ and $E_*(Y)$ is smaller than κ, and therefore the cardinality of $E_*(X \vee Y)$ is also smaller than κ, since $E_*(X \vee Y) \cong E_*(X) \oplus E_*(Y)$.

Let B be a set of representatives of all homeomorphism classes of zig-zags $(f_\alpha, s_\alpha)\colon X \to Z_\alpha \leftarrow Y$ where Z_α has less than κ cells and s_α is an E_*-equivalence. Suppose given $(f, s)\colon X \to Z \leftarrow Y$ where s is an E_*-equivalence. If W_0 denotes the image of the map $X \vee Y \to Z$ induced by (f, s), then the homomorphism $\varphi_0\colon E_*(W_0) \to E_*(Z)$ induced by the inclusion $W_0 \subset Z$ is an epimorphism. Since

E_* commutes with filtered colimits, every homology class in $\ker\varphi_0$ vanishes on some subcomplex of Z obtained by adding finitely many cells to W_0. Hence we can choose a subcomplex W_1 of Z with less than κ cells such that $W_0 \subset W_1 \subset Z$ and the homomorphism $E_*(W_0) \longrightarrow E_*(W_1)$ sends all the elements of $\ker\varphi_0$ to zero. The inclusion $W_1 \subset Z$ induces again an epimorphism on E_*-homology and we can iterate the same construction in order to obtain a nested sequence of subcomplexes W_n such that if $W = \cup_{n=1}^{\infty} W_n$ then W has still less than κ cells and the inclusion $W \subset Z$ is now an E_*-equivalence. Moreover, the composite map $X \vee Y \longrightarrow W$ yields maps $g\colon X \longrightarrow W$ and $t\colon Y \longrightarrow W$ whose composites with the inclusion $W \subset Z$ are equal to f and s respectively. Hence (g, t) is homeomorphic to an element of B, and condition (B) is fulfilled. □

In this proof, the fact that E_* is a covariant functor that commutes with filtered colimits is essential. Thus, while this approach works well for homology theories, there seems to be no way to check in ZFC that condition (B) holds in the case of cohomology theories. Nevertheless, as we explain in Section 2, condition (B) does hold for the class of E^*-equivalences for any spectrum E if we assume the existence of sufficiently large cardinals —indeed, too large to be available in ZFC.

2 Set-theoretical reflection

2.1 Cardinality and rank

In ZFC set theory, no set can be an element of itself and no descending sequence for the membership relation can be infinite. The *rank* of a set X is defined inductively as the smallest ordinal greater than the ranks of all the elements of X. In particular, the rank of every ordinal is equal to itself.

Cardinality and rank are different concepts. For example, the set $\mathbb{R}$ of real numbers has rank $\omega + 1$ (where ω is the first infinite ordinal) but uncountable cardinality, and the set $\{\mathbb{R}\}$ has cardinality 1 but rank $\omega + 2$. In what follows, the cardinality of a set X will be denoted by $|X|$.

A set X is called *transitive* if every element of X is also a subset of X, that is, if X has the property that whenever $a \in X$ and $b \in a$ then $b \in X$. The *cumulative hierarchy* of sets is defined by transfinite recursion as $V_0 = \emptyset$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ where $\mathcal{P}$ denotes power set, and $V_\lambda = \cup_{\alpha<\lambda} V_\alpha$ if λ is a limit ordinal. It follows by induction that each V_α is transitive and $V_\alpha \subset V_\beta$ if $\alpha < \beta$. Every set X is a member of some set in this hierarchy [19, Lemma 6.3], and the rank of X is the smallest ordinal α such that $X \in V_{\alpha+1}$. Hence V_α is the set of all sets of rank smaller than α. The union $V = \cup_\alpha V_\alpha$ is the set-theoretical universe or *von Neumann universe*. If κ is an inaccessible cardinal then V_κ is a model for ZFC set theory, and $|V_\kappa| = \operatorname{rank}(V_\kappa) = \kappa$.

2.2 Structures

A summary of terminology and basic facts about languages, structures and theories can be found in [1, Ch. 5], [6, § 1] or [19, Ch. 12], among many other places.

For a regular cardinal λ and a set S, a *λ-ary S-sorted signature* consists of a set of *operation symbols*, a set of *relation symbols* and an *arity* function that assigns to each operation symbol an ordinal $\alpha < \lambda$, a sequence of *input sorts* $\langle s_i \mid i < \alpha\rangle$ and an *output sort* $s \in S$ (then we denote the corresponding operation symbol by $\Pi_{i<\alpha} s_i \to s$), and to each relation symbol an ordinal $\beta < \lambda$ and a sequence of sorts $\langle s_j \mid j < \beta\rangle$. An operation symbol of arity 0 is called a *constant*.

A *σ-structure* for a signature σ is an S-sorted set $X = \{X_s \mid s \in S\}$ equipped with an *interpretation* of σ, that is, a function $\Pi_{i<\alpha} X_{s_i} \to X_s$ for each operation symbol $\Pi_{i<\alpha} s_i \to s$ (including a distinguished element in X_s for each constant of sort s) and a relation on X for each relation symbol. A *homomorphism* of σ-structures is an S-sorted function that preserves operations and relations.

The *language* of a λ-ary signature σ is made of a set of *variables* of cardinality λ and *formulas* involving variables, operations and relations in σ, equality, negation, implication, conjunctions and disjunctions of cardinality smaller than λ, and finitely many quantifiers over sets of variables of cardinality smaller than λ. Languages with $\lambda = \aleph_0$ are called *finitary*. For example, the language of ZFC set theory is one-sorted and finitary with a binary relation symbol $\in$, which is interpreted as membership.

Variables that appear unquantified in a formula are called *free*. Formulas without free variables are called *sentences*, and a set of sentences is called a *theory*.

For each language, a *satisfaction relation* is defined inductively for formulas with an assignment for their free variables. Thus, for a formula $\varphi(x_i)_{i\in I}$ with free variables x_i, we say that the sentence $\varphi(a_i)_{i\in I}$ *holds* in a σ-structure X if $a_i \in X$ for all $i \in I$ and φ is satisfied in X under the variable assignment $x_i \mapsto a_i$. A σ-structure X is called a *model* for a set of formulas if each of these formulas holds in X under any variable assignment.

For example, the signature of the theory of pointed simplicial sets is ω-sorted with unary operations d_i^n of sorts $n \to n-1$ (faces) and s_i^n of sorts $n \to n+1$ (degeneracies) for $0 \le i \le n$, and a constant of sort 0 (the basepoint), and the axioms of this theory are the simplicial identities [23]. Homomorphisms between models are basepoint-preserving simplicial maps.

A *parameter* in a formula is a set which is fixed in every variable assignment. Every formula φ of the language of ZFC set theory with a parameter p defines a class $C = \{X \mid \varphi(X,p)\}$, meaning that C consists of all sets X for which $\varphi(X,p)$ holds in V. A formula φ with a parameter p is called *absolute* for a set or a proper class M with $p \in M$ if φ holds in V if and only if it holds with its quantifiers relativized to M; then one also says that M is *elementary* for φ.

An *elementary embedding* between σ-structures is a function $f\colon X \to Y$ that preserves and reflects truth, that is, for every formula $\varphi(x_i)_{i\in I}$ of the language of σ with free variables x_i, and for all $a_i \in X$, the sentence $\varphi(a_i)_{i\in I}$ holds in X if and only if $\varphi(f(a_i))_{i\in I}$ holds in Y. Thus, every elementary embedding $X \to Y$ is, in particular, an injective homomorphism of σ-structures.

2.3 Reflection principles

The Löwenheim–Skolem theorem is a central result in first-order logic. Its simplest form states that every infinite model for a countable language has a countable elementary submodel [19, Theorem 12.1]. More generally, for every infinite σ-structure X and every infinite cardinal $\kappa \geq |\sigma|$, if $\kappa < |X|$ then there exists an elementary substructure $Y \subset X$ of cardinality κ (downward version) and if $\kappa > |X|$ then there exists an elementary extension $X \subset Y$ of cardinality κ (upward version).

Another version specializes to a finite set of formulas (or equivalently one formula) and reads as follows. Given any formula φ of ZFC set theory and given an infinite cardinal κ, there is a set M of cardinality κ which is elementary for φ. Moreover, M can be chosen as an extension of any given set of cardinality κ, and it can be chosen transitive if we remove the restriction on its cardinality [19, Theorem 12.14]. In this situation, one says that the formula φ is *reflected* by M. This is called the *reflection principle* and it is usually referred to by saying that every formula that holds in V already holds in V_α for some α; see [5] for more details and a historical perspective.

The following variant, called *structural reflection*, was used in [5, 6]. A class C of σ-structures closed under isomorphic images is *reflected* by a cardinal κ if every $X \in C$ has an elementary substructure $Y \in C$ of cardinality smaller than κ. The Löwenheim–Skolem theorem implies that every isomorphism-closed class of σ-structures defined by a formula that is absolute for transitive classes is reflected by any uncountable cardinal larger than $|\sigma|$. In fact it is sufficient that the formula be *upward* absolute, in a sense that we next discuss.

2.4 The Lévy hierarchy

The existence of a cardinal reflecting a class C depends on the *Lévy complexity* [22] of a formula defining C. A formula φ is called Σ_0 or Π_0 if it does not contain unbounded existential quantifiers $\exists$ nor unbounded universal quantifiers $\forall$, that is, all quantifiers in φ are of the form $\exists x \in a$ or $\forall x \in a$ where a is some set. For $n \geq 1$, a formula is called Σ_n if it has the form $\exists x\, \varphi(x)$ where φ is Π_{n-1}, and it is called Π_n if it has the form $\forall x\, \varphi(x)$ where φ is Σ_{n-1}.

One of the consequences of the reflection principle is that for every n there exist arbitrarily large cardinals α such that V_α is a Σ_n-elementary substructure of V, that is, a Σ_n formula with parameters in V_α holds in V_α if and only if it holds in V.

Every Σ_0 formula is absolute for transitive classes; see [19, Lemma 12.9]. Likewise, Σ_1 formulas are upward absolute while Π_1 formulas are downward absolute. The reason is that if in a transitive model M there exists a set x with a property expressed by a Σ_0 formula, then every model containing M also has a set x with that property (in fact, the same x), and if every x in M has a property expressed by a Σ_0 formula then the same holds in each transitive submodel of M.

As an example, the clause $a \subseteq b$ is formalized by the Σ_0 formula $\forall x \in a\ (x \in b)$, which is absolute between two models $M \subset N$ if M is transitive and $a, b \in M$ (we need transitivity to ensure that $x \in a$ implies $x \in M$). The claim "a is the set of all subsets of b" can be formalized with the Π_1 formula $\forall x\,(x \in a \leftrightarrow x \subseteq b)$, and its truth is not preserved upwards, since the set $\mathcal{P}(\mathbb{N})$ of all subsets of $\mathbb{N}$ is countable in any countable transitive model of ZFC but uncountable in V.

As another example, the claim "x is finite" can be formalized with a Σ_1 formula stating that there is a bijection between x and some finite ordinal, and the assertion that "a is the set of finite subsets of b" can be expressed as follows:

$$x \in a \leftrightarrow \exists n < \omega \; \exists f \, (f \text{ is a function from } n \text{ to } b \text{ with image } x). \tag{1}$$

Moreover, (1) can be rewritten by stating that there is a transitive model of a sufficiently large finite fragment of ZFC containing a and b in which at least the pairing and union axioms hold and in which (1) is true, as in [6, Example 2.3]. This is a Σ_1 statement. Consequently, quantifying over finite subsets of some given set b can be done by means of Σ_1 formulas with b as a parameter.

2.5 Existence of localizations

For fixed simplicial sets X and Y, a simplicial set Z equipped with pointed maps $X \to Z$ and $Y \to Z$ can be viewed as a model of a theory over an ω-sorted signature σ_{XY} consisting of unary operations $n \to n-1$ for $n \geq 1$ and $n \to n+1$ for $n \geq 0$ plus a constant of sort 0 (to be interpreted as faces, degeneracies, and basepoint in Z) and, in addition, for all n, a constant of sort n for each element of X_n and another constant of sort n for each element of Y_n, to be interpreted as images in Z of the simplices of X and Y. This signature σ_{XY} need no longer be countable but $|\sigma_{XY}| = |X| + |Y|$. The axioms of the theory are the simplicial identities for Z together with a statement that the functions $f\colon X \to Z$ and $s\colon Y \to Z$ determined by the constants are simplicial maps, and we need to add as another axiom that the Kan condition holds for the simplicial set Z. Our choices guarantee that homomorphisms $g\colon Z \to Z'$ of σ_{XY}-structures satisfy $g \circ f = f'$ and $g \circ s = s'$, since constants are preserved by homomorphisms.

Theorem 2 has the same statement as Theorem 1, but a very different proof. It has been written using and adapting results from [6].

Theorem 2 (Existence of homological localizations: set-theoretical proof)
E_*-localization exists for every spectrum E.

Proof Our aim is to infer that condition (B) from Section 1 holds for the class $\mathcal{E}_{XY}$ of Kan simplicial sets Z equipped with pairs of pointed maps $f\colon X \to Z$ and $s\colon Y \to Z$, where X, Y are fixed simplicial sets and s is an E_*-equivalence. For this, we view each such Z as a σ_{XY}-structure for the signature σ_{XY} defined above.

All the terms in a definition of $\mathcal{E}_{XY}$ are absolute, except for a formula stating the fact that s is an E_*-equivalence, which we next analyze following [6, Theorem 9.2]. A map is an E_*-equivalence if and only if its cofibre is E_*-acyclic, and a space

A is E_*-acyclic if and only if the spectrum $E \wedge A$ is weakly contractible, i.e., all its homotopy groups vanish. As detailed in the proof of [6, Proposition 9.1], this fact can be expressed with formulas that contain quantifiers involving finite sets of simplices of the spaces $E_n \wedge A$, where $\{E_n\}$ is the set of constituents of E and fibrant replacements are used when needed. Hence we can write a Σ_1 formula φ with $p = \{X, Y, E\}$ as a set of parameters such that $\mathcal{E}_{XY} = \{Z \mid \varphi(Z, p)\}$.

Pick an uncountable cardinal κ larger than the ranks of X, Y and E and such that $|V_\kappa| = \kappa$. Given any $Z \in \mathcal{E}_{XY}$, the reflection principle ensures that there is a regular cardinal $\lambda > \kappa$ such that $Z \in V_\lambda$ and $\varphi(Z, p)$ holds in V_λ. By the Löwenheim–Skolem theorem, there is an elementary submodel $N \subset V_\lambda$ with $|N| < \kappa$ such that $Z \in N$ and the transitive closure of $\{X, Y, E\}$ is also in N. By elementarity, $\varphi(Z, p)$ holds in N. However, Z need not be a *subset* of N, since N is not transitive.

Now let $\pi\colon N \to M$ be the unique isomorphism where M is transitive —this uses the Mostowski collapse; see [19, Theorem 6.15] for details— and let $j\colon M \to N$ be the inverse isomorphism. If we pick $z = \pi(Z)$, then z is also a σ_{XY}-structure since j is an isomorphism and $j(\sigma_{XY}) = \sigma_{XY}$ because the transitive closure of $\{X, Y\}$ is in N, and $z \in V_\kappa$ since $z \subset M$ and $|M| < \kappa$. Moreover, $\varphi(z, p)$ holds in M because $\varphi(j(z), j(p))$ holds in N, as $j(p) = p$ since the transitive closure of p is in N. Using the fact that Σ_1 formulas are upward absolute, we infer that $\varphi(z, p)$ holds in V, which means that $z \in \mathcal{E}_{XY}$. Moreover, the restriction $j|z\colon z \to Z$ is an elementary embedding of σ_{XY}-structures. This means that there is an injective map $z \to Z$ with $z \in \mathcal{E}_{XY} \cap V_\kappa$, so condition (B) holds. □

In the case of formulas of higher Lévy complexity, the Löwenheim–Skolem theorem is not sufficient to ensure reflectivity. Instead, elementary embeddings from the universe V into convenient models are needed [5].

The *critical point* of an elementary embedding $j\colon V \to M$ is the smallest cardinal κ such that $j(\kappa) \neq \kappa$. Then all sets of rank smaller than κ are fixed by j, and $j(\kappa) > \kappa$. The existence of nontrivial elementary embeddings of V cannot be proved in ZFC, since if there is one then its critical point is a measurable cardinal. Indeed, many kinds of large cardinals are defined as critical points of elementary embeddings $V \to M$ with suitable conditions on M.

A cardinal κ is called *supercompact* if for every ordinal λ there exists an elementary embedding $j\colon V \to M$ with M transitive and with critical point κ such that $j(\kappa) > \lambda$ and M is closed under sequences of length λ; see [19, 21]. As evidenced by the following result, which is based on [6, Theorem 5.2], supercompact cardinals yield structural reflection for Σ_2 formulas.

Theorem 3 (Existence of cohomological localizations)
E^*-localization exists for every E if arbitrarily large supercompact cardinals exist.

Proof Now we aim to prove that condition (B) from Section 1 holds for the class $\mathcal{E}^{XY}$ of Kan simplicial sets Z with pairs of pointed maps $f\colon X \to Z$ and $s\colon Y \to Z$, where X, Y are simplicial sets and s is an E^*-equivalence.

For this, we need to prove that the class of E^*-equivalences for a spectrum E can be defined by means of a Σ_2 formula. This was done in [6, Theorem 9.3] and it is summarized as follows. A map of pointed simplicial sets is an E^*-equivalence if and only if its cofibre A is E^*-acyclic, and this means that the function spaces $\mathrm{map}(A, E_n)$ are weakly contractible for all n, where $E = \{E_n\}$ and we assume that E is an Ω-spectrum without loss of generality. In order to formalize the fact that $\mathrm{map}(A, E_n)$ is weakly contractible for all n, we write the following Σ_2 statement: "A is a simplicial set and there exists a function f with domain $\mathbb{N}$ such that, for all $n \in \mathbb{N}$, $f(n)$ is a simplicial set and, for every x and every $k \in \mathbb{N}$, x is a k-simplex of $f(n)$ if and only if x is a simplicial map $A \wedge \Delta[k]_+ \longrightarrow E_n$, and $f(n)$ is weakly contractible". Here $\Delta[k]_+$ denotes a standard k-simplex with a disjoint basepoint.

Thus, let φ denote a Σ_2 formula defining the class $\mathcal{E}^{XY}$ with $p = \{X, Y, E\}$ as a set of parameters, and let κ be a supercompact cardinal larger than the rank of p. Given $Z \in \mathcal{E}^{XY}$, pick a regular cardinal λ bigger than κ such that $Z \in V_\lambda$ and V_λ is Σ_2-elementary in V (here we use again the reflection principle). Let $j\colon V \longrightarrow M$ be an elementary embedding with M transitive and critical point κ such that $j(\kappa) > \lambda$ and M is closed under sequences of length λ. This implies that M contains V_λ.

Next, observe that V_λ is Σ_1-elementary in M, since every Σ_1 formula that holds in M also holds in V, and V_λ is Σ_2-elementary in V. Consequently, Σ_2 formulas are upward absolute between V_λ and M. Since φ is a Σ_2 formula, $\varphi(Z, p)$ holds in V_λ and hence it holds in M. Since Z and $j(Z)$ are in M and $Z \in V_\lambda$ and M is closed under λ-sequences, the restriction $j|Z\colon Z \longrightarrow j(Z)$ is in M. Furthermore, Z is a σ_{XY}-structure in M since $j(\sigma_{XY}) = \sigma_{XY}$, and $j|Z$ is an elementary embedding of σ_{XY}-structures.

We have that $\mathrm{rank}(Z) < \lambda < j(\kappa)$ in V and also in M, since M contains V_λ. Therefore, as witnessed by Z, in M there exists a σ_{XY}-structure z of rank smaller than $j(\kappa)$ for which $\varphi(z, j(p))$ holds, and there is an elementary embedding $z \longrightarrow Z$. By elementarity of j, the corresponding statement is true in V; that is, there exists a σ_{XY}-structure z of rank smaller than κ and $\varphi(z, p)$ holds in V, and there is an elementary embedding $z \longrightarrow Z$. In other words, every $Z \in \mathcal{E}^{XY}$ has a substructure in $\mathcal{E}^{XY} \cap V_\kappa$. Hence the elements of $\mathcal{E}^{XY} \cap V_\kappa$ form a solution set, and condition (B) holds, as needed. □

If no bound is imposed on the complexity of a formula defining a class $\mathcal{C}$, then the existence of a cardinal reflecting $\mathcal{C}$ follows from the Vopěnka principle. We omit the details of this claim and refer to [6].

3 Conclusions

In practice, in order to construct localizations with respect to proper classes of maps, sometimes one assumes the existence of *Grothendieck universes*, and moves to a higher universe whenever the construction of a category of fractions requires it. However, the assumption that for each set X there exists a Grothendieck universe $\mathcal{U}$ with $X \in \mathcal{U}$ is equivalent to the assumption that for each cardinal κ there exists an inaccessible cardinal $\lambda > \kappa$.

What we have proved in Theorem 3 is that, if there is a supercompact cardinal larger than a given spectrum E and larger than two given simplicial sets X, Y, then $\mathcal{S}^{-1}\mathcal{H}(X, Y)$ is a set (not a proper class), where $\mathcal{H}$ is the homotopy category of pointed simplicial sets and $\mathcal{S}^{-1}\mathcal{H}$ denotes the category of fractions for the class $\mathcal{S}$ of E^*-equivalences. In consequence, we obtain an E^*-localization functor without having to "pass to a higher universe", since the category $\mathcal{S}^{-1}\mathcal{H}$ happens to be locally small, which was the only pending requirement for Adams' argument in [4] to work. It is also remarkable that Adams' construction of E_*-localizations could have been made precise at that time by just using the reflection principle and the Löwenheim–Skolem theorem to infer that $\mathcal{S}^{-1}\mathcal{H}(X, Y)$ is a set for all X and Y (Theorem 2).

In conclusion, what is the situation now? We know that the existence of a proper class of supercompact cardinals ensures the existence of arbitrary cohomological localizations, but we do not know the precise logical strength of the claim that cohomological localizations exist. There are two possibilities:

1. It can be proved in ZFC that cohomological localizations exist.
2. The claim that cohomological localizations exist is itself a large-cardinal principle.

We do not consider the possibility that somebody finds a counterexample in ZFC, since this would imply that the existence of a proper class of supercompact cardinals is inconsistent with ZFC. This would make inconsistent an enormous segment of the large-cardinal hierarchy. Although this is not impossible, it is extremely unlikely.

Possibility 1 cannot be discarded, although this has remained a challenge for almost fifty years. Hence, possibility 2 seems the most plausible one.

One approach to try to prove that possibility 1 is the winning one would be to verify Hovey's conjecture [18], according to which for every spectrum E there is another spectrum F such that the class of E^*-equivalences is equal to the class of F_*-equivalences. If this were true, then the existence of cohomological localizations would be provable in ZFC by Theorem 2 above.

However, a solution of Hovey's conjecture seems still out ot reach, in spite of the fact that it has been shown to be false in derived categories of rings [31]. It is not even known whether the collection of cohomological Bousfield classes of spectra is a set or a proper class, while it is known since Ohkawa's work in [25] that there is only a set of distinct homological Bousfield classes.

Acknowledgements This article is dedicated to Dr. Catriona Byrne on her retirement, with warm thanks for three decades of friendly collaboration and, above all, for her lifelong support of the mathematical community. I also thank Joan Bagaria for revising and correcting the set-theoretical content of the manuscript.

References

1. J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, London Math. Soc. Lecture Note Ser., vol. 189, Cambridge University Press, Cambridge, 1994.
2. J. F. Adams, The sphere, considered as an H-space mod p, *Quart. J. Math.* 12 (1961), 52–60.

3. J. F. Adams, Idempotent functors in homotopy theory, in: *Manifolds – Tokyo 1973*, Univ. of Tokyo Press, Tokyo, 1975, 247–253.
4. J. F. Adams, Localisation and completion, with an addendum on the use of Brown–Peterson homology in stable homotopy, Lecture notes by Z. Fiedorowicz on a course given at The University of Chicago in Spring 1973. Revised and supplemented by Z. Fiedorowicz, 2010, arXiv:1012.5020.
5. J. Bagaria, Large cardinals as principles of structural reflection, 2021, arXiv:2107.01580.
6. J. Bagaria, C. Casacuberta, A.R.D. Mathias, and J. Rosický, Definable orthogonality classes in accessible categories are small, *J. Eur. Math. Soc.* **17** (2015), 549–589.
7. A. K. Bousfield, The localization of spaces with respect to homology, *Topology* **14** (1975), 133–150.
8. A. K. Bousfield, The localization of spectra with respect to homology, *Topology* **18** (1979), 257–281.
9. A. K. Bousfield, Cohomological localizations of spaces and spectra, 1979, unpublished.
10. A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math., vol. 304, Springer-Verlag, Berlin, Heidelberg, 1972.
11. E. H. Brown, Abstract homotopy theory, *Trans. Amer. Math. Soc.* **119** (1965), 79–85.
12. E. H. Brown and F. P. Peterson, A spectrum whose $\mathbb{Z}_p$-cohomology is the algebra of reduced p-th powers, *Topology* **5** (1966), 149–154.
13. C. Casacuberta, D. Scevenels, and J. H. Smith, Implications of large-cardinal principles in homotopical localization, *Adv. Math.* **197** (2005), 120–139.
14. A. Deleanu, Existence of the Adams completion for CW-complexes, *J. Pure Appl. Algebra* **4** (1974), 299–308.
15. S. Eilenberg and N. E. Steenrod, Axiomatic approach to homology theory, *Proc. Natl. Acad. Sci. USA* **31** (1945), 117–120.
16. P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*, Ergeb. Math. Grenzgeb., vol. 35, Springer-Verlag, Berlin, Heidelberg, 1967.
17. P. Hilton, G. Mislin, and J. Roitberg, *Localization of Nilpotent Groups and Spaces*, North-Holland Math. Studies, vol. 15, North-Holland, Amsterdam, 1975.
18. M. Hovey, Cohomological Bousfield classes, *J. Pure Appl. Algebra* **103** (1995), 45–59.
19. T. Jech, *Set Theory. The Third Millenium Edition, Revised and Expanded*, Springer Monographs in Math., Springer-Verlag, Berlin, Heidelberg, 2003.
20. D. C. Johnson and W. S. Wilson, BP operations and Morava's extraordinary K-theories, *Math. Z.* **144** (1975), 55–75.
21. A. Kanamori, *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, Heidelberg, 1994.
22. A. Lévy, *A Hierarchy of Formulas in Set Theory*, Mem. Amer. Math. Soc., vol. 57, Amer. Math. Soc., Providence, 1965.
23. J. P. May, *Simplicial Objects in Algebraic Topology*, The Univ. of Chicago Press, Chicago, 1967.
24. M. Mimura, G. Nishida, and H. Toda, Localization of CW-complexes, *J. Math. Soc. Japan* **23** (1971), 593–624.
25. T. Ohkawa, The injective hull of homotopy types with respect to generalized homology functors, *Hiroshima Math. J.* **19** (1989), 631–639.
26. A. J. Przeździecki, Homotopical localizations at a space, *Topology Appl.* **126** (2002), 131–143.
27. D. G. Quillen, *Homotopical Algebra*, Lecture Notes in Math., vol. 43, Springer-Verlag, Berlin, Heidelberg, 1967.
28. D. G. Quillen, Rational homotopy theory, *Ann. of Math. (2)* **90** (1969), 205–295.
29. D. Ravenel, Localizations with respect to certain periodic homology theories, *Amer. J. Math.* **106** (1984), 351–414.
30. J.-P. Serre, Groupes d'homotopie et classes de groupes abéliens, *Ann. of Math. (2)* **58** (1953), 258–294.
31. G. Stevenson, Derived categories of absolutely flat rings, *Homol. Homotop. Appl.* **16** (2014), 45–64.
32. D. Sullivan, Genetics of homotopy theory and the Adams conjecture, *Ann. of Math. (2)* **100** (1974), 1–79.
33. D. Sullivan, Infinitesimal computations in topology, *Publ. Math. IHÉS* **47** (1977), 269–331.