Homotopy reflectivity is equivalent to the weak Vopěnka principle

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Abstract

Homotopical localizations with respect to (possibly proper) classes of maps are known to exist in suitably structured model categories assuming the validity of Vopěnka's principle from set theory. In this article we prove that the existence of localization with respect to arbitrary classes of maps in the homotopy category of simplicial sets or in the homotopy category of spectra is equivalent to the weak Vopěnka principle. This result is inferred using the equivalence between the latter and the semi-weak Vopěnka principle, a long-standing open problem which was solved in 2020. Implications within triangulated categories and about Bousfield localizations of model categories are provided.

Introduction

One way to state the Vopěnka principle (VP) is that the ordered set Ord of all the ordinals cannot be fully embedded into any locally presentable category. In other words, there is no sequence of objects $\langle X_i \mid i \in \mathrm{Ord}\rangle$ indexed by all the ordinals in a locally presentable category $\mathcal{C}$ such that $\mathcal{C}(X_i, X_j)$ is a singleton if $i \leq j$ and the empty set if $i > j$. This statement is a large-cardinal principle that cannot be proved in ZFC. As explained in [2], it has many important consequences in category theory.

For example, the Vopěnka principle implies that full subcategories of locally presentable categories are reflective if they are closed under limits, and they are coreflective if they are closed under colimits. However, as shown in [2, § 6.D], the reflectivity of limit-closed full subcategories is equivalent to the statement that there is no full embedding of $\mathrm{Ord}^{\mathrm{op}}$ into any locally presentable category, which is a weaker condition than Vopěnka's principle. This is called *weak Vopěnka principle* (WVP). It can be rephrased by saying that there is no sequence of objects $\langle X_i \mid i \in \mathrm{Ord}\rangle$ indexed by all the ordinals in a locally presentable category $\mathcal{C}$ such that $\mathcal{C}(X_i, X_j)$ is a singleton if $i \geq j$ and the empty set if $i < j$.

In [1, Definition I.11], Adámek and Rosický considered another variation with the name of *semi-weak Vopěnka principle* (SWVP), by replacing the condition that $\mathcal{C}(X_i, X_j)$ be a singleton if $i \geq j$ by the condition that $\mathcal{C}(X_i, X_j)$ be nonempty if $i \geq j$. As shown in [1], SWVP is equivalent to the statement that every injectivity class in a locally presentable category is weakly reflective.

While it is easy to see that there are implications VP $\Rightarrow$ SWVP $\Rightarrow$ WVP, whether the reverse implications are true or not remained as an open problem until Wilson proved in [33] that WVP is equivalent to SWVP but not to VP. In fact the weak Vopěnka principle turned out to be equivalent to the statement that the class of all ordinals is Woodin —or, equivalently, that for every class C there exists a C-strong cardinal; see [34]. This determines the precise place of the weak Vopěnka principle in the large-cardinal hierarchy, and shows that it is very far below Vopěnka's principle, even far below the existence of supercompact cardinals.

It was shown in [15] that the statement that all homotopical localizations of simplicial sets are f-localizations for some map f is implied by Vopěnka's principle, and Przeździecki proved in [26] that the converse is true, that is, the statement that every homotopical localization of simplicial sets is an f-localization for a single map f is equivalent to Vopěnka's principle. Przeździecki also proved in [26] that the weak Vopěnka principle is equivalent to the statement that every orthogonality class in the category of groups is reflective, and he later proved in [27] that the same claim is true for the category of abelian groups.

Using arguments from [14], we prove in this article that SWVP implies that localization with respect to any class of maps exists in the homotopy category of (pointed or unpointed) simplicial sets, and next we infer from Przeździecki's results that the latter claim implies WVP. Therefore, since SWVP and WVP are known to be equivalent by [33], the claim that every (simplicially enriched) orthogonality class in simplicial sets or in spectra is associated with a homotopical localization turns out to be equivalent to WVP.

We also show that WVP is equivalent to the claim that every full subcategory closed under products and fibres in any triangulated category with locally presentable models is reflective.

As a consequence of our results, the existence of *cohomological* localizations of simplicial sets or spectra is implied by the weak Vopěnka principle. This improves substantially a conclusion from [3], where a similar result was obtained assuming the existence of a proper class of supercompact cardinals.

One could ask if the weak Vopěnka principle is also sufficient to infer the existence of left Bousfield localizations of left proper combinatorial model categories with respect to arbitrary classes of maps. This is known to be true assuming Vopěnka's principle, by [30, Theorem 2.3] or [12, Lemma 1.4]. In the last section of the article we prove that WVP implies indeed the existence of left Bousfield localizations with respect to every class of maps $\mathcal{S}$ in a combinatorial model category $\mathcal{M}$, assuming that idempotents split in the homotopy category $\mathrm{Ho}(\mathcal{M})$ but with two necessary shortcomings; namely, in general we can only guarantee the existence of a (right) *semi-model* category structure on the localized category $\mathcal{M}_{\mathcal{S}}$, and moreover factorizations in this semi-model category structure need not be functorial. Although Quillen's small object argument normally produces functors on model categories, our existence results in this article —based on SWVP— make no use of the small object argument and therefore yield only functors up to homotopy.

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1 Preliminaries

In this article, a *graph* is a pair (X, R) where X is a set and R is a binary relation $R \subseteq X \times X$. Thus, elements of X are vertices and there is a directed edge from x_1 to x_2 if and only if $(x_1, x_2) \in R$. We denote by Gra the category of graphs with relation-preserving functions, and we let Ord be the class of all ordinals, viewed as a category with a unique morphism $\lambda \to \mu$ if and only if $\lambda \leq \mu$.

A category $\mathcal{C}$ is *locally presentable* if it is cocomplete (i.e., it has all set-indexed colimits) and there is a regular cardinal λ and a set $\mathcal{A}$ of λ-presentable objects such that every object of $\mathcal{C}$ is a λ-filtered colimit of objects from $\mathcal{A}$. An object X is λ-*presentable* if the functor $\mathcal{C}(X, -)$ preserves λ-filtered colimits; see [19, §6.1] for further information about presentability. By [2, Corollary 1.28], locally presentable categories are also complete. Every category of structures is locally presentable [2, 5.1(5)], and the forgetful functor from structures into sets creates limits and colimits.

1.1 Higher orthogonality

A morphism $f\colon A \to B$ and an object X in a category $\mathcal{C}$ are called *orthogonal* if the function $\mathcal{C}(B,X) \to \mathcal{C}(A,X)$ sending every $g\colon B \to X$ to $g \circ f$ is a bijection of sets. The corresponding higher-categorical notion is as follows: A morphism $f\colon A \to B$ and an object X in an ∞-category $\mathcal{C}$ are orthogonal if the induced map of ∞-groupoids $\operatorname{map}(B,X) \to \operatorname{map}(A,X)$ is an equivalence.

To state this definition more accurately, we let $\mathcal{C}$ be any simplicially enriched category such that the hom simplicial set $\operatorname{map}(X,Y)$ is a Kan complex for all X and Y. The *underlying category* of $\mathcal{C}$, which we denote with the same letter for simplicity of notation, has the same objects as $\mathcal{C}$ and the set of morphisms $\mathcal{C}(X,Y)$ is the set of vertices of $\operatorname{map}(X,Y)$, i.e., morphisms $X \to Y$ are represented by simplicial maps $\Delta[0] \to \operatorname{map}(X,Y)$, where $\Delta[n]$ denotes the standard n-simplex for $n \geq 0$. The *homotopy category* $\mathrm{h}\mathcal{C}$ has the same objects as $\mathcal{C}$ and the set of morphisms $\mathrm{h}\mathcal{C}(X,Y)$ is the set $\pi_0(\operatorname{map}(X,Y))$, which we denote by $[X,Y]$. A morphism $f\colon A \to B$ and an object X in $\mathcal{C}$ are called *orthogonal* if the composite

$$\operatorname{map}(B,X) \times \Delta[0] \xrightarrow{\mathrm{id}\times f} \operatorname{map}(B,X) \times \operatorname{map}(A,B) \xrightarrow{\ \circ\ } \operatorname{map}(A,X),$$

which we abbreviate as

$$f^*\colon \operatorname{map}(B,X) \longrightarrow \operatorname{map}(A,X),$$

is a weak equivalence of simplicial sets. We denote this fact by writing $f \perp X$.

For a class $\mathcal{S}$ of morphisms in $\mathcal{C}$, its *orthogonal* class of objects is

$$\mathcal{S}^{\perp} = \{X \in \mathcal{C} \mid f \perp X \text{ for all } f\colon A \to B \text{ in } \mathcal{S}\},$$

and, similarly, given a class $\mathcal{D}$ of objects, we denote

$$\mathcal{D}^{\perp} = \{f\colon A \to B \mid f \perp X \text{ for all } X \text{ in } \mathcal{D}\}.$$

For a class of morphisms $\mathcal{S}$ in $\mathcal{C}$, an *$\mathcal{S}$-localization* of an object X in $\mathcal{C}$ is a morphism $\ell_X\colon X \to LX$ in $\mathcal{S}^{\perp\perp}$ with $LX \in \mathcal{S}^{\perp}$. The objects in $\mathcal{S}^{\perp}$ are called *$\mathcal{S}$-local* and the morphisms in $\mathcal{S}^{\perp\perp}$ are called *$\mathcal{S}$-equivalences.*

Higher orthogonality in $\mathcal{C}$ yields orthogonality in $\mathrm{h}\mathcal{C}$, since a weak equivalence $\operatorname{map}(B,X) \simeq \operatorname{map}(A,X)$ yields a bijection $[B,X] \cong [A,X]$. Although the converse is not true, $\mathcal{S}$-localizations in $\mathcal{C}$ are paired with reflections in $\mathrm{h}\mathcal{C}$, as we next show. For the validity of this result, we need to assume that $\mathcal{C}$ is not only simplicially enriched but also cotensored over simplicial sets. A morphism $r\colon X \to D$ in a category $\mathcal{H}$ is a *reflection* onto a class $\mathcal{L}$ of objects if $D \in \mathcal{L}$ and r induces a bijection $\mathcal{H}(D,E) \cong \mathcal{H}(X,E)$ for every $E \in \mathcal{L}$.

Proposition 1.1. *Let $\mathcal{S}$ be a class of morphisms in a category $\mathcal{C}$ enriched in Kan complexes and cotensored over simplicial sets, and let X be any object of $\mathcal{C}$. A morphism $r\colon X \to D$ is an $\mathcal{S}$-localization of X if and only if r is a reflection onto $\mathcal{S}^{\perp}$ in the homotopy category $\mathrm{h}\mathcal{C}$.*

Proof. Suppose first that $r\colon X \to D$ is an $\mathcal{S}$-localization. Then for every $E \in \mathcal{S}^{\perp}$ the morphism r induces a weak equivalence

$$\mathrm{map}(D, E) \simeq \mathrm{map}(X, E),$$

which yields a bijection $[D, E] \cong [X, E]$ on π_0. This means precisely that r is a reflection onto $\mathcal{S}^{\perp}$ in $\mathrm{h}\mathcal{C}$. For the converse, we let $r\colon X \to D$ be a reflection onto $\mathcal{S}^{\perp}$ and need to prove that $r \in \mathcal{S}^{\perp\perp}$. Hence we need to prove that for every $E \in \mathcal{S}^{\perp}$ the induced map

$$r^*\colon \mathrm{map}(D, E) \longrightarrow \mathrm{map}(X, E) \tag{1.1}$$

is a weak equivalence. For this, we prove that (1.1) induces bijections

$$[W, \mathrm{map}(D, E)] \cong [W, \mathrm{map}(X, E)]$$

for every simplicial set W. Using the cotensoring of $\mathcal{C}$, we have

$$[W, \mathrm{map}(D, E)] \cong \pi_0\big(\mathrm{map}(D, E)^W\big) \cong \pi_0\big(\mathrm{map}\big(D, E^W\big)\big) \cong \big[D, E^W\big]. \tag{1.2}$$

Next we show that $E^W \in \mathcal{S}^{\perp}$. If $f\colon A \to B$ is in $\mathcal{S}$, then, since $E \in \mathcal{S}^{\perp}$, the morphism f induces a weak equivalence $\mathrm{map}(B, E) \simeq \mathrm{map}(A, E)$. Therefore, for every simplicial set V we have

$$\begin{aligned}\big[V, \mathrm{map}\big(B, E^W\big)\big] &\cong \pi_0\Big(\mathrm{map}\big(B, E^W\big)^V\Big) \cong \pi_0\big(\mathrm{map}\big(B, E^{V\times W}\big)\big)\\ &\cong \big[B, E^{V\times W}\big] \cong [V \times W, \mathrm{map}(B, E)]\\ &\cong [V \times W, \mathrm{map}(A, E)] \cong \big[V, \mathrm{map}\big(A, E^W\big)\big].\end{aligned}$$

Hence f induces a weak equivalence $\mathrm{map}\big(B, E^W\big) \simeq \mathrm{map}\big(A, E^W\big)$, as needed. Now, going back to (1.2), we conclude that r induces

$$[W, \mathrm{map}(D, E)] \cong \big[D, E^W\big] \cong \big[X, E^W\big] \cong [W, \mathrm{map}(X, E)],$$

which completes the argument. □

We emphasize, however, that it is not true that every reflection (onto an arbitrary class $\mathcal{L}$) on a homotopy category $\mathrm{h}\mathcal{C}$ is an $\mathcal{S}$-localization for some class $\mathcal{S}$ of morphisms in $\mathcal{C}$. A counterexample is shown in Example 4.3 below.

If $\mathcal{M}$ is a simplicial model category [20, 21, 28], then the full subcategory $\mathcal{C}$ spanned by the fibrant-cofibrant objects of $\mathcal{M}$ is enriched in Kan complexes. Therefore, the concepts of orthogonality and localization apply to $\mathcal{C}$, and can be extended over $\mathcal{M}$ by closing the orthogonality relation under weak equivalences. The homotopy category $\mathrm{h}\mathcal{C}$ coincides with the homotopy category $\mathrm{Ho}(\mathcal{M})$ in the usual sense, and Proposition 1.1 also holds in $\mathcal{M}$. We state this fact for later reference.

Proposition 1.2. *Let $\mathcal{S}$ be a class of morphisms in a simplicial model category $\mathcal{M}$, and let X be an object of $\mathcal{M}$. A morphism $r\colon X \to D$ is an $\mathcal{S}$-localization of X if and only if r is a reflection onto $\mathcal{S}^{\perp}$ on the homotopy category* $\mathrm{Ho}(\mathcal{M})$.

Proof. The argument is the same as for Proposition 1.1, by replacing X and D by weakly equivalent fibrant-cofibrant objects. When considering $E \in \mathcal{S}^{\perp}$, pick it fibrant and use the fact that cotensoring with a simplicial set is a right Quillen endofunctor of $\mathcal{M}$ and therefore it preserves fibrant objects. □

By a *spectrum* we mean an object of any simplicial model category Sp whose underlying category is locally presentable and whose homotopy category is equivalent to the classical stable homotopy category —for example, the Bousfield–Friedlander category [9] or the category of symmetric spectra over simplicial sets [22].

Given spectra X and Y, the homotopy groups of the hom simplicial set $\mathrm{map}(X,Y)$ coincide in nonnegative degrees with those the internal hom in Sp. Therefore, if we consider the derived function spectrum $F(X,Y)$ —that is, the internal hom in Sp composed with a cofibrant replacement of the domain and a fibrant replacement of the codomain— and denote by $F^c(X,Y)$ its connective cover, then orthogonality of spectra in the sense of this section can be alternatively formulated as follows: a map $f\colon A \to B$ and a spectrum X are orthogonal if and only if the map

$$F^c(B,X) \longrightarrow F^c(A,X)$$

induced by f is a weak equivalence of spectra, i.e., it induces isomorphisms of all homotopy groups; see [8, 13] for additional information.

If a simplicial model category $\mathcal{M}$ is stable, then every class of the form $\mathcal{S}^{\perp}$ is *semicolocalizing*, that is, $\mathcal{S}^{\perp}$ is closed under fibres, products, and extensions, hence also under retracts and desuspension [14, § 1.2]. Moreover, if $\mathcal{S}$ is closed under desuspension (or, equivalently, $\mathcal{S}^{\perp}$ is closed under suspension), then $\mathcal{S}^{\perp}$ is *colocalizing*, i.e., triangulated and closed under products. For an arbitrary simplicial model category $\mathcal{M}$, each class of the form $\mathcal{S}^{\perp}$ is closed under homotopy limits and each class of the form $\mathcal{S}^{\perp\perp}$ is closed under homotopy colimits; see [20] for a detailed proof. Moreover, both $\mathcal{S}^{\perp}$ and $\mathcal{S}^{\perp\perp}$ are closed under homotopy retracts.

2 From weak reflections to reflections

A subcategory $\mathcal{D}$ of a category $\mathcal{M}$ is *weakly reflective* if for every object $X \in \mathcal{M}$ there is a morphism $f\colon X \to X^*$ with $X^* \in \mathcal{D}$ such that, for every morphism $g\colon X \to D$ with $D \in \mathcal{D}$ there is a morphism $h\colon X^* \to D$ (not necessarily unique) such that $h \circ f = g$. If a weakly reflective subcategory is closed under retracts, then it is closed under products [2, Remark 4.5(3)].

The following result was proved in [1, Theorem I.9] for classes closed under products and retracts assuming Vopěnka's principle, and it was pointed out in [1, Remark I.10] that the semi-weak Vopěnka principle was sufficient for the validity of the proof. In the next proposition, We repeat the argument from [1] with minor changes and without assuming closedness under retracts, similarly as in [14, Theorem 2.1].

Proposition 2.1. *Suppose that the semi-weak Vopěnka principle holds. If $\mathcal{M}$ is any locally presentable category, then every full subcategory of $\mathcal{M}$ closed under products is weakly reflective.*

Proof. Suppose given a full subcategory $\mathcal{D}$ of $\mathcal{M}$ closed under products. Hence, in particular, $\mathcal{D}$ contains the terminal object of $\mathcal{M}$, and we implicitly assume that $\mathcal{D}$ is closed under isomorphisms.

Let $V = \cup_{i\in \mathrm{Ord}} V_i$ denote the cumulative hierarchy of sets in ZFC; see [23] for a precise definition. For each ordinal i, let $\mathcal{D}_i = \mathcal{D} \cap V_i$ be the set of all objects in $\mathcal{D}$ whose rank is smaller than i, and let $\overline{\mathcal{D}}_i$ be the closure of $\mathcal{D}_i$ under products and isomorphisms. Hence $\mathcal{D}_i \subseteq \mathcal{D}_j$ if $i \le j$, and

$$\mathcal{D} = \bigcup_{i\in \mathrm{Ord}} \mathcal{D}_i = \bigcup_{i\in \mathrm{Ord}} \overline{\mathcal{D}}_i.$$

For each object X of $\mathcal{M}$ and every ordinal i, let $\mathcal{F}_i^X = (X \downarrow \mathcal{D}_i)$ denote the comma category of X over $\mathcal{D}_i$, whose objects are morphisms $X \to D$ with $D \in \mathcal{D}_i$ and whose morphisms are commutative triangles. Let $f_i\colon X \to X_i$ be the product of all the objects of $\mathcal{F}_i^X$ (in case that $\mathcal{F}_i^X = \emptyset$, we let X_i be the terminal object of $\mathcal{M}$).

Every object $Y \in \overline{\mathcal{D}}_i$ is isomorphic to a product $\prod_{\lambda\in\Lambda} D_\lambda$ with $D_\lambda \in \mathcal{D}_i$ (not necessarily distinct), and each morphism $g\colon X \to Y$ is determined by a collection of morphisms $\langle \delta_\lambda \colon X \to D_\lambda \mid \lambda \in \Lambda \rangle$. Here each δ_λ is in $\mathcal{F}_i^X$ and therefore it can be factored as

$$X \xrightarrow{f_i} X_i \xrightarrow{p_\lambda} D_\lambda,$$

where the second arrow is a projection. The morphisms p_λ (possibly repeated) jointly yield a morphism $g_i \colon X_i \to Y$ such that $g_i \circ f_i = g$. Hence f_i is a weak reflection of X onto $\overline{\mathcal{D}}_i$.

Moreover, since $\mathcal{F}_i^X$ is a subcategory of $\mathcal{F}_j^X$ if $i \leq j$, there is a projection $p_{ji} \colon X_j \to X_i$ such that $p_{ji} \circ f_j = f_i$ whenever $i \leq j$. Therefore, the class $\langle f_i \colon X \to X_i \mid i \in \mathrm{Ord} \rangle$ can be viewed as a sequence of objects in the comma category $(X \downarrow \mathcal{M})$ equipped with morphisms $p_{ji} \colon f_j \to f_i$ if $i \leq j$.

In order to obtain a weak reflection of X onto $\mathcal{D}$, it suffices to find an ordinal i such that, for all $j \geq i$, the map f_j can be factorized as $f_j = q_{ij} \circ f_i$ for some $q_{ij} \colon X_i \to X_j$. Suppose the contrary. Then there are ordinals $i_0 < i_1 < \cdots < i_s < \cdots$, where s ranges over all the ordinals, such that, if $s < t$, then

$$(X \downarrow \mathcal{M})(f_{i_s}, f_{i_t}) = \emptyset. \tag{2.1}$$

Since $\mathcal{M}$ is locally presentable, $(X \downarrow \mathcal{M})$ is also locally presentable by [2, Proposition 1.57], and therefore (2.1) is incompatible with the semi-weak Vopěnka principle, as $(X \downarrow \mathcal{M})(f_{i_s}, f_{i_t}) \neq \emptyset$ if $s \geq t$. □

Remark 2.2. It will be useful towards the content of Section 5 to observe here that the weak reflection onto $\mathcal{D}$ constructed in the proof of Proposition 2.1 is in fact functorial, and moreover it is coaugmented, i.e., it comes equipped with a natural transformation from the identity functor. To prove this, for each ordinal i, let us now denote by $f_i^X \colon X \to X_i$ the product of all the objects of the comma category $(X \downarrow \mathcal{D}_i)$ where $\mathcal{D}_i = \mathcal{D} \cap V_i$. Suppose given any map $h \colon X \to Y$. For every ordinal i there is a map $h_i \colon X_i \to Y_i$ such that $h_i \circ f_i^X = f_i^Y \circ h$, which is obtained by projecting $X \to X_i$ onto the product of all the objects $X \to D$ of $(X \downarrow \mathcal{D}_i)$ that factor through h.

Choose an ordinal x such that for all $j \geq x$ there is a map $q_{xj}^X \colon X_x \to X_j$ such that $f_j^X = q_{xj}^X \circ f_x^X$, and choose an ordinal y with the same property for Y. Thus, $f_x^X \colon X \to X_x$ is a weak reflection of X onto $\mathcal{D}$ and $f_y^Y \colon Y \to Y_y$ is a weak reflection of Y.

If $y \leq x$ then let $\tilde{h} \colon X_x \to Y_y$ be defined as $\tilde{h} = p_{xy}^Y \circ h_x$, where p_{xy}^Y is the projection of Y_x onto Y_y. Then

$$\tilde{h} \circ f_x^X = p_{xy}^Y \circ h_x \circ f_x^X = p_{xy}^Y \circ f_x^Y \circ h = f_y^Y \circ h.$$

On the other hand, if $y > x$ then let $\tilde{h} = h_y \circ q_{xy}^X$. Hence,

$$\tilde{h} \circ f_x^X = h_y \circ q_{xy}^X \circ f_x^X = h_y \circ f_y^X = f_y^Y \circ h.$$

In conclusion, for every $h\colon X \to Y$ there is a map $\tilde{h}\colon X_x \to Y_y$ such that $\tilde{h} \circ f_x^X = f_y^Y \circ h$.

Moreover, given two composable maps $h\colon X \to Y$ and $g\colon Y \to Z$, we find that for every ordinal i the composite of the maps $h_i\colon X_i \to Y_i$ and $g_i\colon Y_i \to Z_i$ is equal to $(g \circ h)_i\colon X_i \to Z_i$, and it follows that the composite of the maps $\tilde{h}\colon X_x \to Y_y$ and $\tilde{g}\colon Y_y \to Z_z$ is equal to the map $X_x \to Z_z$ induced by $g \circ h$. Therefore, the assignment $X \mapsto X_x$ is indeed functorial, and coaugmented by f_x^X.

From a weak reflection onto a full subcategory $\mathcal{C}$ it is possible to construct a reflection onto $\mathcal{C}$ under suitable assumptions —our reference is [14, Theorem 2.2]. First of all, we need that idempotents split in the category under study, that is, for every morphism $e\colon X \to X$ such that $e \circ e = e$ there are morphisms $f\colon X \to Y$ and $g\colon Y \to X$ for some Y such that $e = g \circ f$ and $f \circ g = \mathrm{id}$. Second, $\mathcal{C}$ should be closed under retracts. Third, most fundamentally, the class $\mathcal{C}$ should have the following closure property: every pair of parallel arrows between two objects in $\mathcal{C}$ admits a weak equalizer in $\mathcal{C}$. This happens in homotopy categories of model categories for classes of objects $\mathcal{C}$ closed under homotopy limits. Hence we formulate the following result in terms of model categories. We omit the assumption that $\mathcal{M}$ be simplicial since every model category admits a simplicial enrichment by means of homotopy function complexes [20, 21], which suffices to define orthogonality.

Proposition 2.3. *Let $\mathcal{M}$ be a model category such that idempotents split in the homotopy category* $\mathrm{Ho}(\mathcal{M})$*. For a class of maps $\mathcal{S}$, if the class of objects $\mathcal{S}^\perp$ is weakly reflective on* $\mathrm{Ho}(\mathcal{M})$*, then $\mathcal{S}^\perp$ is reflective on* $\mathrm{Ho}(\mathcal{M})$*.*

Proof. Every model category is complete and cocomplete, so $\mathcal{M}$ has products and these are also products in $\mathrm{Ho}(\mathcal{M})$. The class $\mathcal{S}^\perp$ is closed under retracts in $\mathcal{M}$ and also in $\mathrm{Ho}(\mathcal{M})$, since homotopy function complexes are homotopy invariant and every retract of a weak equivalence of simplicial sets is a weak equivalence. Moreover, homotopy equalizers in $\mathcal{M}$ are weak equalizers in $\mathrm{Ho}(\mathcal{M})$, and a homotopy equalizer of two maps between objects in $\mathcal{S}^\perp$ is an object of $\mathcal{S}^\perp$, since $\mathcal{S}^\perp$ is closed under homotopy limits in $\mathcal{M}$.

For completeness, we indicate how the reflection is constructed. Since the class $\mathcal{D} = \mathcal{S}^\perp$ is closed under products, Proposition 2.1 implies that $\mathcal{D}$ is weakly reflective in $\mathcal{M}$. Let X be any object of $\mathcal{M}$ and suppose that $f_x^X\colon X \to X_x$ is a weak reflection of X onto $\mathcal{D}$. Let $X_x \to D$ be a (functorial) trivial cofibration into a fibrant object, which is also in $\mathcal{D}$ since $\mathcal{D}$ is closed under weak equivalences. Then the composite $f\colon X \to D$ is a weak reflection in the homotopy category $\mathrm{Ho}(\mathcal{M})$ since for every map $g\colon X \to W$ with W fibrant in $\mathcal{D}$ there is map $h\colon D \to W$ with $h \circ f = g$.

Consider the set $\mathcal{K}$ of all pairs (α_k, β_k) of maps from D to itself such that $\alpha_k \circ f \simeq \beta_k \circ f$. Let $u\colon E \to D$ be a homotopy equalizer in $\mathcal{M}$ of the two maps $D \to \prod_k D$ given by $\prod_k \alpha_k$ and $\prod_k \beta_k$ respectively, with Y fibrant. Then $E \in \mathcal{D}$ since $\mathcal{D}$ is closed under homotopy limits.

Since $u\colon E \to D$ is a homotopy equalizer, there is a map $s\colon X \to E$ such that $u \circ s = f$. As $E \in \mathcal{D}$ and f is a weak reflection of X onto $\mathcal{D}$, there is a map $t\colon D \to E$ such that $t \circ f \simeq s$. Then the pair $(u \circ t, \mathrm{id})$ is in $\mathcal{K}$. Consequently, $u \circ t \circ u \simeq u$, and this implies that $t \circ u$ is idempotent in $\mathrm{Ho}(\mathcal{M})$. Hence $t \circ u$ splits, so there are maps $t'\colon E \to Z$ and $u'\colon Z \to E$ with Z fibrant such that $u' \circ t' \simeq t \circ u$ and $t' \circ u' \simeq \mathrm{id}$. It then follows that $Z \in \mathcal{D}$ since Z is a homotopy retract of E. Then the map $r = t' \circ s$ from X to Z is a reflection of X onto $\mathcal{D}$ in $\mathrm{Ho}(\mathcal{M})$, as shown in detail within the proof of [14, Theorem 2.2]. □

Corollary 2.4. *Suppose that the semi-weak Vopěnka principle holds. Let $\mathcal{M}$ be a simplicial model category whose underlying category is locally presentable and such that idempotents split in the homotopy category* $\mathrm{Ho}(\mathcal{M})$. *Then $\mathcal{S}$-localization exists in $\mathcal{M}$ for every class of maps $\mathcal{S}$.*

Proof. If $r\colon X \to Z$ is a reflection of X onto $\mathcal{S}^{\perp}$ on $\mathrm{Ho}(\mathcal{M})$ as given by Proposition 2.3, then Proposition 1.2 implies that r is in $\mathcal{S}^{\perp\perp}$. Hence for every X in $\mathcal{C}$ there is a map $r\colon X \to Z$ in $\mathcal{S}^{\perp\perp}$ with $Z \in \mathcal{S}^{\perp}$. □

For the validity of Corollary 2.4, it is not necessary to assume that $\mathcal{M}$ be cofibrantly generated nor left proper —these are standard assumptions on a model category for the purpose of constructing $\mathcal{S}$-localizations by means of the small object argument when $\mathcal{S}$ is a set. Instead of the small object argument, our proof is based on the assumption that idempotents split. Since idempotents split in the category of sets, they do in $\mathrm{Ho}(\mathcal{M})$ if Brown representability holds in $\mathrm{Ho}(\mathcal{M})$.

The assumptions that $\mathcal{M}$ be cofibrantly generated and left proper would however serve to remove the condition that $\mathcal{M}$ be simplicial in Corollary 2.4, since every left proper combinatorial model category is Quillen equivalent to a simplicial model category; see [16, 29]. Likewise, every proper cofibrantly generated stable model category is Quillen equivalent to a simplicial model category, according to [29, Proposition 1.3].

Corollary 2.5. *If the semi-weak Vopěnka principle holds, then $\mathcal{S}$-localization exists for every class $\mathcal{S}$ of maps between spectra.*

Proof. Corollary 2.4 applies to the category of Bousfield–Friedlander spectra [9] or to the category of symmetric spectra [22], since they are locally presentable and idempotents split in their homotopy category. In fact, idempotents split in any triangulated category with countable products [25]. □

As for simplicial sets, it is neither true that idempotents split in the homotopy category Ho(sSet) nor in the pointed homotopy category Ho($sSet_*$), but they do for pointed *connected* simplicial sets [18]. The proof of Proposition 2.3 can be amended to deal with this circumstance. We provide details in order to highlight the necessary changes.

As shown in [31], if some map in $\mathcal{S}$ is not bijective on π_0 then $\mathcal{S}$-local spaces are contractible. Hence we may assume that all maps in $\mathcal{S}$ induce bijections of connected components. If $f\colon A \to B$ is any such map and we denote by $\{f_\alpha\colon A_\alpha \to B_\alpha\}$ the collection of its restrictions to connected components of A and B, then a space X is f-local if and only if X is f_α-local for all α. Therefore we may also assume that the class $\mathcal{S}$ consists of maps between connected simplicial sets (by replacing each map in the given class by the collection of its restrictions to connected components).

We also use the fact that, if we choose basepoints so that each map in $\mathcal{S}$ is basepoint-preserving, then the class of connected $\mathcal{S}$-local spaces in the pointed category Ho($sSet_*$) is the same as the corresponding class in the unpointed category Ho(sSet) by forgetting the basepoints. This follows, as observed in [17, A.1], from the fact that for all pointed connected simplicial sets A and Y there is a fibration

$$\mathrm{map}_*(A,Y) \longrightarrow \mathrm{map}(A,Y) \longrightarrow Y$$

where $\mathrm{map}_*(A,Y)$ is the hom simplicial set in $sSet_*$ and the right-hand arrow is evaluation at the basepoint of A.

Consequently, for the proof of the next result we choose to work in the pointed category $sSet_*$, and there is no loss of generality if we restrict ourselves to connected spaces, since an $\mathcal{S}$-localization of an arbitrary space is the disjoint union of the $\mathcal{S}$-localizations of its connected components after choosing arbitrary basepoints in them.

Theorem 2.6. *If the semi-weak Vopěnka principle holds, then $\mathcal{S}$-localization exists for every class $\mathcal{S}$ of maps between pointed or unpointed simplicial sets.*

Proof. By our previous remarks, it suffices to construct an $\mathcal{S}$-localization in the pointed category $sSet_*$ for every pointed *connected* simplicial set X, by assuming that maps in $\mathcal{S}$ are basepoint-preserving maps between connected simplicial sets. The proof of this result proceeds in the same way as the proof of Proposition 2.3. Let $\mathcal{D} = \mathcal{S}^{\perp}$. By Proposition 2.1, there is a weak reflection $f\colon X \to D$ of X onto $\mathcal{D}$, since $sSet_*$ is locally presentable. We may assume that D is fibrant and view f as a weak reflection in the homotopy category Ho($sSet_*$). Since X is connected, the image of f is contained in the basepoint component D_0 of D, and the codomain restriction $f_0\colon X \to D_0$ is still a weak reflection of X onto $\mathcal{D}$ in Ho($sSet_*$).

Consider the set $\mathcal{K}$ of all pairs (α_k, β_k) of maps from D_0 to itself such that $\alpha_k \circ f_0 \simeq \beta_k \circ f_0$. Let $u\colon E \to D_0$ be a homotopy equalizer in sSet_* of the two maps $D_0 \to \prod_k D_0$ given by $\prod_k \alpha_k$ and $\prod_k \beta_k$ respectively, with E fibrant. Then $E \in \mathcal{D}$ since $\mathcal{D}$ is closed under homotopy limits. Pick the basepoint component E_0 of E, which is also in $\mathcal{D}$ because it is a retract of E.

Since $u\colon E \to D_0$ is a homotopy equalizer, there is a map $s\colon X \to E$ such that $u \circ s = f_0$, and s factors through a map $s_0\colon X \to E_0$ since X is connected; that is, $s = i \circ s_0$, where $i\colon E_0 \to E$ is the inclusion. As $E_0 \in \mathcal{D}$ and f_0 is a weak reflection of X onto $\mathcal{D}$, there is a map $t\colon D_0 \to E_0$ such that $t \circ f_0 \simeq s_0$. Then the pair $(u \circ i \circ t, \mathrm{id})$ is in $\mathcal{K}$ and, consequently, $u \circ i \circ t \circ u \simeq u$. This implies that $t \circ u \circ i$ is idempotent in $\mathrm{Ho}(\mathrm{sSet}_*)$.

Now we use the fact that, according to [18], idempotents split for pointed connected simplicial sets. Thus, since E_0 is connected, $t \circ u \circ i$ splits, so there are maps $t'\colon E_0 \to Z$ and $u'\colon Z \to E_0$ with Z fibrant such that $u' \circ t' \simeq t \circ u \circ i$ and $t' \circ u' \simeq \mathrm{id}$. It then follows that $Z \in \mathcal{D}$ (and Z is connected) since Z is a retract of E_0. The argument showing that the map $r = t' \circ s_0$ from X to Z is a reflection of X onto $\mathcal{D}$ in $\mathrm{Ho}(\mathrm{sSet}_*)$ is the same as in the proof of [14, Theorem 2.2].

□

We emphasize the following special case.

Corollary 2.7. *Cohomological localizations of simplicial sets or spectra exist under the semi-weak Vopěnka principle.*

Proof. For a generalized cohomology theory E^* defined on simplicial sets or on spectra, let $\mathcal{S}$ be the class of E^*-equivalences, that is, maps $X \to Y$ such that the induced homomorphisms $E^n(Y) \to E^n(X)$ are isomorphisms for all $n \in \mathbb{Z}$. Then the reflectivity of $\mathcal{S}^\perp$ follows from Corollary 2.5 in the case of spectra and from Theorem 2.6 in the case of simplicial sets. □

This result is a substantial improvement with respect to the state of the art about the existence of cohomological localizations, which is an open problem in ZFC and it was shown to be implied by the existence of a proper class of supercompact cardinals in [3].

3 Reverse implications

In this section, we address the converse of Corollary 2.5 and Theorem 2.6 using results by Przeździecki, who proved that, if the weak Vopěnka principle is false, then there exist non-reflective orthogonality classes of groups [26] and non-reflective orthogonality classes of abelian groups [27].

In the argument that follows, we use, as in [26], the fact that that sending every group G to an Eilenberg–Mac Lane space $K(G,1)$ is a full embedding of the category of groups into the homotopy category of pointed simplicial sets, since for all groups G and H there is a natural bijective correspondence between the set of pointed homotopy classes of maps $[K(G,1),K(H,1)]$ and the set of group homomorphisms $\mathrm{Hom}(G,H)$.

Moreover, a homomorphism $\varphi\colon P\to Q$ is orthogonal to a group G if and only if the map $K(P,1)\to K(Q,1)$ induced by φ is orthogonal to a $K(G,1)$, since the space $\mathrm{map}_*(K(P,1),K(G,1))$ is discrete and its set of connected components is in bijective correspondence with $\mathrm{Hom}(P,G)$, and similarly with Q.

Theorem 3.1. *If $\mathcal{S}$-localization exists in the homotopy category of pointed simplicial sets for every class of maps $\mathcal{S}$, then the weak Vopěnka principle holds.*

Proof. Suppose that the weak Vopěnka principle is false. Then, according to [26, Proposition 8.7], there exists a non-reflective orthogonality class $\mathcal{G}$ of groups. The fact that $\mathcal{G}$ is an orthogonality class implies that $\mathcal{G}=\mathcal{G}^{\perp\perp}$.

Let $\mathcal{K}$ be the class of Eilenberg–Mac Lane spaces $K(G,1)$ with $G\in\mathcal{G}$. By assumption, the class $\mathcal{K}^{\perp\perp}$ is reflective in $\mathrm{Ho}(\mathrm{sSet}_*)$. Let L be a reflector. We next show that every *connected* simplicial set $X\in\mathcal{K}^{\perp\perp}$ is in $\mathcal{K}$. First of all, the map $S^2\to *$ is in $\mathcal{K}^{\perp}$, since $\mathrm{map}_*(S^2,K(G,1))=\Omega^2K(G,1)$ is contractible for every group G. If X is a connected simplicial set in $\mathcal{K}^{\perp\perp}$, then X is orthogonal to $S^2\to *$ and hence $\mathrm{map}_*(S^2,X)$ is contractible. This implies that $\pi_n(X)=0$ for $n\geq 2$, so X is indeed an Eilenberg–Mac Lane space. There remains to show that $\pi_1(X)\in\mathcal{G}$, which is equivalent to the statement that $\pi_1(X)\in\mathcal{G}^{\perp\perp}$. Let $\varphi\colon P\to Q$ be any homomorphism in $\mathcal{G}^{\perp}$. Then the induced map $K(P,1)\to K(Q,1)$ is in $\mathcal{K}^{\perp}$. Since $X\in\mathcal{K}^{\perp\perp}$, the map $K(P,1)\to K(Q,1)$ is orthogonal to X, and this implies that $\pi_1(X)$ is orthogonal to φ, as needed.

Now let G be any group. Consider the localization $\ell\colon K(G,1)\to LK(G,1)$. Here $LK(G,1)$ is connected since every localization of a connected space is connected [31]. Let us consider the induced group homomorphism $\ell_*\colon G\to H$ where $H=\pi_1(LK(G,1))$. Thus, $LK(G,1)=K(H,1)$ with $H\in\mathcal{G}$ since $LK(G,1)\in\mathcal{K}^{\perp\perp}$ and it is connected.

If J is any group in $\mathcal{G}$, then the corresponding $K(J,1)$ is in $\mathcal{K}$ and hence it is orthogonal to ℓ. This means precisely that J is orthogonal to ℓ_* and therefore ℓ_* is a reflection of G onto $\mathcal{G}$. Hence the class $\mathcal{G}$ is reflective, which is a contradiction. □

Corollary 3.2. *The statement that $\mathcal{S}$-localization exists in the homotopy category of pointed simplicial sets for every class of maps $\mathcal{S}$ is equivalent to the weak Vopěnka principle.*

Proof. We have proved that the semi-weak Vopěnka principle implies the existence of arbitrary $\mathcal{S}$-localizations of pointed simplicial sets in Section 2, and Theorem 3.1 says that the latter implies the weak Vopěnka principle. Hence our result follows from the fact that the weak Vopěnka principle is equivalent to the semi-weak Vopěnka principle, as proved in [33]. □

The stable analogue is similar. In the next result, we use the fact that that sending every abelian group A to an Eilenberg–Mac Lane spectrum HA with a single nonzero homotopy group isomorphic to A in dimension 0 is a full embedding of the category of abelian groups into the homotopy category of spectra. Indeed, the derived function spectrum $F(HA, HB)$ has two nonzero homotopy groups in general, namely

$$\begin{aligned} \pi_0(F(HA, HB)) &\cong \mathrm{Hom}(A, B) \\ \pi_{-1}(F(HA, HB)) &\cong [HA, \Sigma HB] \cong \mathrm{Ext}(A, B). \end{aligned}$$

Therefore the connective cover $F^c(HA, HB)$ is an Eilenberg–Mac Lane spectrum whose π_0 is isomorphic to $\mathrm{Hom}(A, B)$, and this implies that a homomorphism of abelian groups $\varphi\colon A \to B$ is orthogonal to an abelian group C if and only if the induced map $HA \to HB$ is orthogonal to HC.

Theorem 3.3. *If $\mathcal{S}$-localization exists in the homotopy category of spectra for every class of maps $\mathcal{S}$, then the weak Vopěnka principle holds.*

Proof. Under the negation of the weak Vopěnka principle, there is a non-reflective orthogonality class $\mathcal{A}$ of abelian groups by [27, Proposition 6.8]. Thus, $\mathcal{A} = \mathcal{A}^{\perp\perp}$.

Let $\mathcal{H}$ be the class of Eilenberg–Mac Lane spectra HA with $A \in \mathcal{A}$. By assumption, the class $\mathcal{H}^{\perp\perp}$ is associated with a localization L on the homotopy category of spectra. Similarly as in the proof of Proposition 3.1, we next show that every *connective* spectrum $X \in \mathcal{H}^{\perp\perp}$ is in $\mathcal{H}$. If S denotes the sphere spectrum, then the map $\Sigma S \to 0$ is in $\mathcal{H}^{\perp}$, since $F^c(\Sigma S, HA) = 0$ for every A. Since X is in $\mathcal{H}^{\perp\perp}$, we have that X is orthogonal to $\Sigma S \to 0$ and hence $F^c(\Sigma S, X) = 0$. This implies that $\pi_n(X) = 0$ for $n \geq 1$, so $X \simeq HE$ since X is connective. There remains to show that $E \in \mathcal{A}$, that is, $E \in \mathcal{A}^{\perp\perp}$. For this, let $\varphi\colon P \to Q$ be any homomorphism in $\mathcal{A}^{\perp}$. Then the induced map $HP \to HQ$ is in $\mathcal{H}^{\perp}$. It follows that the map $HP \to HQ$ is orthogonal to X, and this implies that E is orthogonal to φ, as needed.

Let A be any abelian group. Consider the localization $\ell\colon HA \to LHA$ and the induced group homomorphism $\ell_*\colon A \to B$ where $B = \pi_1(LHA)$. Since ℓ is also a localization of HA with respect to the set $\{\ell\}$, it follows from [13, Theorem 5.6] that

$$LHA \simeq HB \times \Sigma HC$$

for some abelian group C. Therefore, LHA is connective. Since $LHA \in \mathcal{H}^{\perp\perp}$, we may infer from the above argument that $LHA \in \mathcal{H}$, and consequently $C = 0$ and $B \in \mathcal{A}$.

We finally show that $\ell_*\colon A \to B$ is a reflection of A onto $\mathcal{A}$ and hence the class $\mathcal{A}$ is reflective, which is a contradiction. If E is any group in $\mathcal{A}$, then HE is in $\mathcal{H}$ and hence it is orthogonal to ℓ. If $\varphi\colon P \to Q$ is any homomorphism in $\mathcal{A}^{\perp}$, then the induced map $HP \to HQ$ is in $\mathcal{H}^{\perp}$, and it follows that the map $HP \to HQ$ is orthogonal to HE, which implies that E is orthogonal to φ, as needed. □

Corollary 3.4. *The statement that $\mathcal{S}$-localization exists in the homotopy category of spectra for every class of maps $\mathcal{S}$ is equivalent to the weak Vopěnka principle.*

Proof. One implication is given by Corollary 2.5, and the converse follows from Theorem 3.3. □

It was shown in [14, Theorem 2.4] that, if Vopěnka's principle holds, then every full subcategory closed under products and fibres of the homotopy category of a stable locally presentable model category is reflective. Here we improve this result as follows.

Corollary 3.5. *The statement that every full subcategory closed under products and fibres of the homotopy category of a stable locally presentable model category is reflective is equivalent to the weak Vopěnka principle.*

Proof. Assuming the weak Vopěnka principle, Proposition 2.3 tells us that every full subcategory closed under products and fibres of the homotopy category of a stable locally presentable model category is reflective, since every model category is cocomplete and hence, assuming it stable, idempotents split in its homotopy category. Moreover, a fibre of the difference $f - g$ of two parallel maps f and g is a weak equalizer.

To prove the converse, suppose that the weak Vopěnka principle does not hold. Then, as in the proof of Theorem 3.3, there exists a non-reflective orthogonality class $\mathcal{A}$ of abelian groups. If $\mathcal{H}$ denotes the class of Eilenberg–Mac Lane spectra HA with $A \in \mathcal{A}$, then $\mathcal{H}^{\perp\perp}$ is closed under fibres (although

$\mathcal{H}$ need not be). Therefore a localization $\ell\colon HA \to LHA$ onto $\mathcal{H}^{\perp\perp}$ exists for every abelian group A and this implies that $\mathcal{A}$ is reflective, which is a contradiction. □

Recall that a full triangulated subcategory of a triangulated category is called *colocalizing* if it is closed under products. We do not know if the statement that every colocalizing subcategory of the homotopy category of a stable locally presentable model category is reflective implies the weak Vopěnka principle (or any other large-cardinal principle). The class $\mathcal{H}^{\perp\perp}$ used in the proof of Corollary 3.5 is not closed under suspensions (nor cofibres in general), hence not colocalizing.

4 Examples and counterexamples

Example 4.1. This example shows that the statement that every full subcategory closed under products in a locally presentable category is weakly reflective —which was proved under the semi-weak Vopěnka principle in Proposition 2.1— cannot be proved in ZFC. In [14, Corollary 2.7], a class of spectra closed under products and retracts but not weakly reflective was exhibited assuming the nonexistence of measurable cardinals. That class consists of Eilenberg–Mac Lane spectra HA where A belongs to the closure of the class of groups $\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}$ under products and retracts, where κ runs over all cardinals and $\mathbb{Z}^\kappa$ denotes a product of copies of $\mathbb{Z}$ indexed by κ while $\mathbb{Z}^{<\kappa}$ is the subgroup of those sequences whose support has cardinality smaller than κ.

Example 4.2. It is not true that every full subcategory closed under products an retracts in a locally presentable category is reflective, not even assuming large-cardinal principles. To illustrate this fact, we recall that the class $\mathcal{C}$ of 1-connected simplicial sets is closed under products and retracts but it is not reflective in $\mathrm{Ho}(\mathrm{sSet}_*)$. The following argument is due to Mislin [17, A.1.3]. Suppose that a map $\ell\colon \mathbb{R}P^2 \to X$ is a reflection onto $\mathcal{C}$, where $\mathbb{R}P^2$ denotes the real projective plane. Then ℓ induces an isomorphism

$$[X, K(\mathbb{Z}, 2)] \cong [\mathbb{R}P^2, K(\mathbb{Z}, 2)].$$

However, $[X, K(\mathbb{Z}, 2)] \cong H^2(\mathbb{R}P^2; \mathbb{Z}) \cong \mathbb{Z}/2$ while

$$[\mathbb{R}P^2, K(\mathbb{Z}, 2)] \cong H^2(X; \mathbb{Z}) \cong \mathrm{Hom}(H_2(X; \mathbb{Z}), \mathbb{Z})$$

is torsion-free for every 1-connected space X. This does not contradict Theorem 2.6 because $\mathcal{C}$ is not of the form $\mathcal{S}^\perp$ for any class of maps $\mathcal{S}$; this is due to the fact that $\mathcal{C}$ is not closed under homotopy limits, since the homotopy fibre of a map between 1-connected spaces need not be 1-connected.

Similarly, the class of spaces whose fundamental group is uniquely radicable and whose higher homotopy groups are $\mathbb{Q}$-vector spaces for every choice of a basepoint is closed under products and retracts but it is not reflective in $\mathrm{Ho}(\mathrm{sSet}_*)$, as shown in [11].

Example 4.3. There are reflections onto full subcategories closed under products and retracts in $\mathrm{Ho}(\mathcal{M})$ for a simplicial model category $\mathcal{M}$ that are not $\mathcal{S}$-localizations for any class of morphisms $\mathcal{S}$. Our main example involves the class $\mathcal{C}$ of connective spectra whose homotopy groups are $\mathbb{Q}$-vector spaces. A reflection onto $\mathcal{C}$ in $\mathrm{Ho}(\mathrm{Sp})$ can be given explicitly as follows. For an arbitrary spectrum X, let $X \wedge H\mathbb{Q}$ be its rationalization. Since $X \wedge H\mathbb{Q}$ splits as a wedge $\bigvee_{k\in\mathbb{Z}} \Sigma^k H(\pi_k(X) \otimes \mathbb{Q})$, we can retract it into

$$LX = \bigvee_{k\geq 0} \Sigma^k H(\pi_k(X) \otimes \mathbb{Q})$$

(or into any other segment). The composite $\ell\colon X \to LX$ is a reflection onto $\mathcal{C}$, since every map $X \to Y$ where $Y \in \mathcal{C}$ factors uniquely through $X \wedge H\mathbb{Q}$ up to homotopy, and $[\Sigma^k HA, Y] = 0$ if $k < 0$, for all A. However, the class $\mathcal{C}$ is not closed under fibres and therefore L is not an $\mathcal{S}$-localization for any class of maps $\mathcal{S}$.

In this example, L can be lifted to an endofunctor on Sp, namely the composite of $X \mapsto X \wedge H\mathbb{Q}$ with passage to the connective cover. However, while the first functor is coaugmented, the second functor is augmented, that is, there is a zig-zag of natural transformations

$$X \longrightarrow X \wedge H\mathbb{Q} \longleftarrow (X \wedge H\mathbb{Q})^c.$$

The second arrow can be reversed in $\mathrm{Ho}(\mathrm{Sp})$, but not in Sp. Indeed, it follows from [12, Theorem 2.2] that there does not exist any natural transformation $\mathrm{Id} \to L$ in Sp lifting the unit ℓ of the reflection L on $\mathrm{Ho}(\mathrm{Sp})$.

Example 4.4. The existence of cohomological localizations is *not* equivalent to the weak Vopěnka principle. The reason is that it follows from [3, Theorem 9.3] that to infer the existence of cohomological localizations it is sufficient to assume that the weak Vopěnka principle holds for classes definable with Σ_2 formulas (with parameters) in the Lévy hierarchy [24]. The weak Vopěnka principle for Σ_2 classes with parameters is equivalent to the existence of a proper class of strong cardinals, as shown in [4].

As of today, it is not known if the existence of cohomological localizations is equivalent to the validity of any large-cardinal principle.

5 Left Bousfield localizations

left Bousfield localization of a model category $\mathcal{M}$ with respect to a class of maps $\mathcal{S}$ is a model structure $\mathcal{M}_{\mathcal{S}}$ on the same underlying category as $\mathcal{M}$ together with a left Quillen functor $\mathcal{M} \to \mathcal{M}_{\mathcal{S}}$ which is initial among left Quillen functors that send maps in $\mathcal{S}$ to weak equivalences. Such a model category is known to exist if $\mathcal{M}$ is combinatorial and left proper and $\mathcal{S}$ is a set. The same conclusion holds for a class $\mathcal{S}$ using Vopěnka's principle, by [30, Theorem 2.3] or by [12, Lemma 1.4]. If a left Bousfield localization $\mathcal{M}_{\mathcal{S}}$ exists, then a fibrant replacement on $\mathcal{M}_{\mathcal{S}}$ yields an $\mathcal{S}$-localization on $\mathcal{M}$ in the sense of the previous sections.

It is natural to ask under which conditions an $\mathcal{S}$-localization L on $\mathrm{Ho}(\mathcal{M})$ can be enhanced into a left Bousfield localization of $\mathcal{M}$. Example 4.4 in the previous section is not an obstruction for this, since the reflection L constructed in that example is not an $\mathcal{S}$-localization for any class of maps $\mathcal{S}$. Yet, Example 4.4 illustrates the fact that it is not possible in general to lift a reflection on $\mathrm{Ho}(\mathcal{M})$ to a coaugmented functor on $\mathcal{M}$.

A *semi-model category* is defined with the same axioms as a model category, except that the lifting axiom and the factorization axiom hold only for morphisms with fibrant codomain (in *right* semi-model categories) or instead with cofibrant domain (in *left* semi-model categories). It was shown in [5] that if the assumption that $\mathcal{M}$ be left proper is omitted, then a Bousfield localization $\mathcal{M}_{\mathcal{S}}$ for a set of maps $\mathcal{S}$ still exists as a left semi-model category. Furthermore, as pointed out in see [5, § 5], an example where left properness fails and left Bousfield localization does not exist as a model category is given in [32, Example 3.48].

The next theorem is based on results from [7] and [10] and states the existence of a right semi-model structure on $\mathcal{M}_{\mathcal{S}}$ under suitable assumptions.

Theorem 5.1. *Suppose given a class of maps $\mathcal{S}$ in a combinatorial model category $\mathcal{M}$ such that idempotents split in* $\mathrm{Ho}(\mathcal{M})$. *Assuming the weak Vopěnka principle, a left Bousfield localization $\mathcal{M}_{\mathcal{S}}$ exists as a right semi-model category with non necessarily functorial factorizations.*

Proof. Since combinatorial model categories are locally presentable, we use the argument given in the proof of Proposition 2.1 to obtain a weak reflection $f_x^X \colon X \to X_x$ onto $\mathcal{S}^{\perp}$ for every object X. Furthermore, for every map $h \colon X \to Y$ in $\mathcal{M}$ there is a map $\tilde{h} \colon X_x \to Y_y$ such that $\tilde{h} \circ f_x^X = f_y^Y \circ h$, as shown in Remark 2.2.

Using the assumption that idempotents split in $\mathrm{Ho}(\mathcal{M})$, we obtain from Proposition 2.3 a reflection $\ell_X \colon X \to LX$ for every X. According to the proof of Proposition 2.3, the map ℓ_X is the composite of two maps $s_X \colon X \to E_X$

and $t'_X \colon E_X \to LX$. The map s_X can be made functorial since homotopy limits are functorial in $\mathcal{M}$, and we also have a strict equality $u_X \circ s_X = f_x^X$. The map t_X can be chosen so that $t_X \circ f_x^X = s_X$, since f_x^X is a weak reflection on $\mathcal{M}$. Moreover, if $LX \to E_X$ is a cofibration then t'_X and u'_X can be chosen so that $u'_X \circ t'_X = t_X \circ u_X$ strictly. Then we may define $Lh \colon LX \to LY$ as $Lh = t'_Y \circ E_f \circ u'_X$, and we obtain that

$$Lh \circ t'_X \circ s_X = t'_Y \circ E_f \circ t_X \circ u_X \circ s_X = t'_X \circ E_f \circ s_X = t'_Y \circ s_Y \circ h.$$

Hence for every map $h \colon X \to Y$ in $\mathcal{M}$ there is a map $Lh \colon LX \to LY$ such that $Lh \circ \ell_X = \ell_Y \circ h$ strictly, not just up to homotopy.

As a consequence, even though L need not be an endofunctor of $\mathcal{M}$, condition A.2 from [7] is fulfilled. Conditions A.3 and A.4 are also fulfilled, since the class of L-equivalences is equal to $\mathcal{S}^{\perp\perp}$, and condition A.5 is checked with a similar argument as in [7, Proposition 6.6].

As pointed out in [10, Proposition 3.13], the fact that (L, ℓ) satisfies conditions A.2–A.5 from [7] ensures the existence of a right semi-model structure $\mathcal{M}_{\mathcal{S}}$ as in [10, Theorem 3.5]. For this to hold, it is not necessary to assume that $\mathcal{M}$ be proper. In this semi-model structure, factorizations need not be functorial, in accordance with the fact that there need not be a fibrant replacement functor on $\mathcal{M}_{\mathcal{S}}$ lifting L. □

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