Groups and spaces
with all localizations trivial

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0 Introduction

The *genus* of a finitely generated nilpotent group G is defined as the set of isomorphism classes of finitely generated nilpotent groups K such that the p-localizations K_p, G_p are isomorphic for all primes p [19]. This notion turns out to be particularly relevant in the study of non-cancellation phenomena in group theory and homotopy theory.

In the above definition, the restriction of finite generation is imposed in order to prevent the genera from becoming too large—in fact, with that restriction, genera are always finite sets. Nevertheless, it is perfectly possible to deal with the so-called *extended genus*, in which the groups involved, though still nilpotent, are no longer asked to be finitely generated. This generalization has been found to be useful [9, 10, 15].

More serious difficulties arise in this context if one attempts to remove the hypothesis of nilpotency. Given any family of idempotent functors $\{E_p\}$ in the category of groups, one for each prime p, extending p-localization of nilpotent groups, one could expect to find groups G such that $E_pG = 1$ for all primes p, that is, belonging to the "genus" of the trivial group. In fact, as shown below, there even exist groups G sharing this property for *every* family $\{E_p\}$ chosen. We call such groups *generically trivial*. In Sections 1 and 2 we exhibit their basic properties and point out several sources of examples.

A group G is called *separable* if for *some* family of idempotent functors $\{E_p\}$ as above, the canonical homomorphism from G to the cartesian product of the groups E_pG is injective. Residually nilpotent groups and many others are separable. In Section 3, we observe that acyclic spaces X with generically trivial fundamental group are relevant because the space $\mathrm{map}_*(X, Y)$ of pointed maps from X to Y is weakly contractible for a very broad class of spaces Y, namely for all those Y such that $\pi_1(Y)$ is separable.

Acyclic spaces X with generically trivial fundamental group deserve to be called *generically trivial* spaces, because, for every family of idempotent functors E_p in the pointed homotopy category of connected CW-complexes extending p-localization of nilpotent CW-complexes, the spaces E_pX are contractible for all p. Familiar examples of generically trivial spaces include all acyclic spaces whose fundamental group is finite.

Note that nilpotent and generically trivial groups (and spaces) form together a class to which p-localization extends in a unique way. The latter provide the obstruction to the existence of localization-completion pullback squares for general groups and spaces.

Finally, Section 4 is devoted to the problem of recognizing generically trivial groups by inspecting their structure. This is in fact rather difficult, and linked to the problem of determining in general the kernel of the universal homomorphism from a given group G to a group in which p'-roots exist and are unique [22].

1 Separable and generically trivial groups

Let P be a set of primes and P' denote its complement. A group G is called *P-local* [16, 20] if the map $x \mapsto x^n$ is bijective in G for all positive integers

n whose prime divisors lie in P' (written $n \in P'$ for simplicity). For every group G there is a universal homomorphism $l \colon G \to G_P$ with G_P P-local [13, 20, 21], which is called *P-localization*. If the set P consists of a single prime p, then we usually write G_p instead of G_P. The properties of the P-localization homomorphism are particularly well understood when G is nilpotent [16].

A group G is called *separable* [22] if the canonical map from G to the cartesian product of its p-localizations

$$\gamma \colon G \longrightarrow \prod_p G_p \tag{1.1}$$

is injective. It is well-known that nilpotent groups are separable [16]. The class of separable groups is in fact much larger. It also contains all groups which are p-local for some prime p, and it is closed under taking subgroups and forming cartesian products; cf. [22, Proposition 6.10]. Thus it is closed under small (inverse) limits. In particular, since every residually nilpotent group embeds in a cartesian product of countably many nilpotent groups, we have

Proposition 1.1 *Residually nilpotent groups are separable.* $\quad\square$

If a group G is not separable, then one cannot expect to recover full information about G from the family of its p-localizations G_p. The worst possible situation occurs, of course, when all these vanish. We introduce new terminology to analyze this case.

Definition 1.2 A group G is called *generically trivial* if $G_p = 1$ for all primes p.

As next shown, it turns out that such groups cannot be detected by *any* idempotent functor extending p-localization of nilpotent groups to all groups. The basic facts about idempotent functors and localization in arbitrary categories are explained in [1, 13].

Lemma 1.3 *Assume given a set of primes P and an idempotent functor E in the category of groups such that $E\mathbf{Z} \cong \mathbf{Z}_P$, the integers localized at P. Then, for every group G and every $n \in P'$, the map $x \mapsto x^n$ is bijective in EG.*

PROOF. This is in fact a stronger form of a result in [13]. Fix an integer $n \in P'$ and denote by $\rho_n \colon \mathbf{Z} \to \mathbf{Z}$ the multiplication by n. For a group K, the function $(\rho_n)^* \colon \mathrm{Hom}(\mathbf{Z}, K) \to \mathrm{Hom}(\mathbf{Z}, K)$ corresponds precisely to the nth power map $x \mapsto x^n$ in K under the obvious bijection $\mathrm{Hom}(\mathbf{Z}, K) \cong K$. Thus we have to prove that $(\rho_n)^*$ is a bijection when $K = EG$ for some G; in other words, that ρ_n is an E-equivalence. By assumption, there is a commutative diagram

$$
\begin{array}{ccccc}
\mathbf{Z} & \xrightarrow{\eta} & E\mathbf{Z} & \xrightarrow{\cong} & \mathbf{Z}_P \\
{\scriptstyle \rho_n}\downarrow & & {\scriptstyle E\rho_n}\downarrow & & {\scriptstyle \psi}\downarrow \\
\mathbf{Z} & \xrightarrow{\eta} & E\mathbf{Z} & \xrightarrow{\cong} & \mathbf{Z}_P,
\end{array} \tag{1.2}
$$

in which η denotes the natural transformation associated to E. Here η cannot be identically zero, because there is a non-trivial homomorphism from $\mathbf{Z}$ to an E-local group—namely, $\mathbf{Z}_P$. Thus, if $x \in \mathbf{Z}_P$ is the image of $1 \in \mathbf{Z}$ under the top composition in (1.2), then $x \neq 0$ and $\psi(x) = nx$. It follows that ψ is multiplication by n and hence an isomorphism. This implies that $E\rho_n$ is also an isomorphism, i.e. that ρ_n is an E-equivalence, as desired. $\qquad \square$

Theorem 1.4 (a) *A group G is generically trivial if and only if $E_p G = 1$ for every prime p and every idempotent functor E_p in the category of groups satisfying $E_p \mathbf{Z} \cong \mathbf{Z}_p$.*

(b) *A group G is separable if and only if the canonical homomorphism $G \to \prod_p E_p G$ is injective for some family $\{E_p\}$ of idempotent functors in the category of groups, one for each prime p, satisfying $E_p \mathbf{Z} \cong \mathbf{Z}_p$.*

PROOF. One implication is trivial in both part (a) and part (b). To prove the converse in (a), note that, for every prime p and every choice of E_p, the homomorphism $G \to 1$ is an E_p-equivalence when $G_p = 1$, because every homomorphism $\varphi \colon G \to K$ with K E_p-local factorizes through G_p by Lemma 1.3. To prove the converse in (b), use again Lemma 1.3 to obtain, for every family $\{E_p\}$, a factorization

$$
G \xrightarrow{\gamma} \prod_p G_p \to \prod_p E_p G. \qquad \square
$$

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Corollary 1.5 *Generically trivial groups are perfect.*

PROOF. Choose E_p to be Bousfield's $H\mathbf{Z}_p$-localization [8]. It follows from Theorem 1.4 that, if G is generically trivial, then $H_1(G; \mathbf{Z}_p) = 0$ for all primes p, and hence $H_1(G; \mathbf{Z}) = 0$. Actually, there is another argument available: It suffices to apply Corollary 2.1 below to the projection of a generically trivial group G onto its abelianization $G \twoheadrightarrow G/[G, G]$, using the fact that a generically trivial abelian group is necessarily trivial. □

In general, a perfect group need not be generically trivial. There are indeed perfect groups which are locally free [3, Lemma 3.1] and hence separable [22, §9]. Further, in Section 4 below, we present an example of a countable perfect group whose localizations contain the localizations of all finitely generated nilpotent groups. However, as next shown, finite perfect groups are generically trivial.

Proposition 1.6 *For a finite group G, the following assertions are equivalent:*

(a) *G is generically trivial;*

(b) *G is perfect;*

(c) *for every prime p, G is generated by p'-torsion elements.*

PROOF. The implications (c)⇒(a)⇒(b) hold for all groups G. If G is finite, then, for each set of primes P, $l \colon G \to G_P$ is an epimorphism onto a P-group, and $\operatorname{Ker} l$ is generated by the set of P'-torsion elements of G; cf. [22, §7]. This shows that (a)⇒(c). To prove that (b)⇒(a), observe that, given a prime p, G_p is perfect because it is a homomorphic image of G, and also nilpotent because it is a finite p-group. This forces $G_p = 1$. □

The implication (a)⇒(c) in Proposition 1.6 still holds if we assume G locally finite, by Lemma 2.5 below. Its failure to be true for arbitrary groups is discussed in Section 4, cf. Theorem 4.7.

One of the most characteristic features of generically trivial groups G is that homomorphisms $\varphi \colon G \to K$ are trivial for a broad class of groups K. It follows

from Corollary 1.5 that this happens whenever K is residually nilpotent. More generally,

Proposition 1.7 *A group G is generically trivial if and only if every homomorphism $\varphi \colon G \to K$ with K separable is trivial.*

PROOF. If G is generically trivial, then the composition

$$G \xrightarrow{\varphi} K \xrightarrow{\gamma} \prod_p K_p \xrightarrow{\text{proj}} K_p$$

is trivial for all p. Hence, $\gamma(\varphi(x)) = 1$ for every $x \in G$, and, if γ is a monomorphism, then $\varphi(x) = 1$ for every $x \in G$. To prove the converse, take $K = G_p$ for each prime p. □

2 Other properties and examples

The class of generically trivial groups is closed under several constructions which we list below in the form of corollaries of Proposition 1.7. Direct proofs of these statements can also be given using basic properties of the P-localization functor [20, 21, 22].

Corollary 2.1 *Every homomorphic image of a generically trivial group is generically trivial.* □

Corollary 2.2 *If $N \rightarrowtail G \twoheadrightarrow Q$ is a group extension in which N and Q are generically trivial, then G is also generically trivial.* □

Corollary 2.3 *The (restricted) direct product of a family of generically trivial groups is generically trivial.* □

Corollary 2.4 *The free product of a family of generically trivial groups is generically trivial.* □

Thus, the class of generically trivial groups is closed under small colimits. This is not a surprise, in view of the next general fact: Since the P-localization functor has a right adjoint—namely, the inclusion of the subcategory of P-local groups in the category of groups—it preserves colimits [18, V.5], that is

Lemma 2.5 *Let F be a diagram of groups, and denote by F_P the diagram of P-local groups induced by functoriality. Then $(\operatorname{colim} F)_P \cong (\operatorname{colim} F_P)_P$.* □

(Note that, to construct colimits in the category of P-local groups, one computes the corresponding colimit in the category of groups and takes its P-localization.)

We next describe other important sources of examples of generically trivial groups. As already observed, groups satisfying condition (c) in Proposition 1.6 are always generically trivial. Among such groups are all strongly torsion generated groups (a group is *strongly torsion generated* [7] if, for every $n \geq 2$, there is an element $x \in G$ of order n whose conjugates generate G). Examples include the subgroup $E(R)$ generated by the elementary matrices within the general linear group $GL(R)$, for an arbitrary associative ring R with 1; see [6].

A simple group G containing elements of each finite order is strongly torsion generated. Interesting examples of this kind are

– the infinite alternating group $\mathcal{A}_\infty$;

– Philip Hall's countable universal locally finite group [14];

– all non-trivial algebraically closed groups [17].

This last example shows, by [17, IV.8.1], that

Theorem 2.6 *Every infinite group G can be embedded in a generically trivial group of the same cardinality as G.* □

By [6], every abelian group is the centre of a generically trivial group. This prompts the question: which groups are normal in generically trivial groups?

Although generically trivial groups are perfect, no further restriction can be made on their integral homology in general, because

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Theorem 2.7 *For any sequence $A_2, A_3, \ldots, A_n, \ldots$ of abelian groups, there exists a generically trivial group G such that $H_n(G; \mathbf{Z}) \cong A_n$ for all $n \geq 2$.*

PROOF. By [7, Theorem 1], one can always find a strongly torsion generated group with this property. $\square$

3 Some implications in homotopy theory

Our setting in this section is the pointed homotopy category $\mathbf{Ho}_*$ of connected CW-complexes. Our main tool will be the functor $(\ )_P$ defined in [11, 12, 13] for a set of primes P. It is an idempotent functor in $\mathbf{Ho}_*$ which is left adjoint to the inclusion of the subcategory of spaces X for which the nth power map $\rho_n \colon \Omega X \to \Omega X$, $\omega \mapsto \omega^n$, is a homotopy equivalence for every $n \in P'$. The map $l \colon X \to X_P$ turns out to be indeed P-localization if X is nilpotent, and, for every space X, the induced homomorphism $l_* \colon \pi_1(X) \to \pi_1(X_P)$ P-localizes in the sense of Section 1. As explained in [13], the universality of Bousfield's $H_*(\ ; \mathbf{Z}_P)$-localization [8] in $\mathbf{Ho}_*$ implies that

$$l_* \colon H_*(X; \mathbf{Z}_P) \cong H_*(X_P; \mathbf{Z}_P) \tag{3.1}$$

for every space X. In fact, the map $l_* \colon H_*(X; A) \to H_*(X_P; A)$ is an isomorphism for a broader class of (twisted) coefficient modules A, namely those which are P-local as $\mathbf{Z}[\pi_1(X_P)]$-modules; see [11].

Theorem 3.1 *The following assertions are equivalent.*

(a) X is acyclic and $\pi_1(X)$ is a generically trivial group.

(b) For every space Y such that $\pi_1(Y)$ is separable, the space $\mathrm{map}_*(X, Y)$ of pointed maps from X to Y is weakly contractible.

(c) $E_p X$ is contractible for every prime p and every idempotent functor E_p in $\mathbf{Ho}_*$ satisfying $E_p S^1 \simeq (S^1)_p$.

(d) X_p is contractible for all primes p.

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For the proof we need to remark the following fact.

Lemma 3.2 *If the space X is acyclic and $\mathrm{Hom}(\pi_1(X), \pi_1(Y))$ consists of a single element, then $\mathrm{map}_*(X, Y)$ is weakly contractible.*

PROOF. Every map $f\colon X \to Y$ can be extended to the cone of X by obstruction theory, because $f_*\colon \pi_1(X) \to \pi_1(Y)$ is trivial and the cohomology groups of X with untwisted coefficients are zero. Thus $[X, Y]$ consists of a single element, and hence $\mathrm{map}_*(X, Y)$ is path-connected. The higher homotopy groups $\pi_k(\mathrm{map}_*(X, Y)) \cong [\Sigma^k X, Y]$, $k \geq 1$, vanish because the suspension of an acyclic space is contractible. $\square$

PROOF OF THEOREM 3.1. The implication (a)$\Rightarrow$(b) follows from Proposition 1.7 and Lemma 3.2. Now assume given a prime p and an idempotent functor E_p as in (c). Then, by the same argument used in the proof of Lemma 1.3, the standard map $\rho_n\colon S^1 \to S^1$ of degree n is an E_p-equivalence if $(n, p) = 1$, and thus, for every space X, the map $(\rho_n)^*\colon [S^1, E_pX] \to [S^1, E_pX]$ is a bijection. This tells us that $\pi_1(E_pX)$ is a p-local group and hence separable. If we assume that (b) holds, then in particular $\eta\colon X \to E_pX$ is nullhomotopic, and the universal property of η forces E_pX to be contractible. This shows that (b)$\Rightarrow$(c). The implication (c)$\Rightarrow$(d) is trivial. Finally, if X_p is contractible then $\pi_1(X)_p = 1$ and, by (3.1), $H_k(X; \mathbf{Z}_p) = 0$ for all $k \geq 1$, where $\mathbf{Z}_p$ denotes the integers localized at p. If this happens for all primes p, then X is acyclic and $\pi_1(X)$ is generically trivial. Thus, (d)$\Rightarrow$(a). $\square$

We call *generically trivial* those spaces satisfying the equivalent conditions of Theorem 3.1. Such spaces are not rare. For example,

Proposition 3.3 *Every acyclic space X with finite fundamental group is generically trivial.*

PROOF. If X is acyclic, then $\pi_1(X)$ is a perfect group. But finite perfect groups are generically trivial by Proposition 1.6. $\square$

Also classifying spaces of generically trivial acyclic groups G are generically trivial spaces. Such groups G exist by Theorem 2.7. One explicit example is Philip Hall's countable universal locally finite group; see [5]. Another example is the general linear group on the cone of a ring [5, 7]. A third example is the universal finitely presented strongly torsion generated acyclic group constructed in [7].

Since many groups are separable (cf. Section 1), Theorem 3.1 provides a good number of examples of mapping spaces which are weakly contractible.

4   Structure of generically trivial groups

In this final section we address the question of how to characterize generically trivial groups in terms of their structure.

Given a group G and a set of primes P, the kernel of $l\colon G \to G_P$ is hard to compute in general. If G is nilpotent, then $\operatorname{Ker} l$ is precisely the set of P'-torsion elements of G [16]. For other groups G, it can be considerably bigger, as we next explain.

An element $x \in G$ is said to be *of type $T_{P'}$* [20, 21] if there exist $a, b \in G$ and an integer $n \in P'$ such that

$$x = ab^{-1}, \qquad a^n = b^n.$$

Note that the P-localization homomorphism kills all elements of type $T_{P'}$. We may recursively define a sequence of normal subgroups of G

$$1 = T_0 \leq T_1 \leq T_2 \leq \dots \tag{4.1}$$

by letting T_{i+1}/T_i be the subgroup of G/T_i generated by all its elements of type $T_{P'}$. Set $T_{P'}(G) = \bigcup_i T_i$. This subgroup of G was first analyzed by Ribenboim [20]. It has, among others, the following elementary properties.

Proposition 4.1 (a) *If G is P-local, then $T_{P'}(G) = 1$.*

(b) *Every homomorphism $\varphi\colon G \to K$ satisfies $\varphi(T_{P'}(G)) \subseteq T_{P'}(K)$.*

(c) *For every group G, P'-roots are unique in $G/T_{P'}(G)$.* $\square$

It follows from (a) and (b) that $T_{P'}(G)$ is always contained in the kernel of $l\colon G \to G_P$. However, $\operatorname{Ker} l$ can be bigger still. To understand this, we introduce the following notion.

Definition 4.2 Given a set of primes P, a monomorphism $\iota\colon G \to K$ is called *P-faithful* if $\iota_P\colon G_P \to K_P$ is also a monomorphism.

By [16, I.3.1] and Lemma 2.5, every embedding into a locally nilpotent group is P-faithful for all sets P. On the other hand, an embedding of the cyclic group of order 3 in the symmetric group Σ_3 is not 3-faithful, since $(\Sigma_3)_3 = 1$.

The authors wish to acknowledge remarks of Derek Robinson helpful to the following example.

Example 4.3 Let R denote the ring $\mathbf{Z} \oplus (\oplus_p \mathbf{F}_p)$ and consider the group $K = M(\mathbf{Q}, R)$ of all upper unitriangular R-matrices with finitely many non-zero off-diagonal entries indexed by the rational numbers. By [4], K is acyclic and locally nilpotent. Now, by Hirsch [23, p. 139], every finitely generated nilpotent group embeds in the direct product of a torsion-free finitely generated nilpotent group and finitely many finite p-groups. By Hall [23, p. 159], the first factor admits a faithful representation by integral unitriangular matrices (of finite size). Also, via its regular representation, a finite p-group of order n embeds in the group of $n \times n$ unitriangular matrices over $\mathbf{F}_p$. Thus every finitely generated nilpotent group embeds in K. By the above remark, such an embedding must be P-faithful. Thus K is a countable, acyclic group such that, for all sets of primes P, K_P contains a copy of G_P for every finitely generated nilpotent group G.

A group is called *P'-radicable* [23] if each of its elements has at least one nth root for every $n \in P'$.

Lemma 4.4 *For every group G there exists a P-faithful embedding $\iota\colon G \hookrightarrow K$ into a group K which is P'-radicable.*

PROOF. This follows from [21, Proposition 5.2]. In fact, ι can be chosen so that ι_P is an isomorphism. □

We denote by $ET_{P'}(G)$ the subset of G of those elements x belonging to $T_{P'}(K)$ for some P-faithful embedding $G \hookrightarrow K$. This is in fact a normal subgroup of G and, moreover, we have

Proposition 4.5 *Assume given a group G and a set of primes P. Then the kernel of $l\colon G \to G_P$ is precisely $ET_{P'}(G)$.*

PROOF. The inclusion $ET_{P'}(G) \subseteq \operatorname{Ker} l$ is clear. To check the converse, choose a P-faithful embedding $G \hookrightarrow K$ with K P'-radicable, as given by Lemma 4.4. Then, by part (c) of Proposition 4.1, the group $K/T_{P'}(K)$ is P-local. Since every homomorphism $\varphi\colon K \to L$ with L P-local satisfies $\varphi(T_{P'}(K)) = 1$, the projection $K \twoheadrightarrow K/T_{P'}(K)$ is a P-equivalence and hence a P-localization. It follows that the kernel of $l\colon G \to G_P$ is $G \cap T_{P'}(K)$, which is contained in $ET_{P'}(G)$. □

Example 4.6 Let the infinite cyclic group $C = \langle \xi \rangle$ act on the abelian group $A = \mathbf{Z}[1/q]$ by $\xi \cdot a = (1/q)a$, for a certain positive integer q. Let S be the semidirect product $A \rtimes C$ with respect to this action. One can directly check that the nth power map is injective in S for all positive integers n, and hence $T_{P'}(S) = 1$ for every set of primes P. Now denote $a = (0, \xi)$, $b = (1, 1)$ in S, and let $G = \langle S, c \mid c^r = b^m a^n \rangle$, where $b^m a^n$ is not a kth power in S for any $k > 1$ dividing r. Then $T_{P'}(G)$ is still trivial. However, $ET_{P'}(G)$ need not be trivial in general. For example, choose $q = 2$, $r = 2$, $m = 1$, $n = -2$, so that $G = \langle a, b, c \mid a^{-1}ba = b^2, \ b = c^2 a^2 \rangle$. (This example is due to B. H. Neumann [2].) Let P be any set of primes such that $2, 3 \notin P$, and $G \hookrightarrow K$ be a P-faithful embedding in which b has a cube root d. Write $T_i = T_i(K)$ as in

(4.1). Then one readily checks that $(a^{-1}da)^3 = d^6$, and hence $a^{-1}dad^{-2} \in T_1$; further $(da^{-1})^2c^{-2} \in T_1$, which implies $d(ca)^{-1} \in T_2$, and so $(ca)^{-3}b \in T_2$. This element belongs to $ET_{P'}(G)$ and is not trivial.

We can now give the following characterization of generically trivial groups.

Theorem 4.7 *A group G is generically trivial if and only if, for every prime p, there exists a p-faithful embedding $G \hookrightarrow K$ such that $G \subseteq T_{p'}(K)$.* □

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