

A universal space for plus-constructions

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Abstract

We exhibit a two-dimensional, acyclic, Eilenberg–Mac Lane space W such that, for every space X , the plus-construction X^+ with respect to the largest perfect subgroup of $\pi_1(X)$ coincides, up to homotopy, with the W -nullification of X ; that is, the natural map $X \rightarrow X^+$ is homotopy initial among maps $X \rightarrow Y$ where the based mapping space $\text{map}_*(W, Y)$ is weakly contractible. Furthermore, we describe the effect of W -nullification for any acyclic W , and show that some of its properties imply, in their turn, the acyclicity of W .

0 Introduction

In recent articles, Bousfield, Dror Farjoun, and others have developed a far-reaching generalization of earlier work on homological localizations and localizations at sets of primes in unstable homotopy theory. The basic material can be found in [9], [10], [13], [15]. Most of it has been collected in Dror Farjoun’s book [14]. The concept of *nullification* plays a central role in this theory. Given any space W , a W -nullification of a space X is a map $X \rightarrow P_W X$ where the based mapping space $\text{map}_*(W, P_W X)$ is weakly contractible and $P_W X$ is initial in the homotopy category of spaces under X with this property.

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Many familiar constructions can be viewed, up to homotopy, as nullification functors; see [14, 1.E]. A basic example is the plus-construction $q: X \rightarrow X^+$ of Quillen [22] with respect to the largest perfect subgroup of $\pi_1(X)$. Indeed, any appropriate representative W of the wedge of “all” acyclic spaces satisfies $P_W X \simeq X^+$ for all X .

In our attempt to choose W as small as possible, we have found that it suffices to pick a two-dimensional Eilenberg–Mac Lane space $W = K(\mathcal{F}, 1)$, where \mathcal{F} is a locally free, perfect group which is uniquely determined up to the following equivalence. Two groups G and G' have the *same nullity* if the class of groups R such that $\text{Hom}(G, R)$ is trivial coincides with the class of those for which $\text{Hom}(G', R)$ is trivial. We prove that there is precisely one nullity class of locally free, perfect groups \mathcal{F} such that $K(\mathcal{F}, 1)$ -nullification is naturally isomorphic to the plus-construction. We can exhibit groups in this nullity class whose cardinality is the continuum (see Example 5.3 below), but we do not know if there is any countable group in the same class.

In a preparatory analysis, we describe the nature of W -nullification for every W acyclic. Given W acyclic and any X , we show that the map $X \rightarrow P_W X$ coincides up to homotopy with the plus-construction on X with respect to a certain perfect normal subgroup of $\pi_1(X)$ which we denote by $T(W, X)$. Moreover, this property implies in its turn the acyclicity of W . If the space W is a CW-complex of dimension at most two, then $T(W, X)$ is the $\pi_1(W)$ -radical of $\pi_1(X)$, a concept discussed in [11]. Thus, another feature of the present paper is to prove that the perfect radical of an arbitrary group G coincides with the \mathcal{F} -radical of G , where \mathcal{F} is the free product of a set of representatives of all isomorphism classes of countable, locally free, perfect groups.

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1 Sweeping subgroups

For any two groups G and H , we denote by $S(G, H)$ the subgroup of H generated by the images of all homomorphisms $G \rightarrow H$, and call it the subgroup of H *swept* by G . Among its basic properties we emphasize the following:

- (a) For all groups G and H , the subgroup $S(G, H)$ is normal in H .
- (b) If $S(G, H) = H$, then $S(H, K) \leq S(G, K)$ for all groups K .
- (c) The quotient $H/S(G, H)$ is initial in the category of groups Q under H such that every composite homomorphism $G \rightarrow H \rightarrow Q$ has a trivial image.

It is important to observe that, after factoring out $S(G, H)$ from H , new homomorphisms from G to the quotient $Q = H/S(G, H)$ may arise; thus, $S(G, Q)$ need not be trivial in general. It is not, for instance, if $G = \mathbf{Z}/p$ and $H = \mathbf{Z}/p^2$. A more interesting example is seen by choosing $G = \mathbf{Z}/p$ and

$$H = \langle x, y, z \mid x^p, y^p, xyz^{-p} \rangle, \tag{1.1}$$

as in this case $S(G, H)$ is the subgroup generated by the p -torsion elements of H and yet the quotient $H/S(G, H)$ is not p -torsion-free.

Given any group G , we say, as in [9] or [11], that a group R is G -reduced if $S(G, R) = \{1\}$, that is, if $\text{Hom}(G, R)$ is trivial. Every group H has a largest G -reduced quotient, which is denoted by $H//G$. The epimorphism $H \twoheadrightarrow H//G$ is referred to as G -*reduction*, and its kernel is called the G -*radical* of H . Although it has previously been denoted by $T_G(H)$, in this paper we shall adopt instead the notation $T(G, H)$ in order to avoid double subscripts.

It follows from (c) above that $S(G, H) \leq T(G, H)$ for all groups G and H . In fact, as explained in [11, Theorem 3.2], $T(G, H)$ can be constructed as the union of a possibly transfinite sequence of normal subgroups $T(\alpha)$ of H , starting with $T(1) = S(G, H)$. If α is a successor ordinal, let $T(\alpha)$ be the inverse image

under the surjection $H \twoheadrightarrow H/T(\alpha - 1)$ of the subgroup $S(G, H/T(\alpha - 1))$. If α is a limit ordinal, define $T(\alpha)$ to be the union of the subgroups $T(\beta)$ with $\beta < \alpha$. Then $T(G, H)$ equals $T(\nu)$ for the smallest ordinal ν such that $T(\nu) = T(\nu + 1)$.

Recall that a group G is *perfect* if its commutator subgroup $[G, G]$ is the whole of G . A free product of perfect groups is also perfect, and every epimorphic image of a perfect group is perfect. Therefore, if G is perfect, then so is $S(G, H)$ for every group H . In addition, every extension of a perfect group by another perfect group is perfect. Hence, if G is perfect, then so is each $T(\alpha)$ in the above construction, and hence $T(G, H)$ is perfect as well, for any H . It follows that, if we denote by $\mathcal{P}H$ the *perfect radical* of H (that is, the largest perfect subgroup), then

$$S(G, H) \leq T(G, H) \leq \mathcal{P}H \tag{1.2}$$

for all groups H , whenever G is perfect.

Now let W be a fixed connected space with basepoint. For every pointed connected space X , we denote by $S(W, X)$ the subgroup of $\pi_1(X)$ generated by the images of all homomorphisms $\pi_1(W) \rightarrow \pi_1(X)$ which are induced by pointed maps $W \rightarrow X$. Accordingly, we refer to $S(W, X)$ as the subgroup of $\pi_1(X)$ *swept* by the space W . It is clear from this definition that

$$S(W, X) \leq S(\pi_1(W), \pi_1(X)) \tag{1.3}$$

for all spaces W and X . However, the inclusion can be strict. For example, choose X to be any finite-dimensional CW-complex whose fundamental group has an element of order p , where p is any prime. Then $S(B\mathbf{Z}/p, X) = \{1\}$ by Miller's main theorem in [21], yet $S(\mathbf{Z}/p, \pi_1(X)) \neq \{1\}$.

In contrast with this fact, if W is homotopy equivalent to a CW-complex of dimension at most two, then every group homomorphism $\pi_1(W) \rightarrow \pi_1(X)$ is induced by some map $W \rightarrow X$; see e.g. [20, ch. 7]. We also recall that if X is a $K(G, 1)$, then the set of pointed homotopy classes of maps $[W, X]$ is in one-to-one correspondence with the set $\text{Hom}(\pi_1(W), G)$. This yields the following.

Lemma 1.1 *Suppose that either W is homotopy equivalent to a CW-complex of dimension at most two, or X is a $K(G, 1)$ for some group G . Then $S(W, X) = S(\pi_1(W), \pi_1(X))$. \square*

Lemma 1.2 *For all connected spaces W and X , the subgroup $S(W, X)$ is normal in $\pi_1(X)$.*

PROOF. The standard action of the group $\pi_1(X)$ on the set $[W, X]$ ensures that, for each map $f: W \rightarrow X$ and each element $x \in \pi_1(X)$, there exists a map $g: W \rightarrow X$ such that $g_* = x^{-1}f_*x$. \square

Observe that, if the fundamental group $\pi_1(W)$ is perfect, then $S(W, X)$ is perfect for every space X . We also remark that if $S(V, W) = \pi_1(W)$, then $S(W, X) \leq S(V, X)$ for every space X . This shows that our construction is functorial, in the sense explained below.

2 Sequences of plus-constructions

Let \mathbf{S} be the category whose objects are connected spaces W , and with one arrow $V \rightarrow W$ if and only if V sweeps $\pi_1(W)$, that is, $S(V, W) = \pi_1(W)$. Then $S(-, -)$ is a contravariant-covariant bifunctor on the product category $\mathbf{S} \times \mathbf{Ho}$, where \mathbf{Ho} denotes the ordinary pointed homotopy category of connected CW-complexes. For each X in \mathbf{Ho} , the functor $S(-, X)$ takes values in the partially ordered set of normal subgroups of $\pi_1(X)$.

Let \mathbf{SP} be the full subcategory of \mathbf{S} whose objects are spaces W with perfect fundamental group. Then the restriction of $S(-, X)$ to \mathbf{SP} takes values in the partially ordered set of perfect normal subgroups of $\pi_1(X)$. This relates to the plus-construction via the following fundamental lemma due to Quillen; see (5.2) in [2] or (3.1) in [17].

Lemma 2.1 *Let X be any connected space. If N is a perfect normal subgroup of $\pi_1(X)$, then any map $h: X \rightarrow Y$ such that $h_*(N)$ is trivial factors through the Quillen map $q: X \rightarrow X_N^+$, uniquely up to homotopy under X . \square*

Corollary 2.2 *Let W be a connected space with perfect fundamental group. Then, for each X , the space $X_{S(W,X)}^+$ together with the corresponding Quillen map is initial in the homotopy category of spaces Y under X such that every composite map $W \rightarrow X \rightarrow Y$ sends $\pi_1(W)$ to $\{1\}$. \square*

However, for the same reason as in (1.1), the group $S(W, X_{S(W,X)}^+)$ need not be trivial in general. Our next aim is to describe the initial space Y under X such that $S(W, Y) = \{1\}$, for a fixed W . It turns out to be the plus-construction defined with respect to a certain larger subgroup of $\pi_1(X)$, which will be denoted by $T(W, X)$. It should be regarded as the appropriate closure of $S(W, X)$.

To this end, we define iteratively a natural, increasing, possibly transfinite, nested sequence of perfect normal subgroups $N(\alpha)$ of $\pi_1(X)$, together with corresponding plus-construction maps $q_\alpha: X \rightarrow X_{N(\alpha)}^+$. First define $N(0) = \{1\}$, so that $X_{N(0)}^+ = X$ and q_0 is the identity map. Given $N(\alpha)$, define $N(\alpha + 1)$ to be the inverse image of $S(W, X_{N(\alpha)}^+)$ in $\pi_1(X)$ under the surjection

$$(q_\alpha)_*: \pi_1(X) \twoheadrightarrow \pi_1(X_{N(\alpha)}^+) \cong \pi_1(X)/N(\alpha).$$

As the inverse image of a normal subgroup, $N(\alpha + 1)$ is also normal. From the perfect-by-perfect group extension

$$\{1\} \rightarrow N(\alpha) \rightarrow N(\alpha + 1) \rightarrow S(W, X_{N(\alpha)}^+) \rightarrow \{1\},$$

we obtain that $N(\alpha + 1)$ is perfect. When α is a limit ordinal, define $N(\alpha)$ to be the union of the subgroups $N(\beta)$ with $\beta < \alpha$. Finally, we take for $T(W, X)$ the union of all subgroups $N(\alpha)$. It is of the form $N(\nu)$ for some ordinal ν with $N(\nu) = N(\nu + 1)$. Hence, $S(W, X_{N(\nu)}^+)$ is trivial.

Proposition 2.3 *The construction $T(-, -)$ is a contravariant-covariant bifunctor on the product category $\mathbf{SP} \times \mathbf{Ho}$, such that, for each X in \mathbf{Ho} , $T(-, X)$ is a functor from \mathbf{SP} to the partially ordered set of perfect normal subgroups of $\pi_1(X)$. The space $X_{T(W,X)}^+$ together with the corresponding Quillen map is initial in the homotopy category of spaces Y under X with $S(W, Y) = \{1\}$.*

PROOF. Naturality with respect to the second variable X is clear. Now, given a morphism $V \rightarrow W$ in \mathbf{SP} , with sequence of perfect normal subgroups $M(\alpha)$ of $\pi_1(X)$ associated with V , we must show inductively that $N(\alpha) \leq M(\alpha)$ for all α . Suppose that this is true for an ordinal α . Given any map $W \rightarrow X_{N(\alpha)}^+$, denote by h the composite

$$W \rightarrow X_{N(\alpha)}^+ \rightarrow X_{M(\alpha)}^+ \rightarrow X_{M(\alpha+1)}^+,$$

which is defined by means of Lemma 2.1. Then, as g ranges through all maps from V to W , the composites $h \circ g$ send all elements of $\pi_1(V)$ to $\{1\}$. Therefore, h sends $S(V, W) = \pi_1(W)$ to $\{1\}$. Hence, by Corollary 2.2, there is a unique map $X_{N(\alpha+1)}^+ \rightarrow X_{M(\alpha+1)}^+$ under $X_{N(\alpha)}^+$, as required. Thus the induction goes through.

For the second claim, we proceed by transfinite induction, considering the fate of a map $f: X \rightarrow Y$ where $S(W, Y) = \{1\}$ as we move through the nested sequence of subgroups $N(\alpha)$. We inductively assume that, for each α , the map f factors uniquely through $q_\alpha: X \rightarrow X_{N(\alpha)}^+$ as $f \simeq f_\alpha \circ q_\alpha$. Since every composite map $W \rightarrow X_{N(\alpha)}^+ \rightarrow Y$ sends $\pi_1(W)$ to $\{1\}$, Corollary 2.2 tells us that f_α factors uniquely through $X_{N(\alpha+1)}^+$. If, on the other hand, α is a limit ordinal, then since each group $f_*(N(\beta))$ with $\beta < \alpha$ is trivial, so too is $f_*(N(\alpha))$. Again, by Lemma 2.1, we obtain unique factorization of f through $q_\alpha: X \rightarrow X_{N(\alpha)}^+$. Hence, as required, f factors uniquely through $X \rightarrow X_{T(W, X)}^+$. \square

Proposition 2.4 *Let W be a connected space whose fundamental group is perfect. Then*

$$T(W, X) \leq T(\pi_1(W), \pi_1(X)).$$

Moreover, if W is homotopy equivalent to a CW-complex of dimension at most two, then equality holds.

PROOF. To prove that $T(W, X)$ is contained in $T(\pi_1(W), \pi_1(X))$, it suffices to show that each subgroup $N(\alpha)$ in the construction of $T(W, X)$ is contained

in the corresponding subgroup $T(\alpha)$ in the construction of $T(\pi_1(W), \pi_1(X))$. For $\alpha = 1$, the subgroup $N(1) = S(W, X)$ is indeed contained in $T(1) = S(\pi_1(W), \pi_1(X))$. The argument continues by transfinite induction as above.

Now suppose that W is a CW-complex of dimension at most two. Assume given a homomorphism $\varphi: \pi_1(W) \rightarrow \pi_1(X)/T(W, X)$. Then there is at least one map $W \rightarrow X_{T(W, X)}^+$ inducing φ on the fundamental group. Since $S(W, X_{T(W, X)}^+) = \{1\}$, the homomorphism φ has to be trivial. This shows that $\pi_1(X)/T(W, X)$ is $\pi_1(W)$ -reduced and hence $T(W, X)$ contains the radical $T(\pi_1(W), \pi_1(X))$. \square

3 Nullification with respect to acyclic spaces

Let W be any connected space. A space X is called *W-null* [10], [15] if the space of pointed maps $\text{map}_*(W, X)$ is weakly contractible. Thus, X is *W-null* if and only if $[\Sigma^n W, X]$ is trivial for all $n \geq 0$. For every space X there is a map $l_X: X \rightarrow P_W X$, called *W-nullification*, which is initial in \mathbf{Ho} among maps from X into *W-null* spaces. It can be constructed by repeatedly attaching mapping cones to X in order to trivialize all maps coming from W and its suspensions; see [9], [13].

If we assume, in addition, that the space W is acyclic (that is, the reduced integral homology groups $\tilde{H}_n(W)$ vanish for all n), then, as we next explain, *W-null* spaces are particularly easy to recognize.

Lemma 3.1 *Suppose that W is acyclic. Then, for a space X , the following statements are equivalent:*

- (i) X is *W-null*.
- (ii) Every map $W \rightarrow X$ is nullhomotopic.
- (iii) The group $S(W, X)$ is trivial.

PROOF. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are obvious. In order to prove that (iii) \Rightarrow (ii), observe that, given any map $g: W \rightarrow X$, the homomorphism

$$g_*: \pi_1(W) \rightarrow \pi_1(X)$$

is trivial by assumption. Since the space W is acyclic, the cohomology groups $H^n(W; \pi_m(X))$ (with untwisted coefficients) vanish for $n \geq 1$ and $m \geq 2$. It follows from obstruction theory that g is nullhomotopic. Finally, (ii) \Rightarrow (i) because the acyclicity of W tells us that the suspensions $\Sigma^n W$ are contractible for $n \geq 1$. \square

We prove below that the converse of Lemma 3.1 is also true, in that the equivalence of (i), (ii), and (iii) implies that W is acyclic. On the other hand, equivalence merely of (i) and (ii) is in general insufficient for this conclusion. This is clear for $W = S^2 \vee S^3 \vee \dots$ (or any other space W having ΣW as a retract).

Proposition 3.2 *Let W be such that the class of W -null spaces coincides with the class of spaces admitting no essential maps from W . If for some $k \geq 1$ the integral cohomology group $H^k(W)$ is nonzero, then $H^m(W)$ is nonzero for all $m \geq k$. In particular, if W is finite-dimensional, then it is acyclic.*

PROOF. Suppose that for some $m \geq k$ the cohomology group $H^m(W)$ vanishes. Then $[W, K(\mathbf{Z}, m)] = 0$, and this implies, by assumption, that the space $K(\mathbf{Z}, m)$ is W -null. Therefore

$$H^k(W) \cong [W, K(\mathbf{Z}, k)] \cong [\Sigma^{m-k}W, K(\mathbf{Z}, m)] = 0,$$

and the result follows. \square

Theorem 3.3 *Let W be any connected space. Then the following statements are equivalent:*

- (i) W is acyclic.

- (ii) *The class of W -null spaces coincides with the class of spaces X such that $S(W, X)$ is trivial.*
- (iii) *For every space X , the W -nullification map $l_X : X \rightarrow P_W X$ coincides, up to homotopy under X , with the plus-construction with respect to the perfect normal subgroup $T(W, X)$ of $\pi_1(X)$.*
- (iv) *For every space X , the map $l_X : X \rightarrow P_W X$ is an integral homology equivalence.*

PROOF. (i) \Rightarrow (ii): This has been shown in Lemma 3.1.

(ii) \Rightarrow (iii): Since $S(W, K(\mathbf{Z}, 2))$ is trivial, it follows from (ii) that $K(\mathbf{Z}, 2)$ is W -null. Therefore, $H^2(W) = [W, K(\mathbf{Z}, 2)]$ and $H^1(W) = [\Sigma W, K(\mathbf{Z}, 2)]$ are zero, whence $\pi_1(W)$ is perfect. (An obvious extension of this argument may be used to infer (i).) Then by (ii) and Proposition 2.3, $X \rightarrow X_{T(W, X)}^+$ and $X \rightarrow P_W X$ are both initial objects in the homotopy category of W -null spaces under X .

(iii) \Rightarrow (iv): This is immediate.

(iv) \Rightarrow (i): By (iv), $l_W : W \rightarrow P_W W$ is an integral homology equivalence. Since the space $P_W W$ is contractible, it follows that W is acyclic. \square

At this point we recall that the condition that the subgroup $S(W, X)$ of $\pi_1(X)$ be trivial is strictly weaker than the more obvious condition that the set $\text{Hom}(\pi_1(W), \pi_1(X))$ be trivial. Here is another example, where the space W involved is acyclic.

Example 3.4 Let A be any locally finite acyclic group (e.g. the McLain group $M(\mathbf{Q}, \mathbf{F}_2)$; see [3]), and let W be its classifying space. Then the 2-skeleton X of W also has A as fundamental group, so that the identity homomorphism on A lies in $\text{Hom}(\pi_1(W), \pi_1(X))$. On the other hand, by [21], the set $[W, X]$ is trivial, as therefore is $S(W, X)$.

We exhibit some further instances where the triviality of $S(W, X)$ is guaranteed for an acyclic space W , and hence X is W -null, by Lemma 3.1.

Example 3.5 Choose W such that $\pi_1(W)$ is an acyclic group which has no finite-dimensional (integral, say) representation. For example, after [4], [1], [7] respectively, it could be any torsion-generated acyclic group, a binate group or a “large” group of automorphisms as in [16]. Then $S(W, X)$ is trivial whenever the perfect radical $\mathcal{P}\pi_1(X)$ is linear (for example, finite).

Example 3.6 Suppose that W is fundamentally torsion-generated (meaning that there exists a map to W from a wedge of $K(\pi_\alpha, 1)$ spaces, with each π_α a finite group, which induces an epimorphism on fundamental groups [6]). Then, as in the proof of Theorem 2.1 in [6], for any X with $\mathcal{P}\pi_1(X)$ complex-linear, $S(W, X)$ is trivial.

4 Nullification and group reduction

In this section, we show that the plus-construction $q: X \rightarrow X^+$ with respect to the largest perfect subgroup $\mathcal{P}\pi_1(X)$ can be described as nullification with respect to a certain acyclic, two-dimensional, Eilenberg–Mac Lane space $K(\mathcal{F}, 1)$, where the group \mathcal{F} does not depend on X .

We rely on the following observation due to Heller; see [18, Lemma 5.7] and Example 5.3 below.

Lemma 4.1 *Assume given a perfect group P and any element $x \in P$. Then there exists a countable, locally free, perfect group D and a homomorphism $\varphi: D \rightarrow P$ containing x in its image. \square*

Note that, since homology commutes with direct limits, a locally free perfect group is necessarily acyclic. Moreover, if D is locally free, then a $K(D, 1)$ can be constructed as the homotopy direct limit of a sequence of wedges of circles, and hence there is a two-dimensional CW-complex in its homotopy type.

Now choose a set of representatives of all isomorphism classes of countable, locally free, perfect groups, and denote by \mathcal{F} their free product (which is still

acyclic). The following result is a direct consequence of Lemma 4.1, which we record for its use elsewhere; cf. [11, § 3], [12].

Proposition 4.2 *For a group H , the following statements are equivalent:*

- (i) *The largest perfect subgroup of H is trivial.*
- (ii) *$\text{Hom}(P, H)$ is trivial for every perfect group P .*
- (iii) *$\text{Hom}(P, H)$ is trivial for every countable perfect group P .*
- (iv) *$\text{Hom}(P, H)$ is trivial for every countable, locally free, perfect group P .*
- (v) *$\text{Hom}(\mathcal{F}, H)$ is trivial. \square*

The equivalence between the statements (i) and (v) tells us precisely that a group H is \mathcal{F} -reduced if and only if its largest perfect subgroup $\mathcal{P}H$ is trivial. We infer the following.

Theorem 4.3 *Let \mathcal{F} be the free product of a set of representatives of all isomorphism classes of countable, locally free, perfect groups. Then the following hold.*

- (a) *$T(\mathcal{F}, H) = \mathcal{P}H$ for all groups H .*
- (b) *$S(B\mathcal{F}, X) = T(B\mathcal{F}, X) = T(\mathcal{F}, \pi_1(X)) = \mathcal{P}\pi_1(X)$ for all spaces X .*
- (c) *The $B\mathcal{F}$ -nullification functor $P_{B\mathcal{F}}$ is naturally isomorphic, in the pointed homotopy category, to the plus-construction with respect to the largest perfect subgroup.*

PROOF. Since $H/T(\mathcal{F}, H)$ is \mathcal{F} -reduced, $\mathcal{P}(H/T(\mathcal{F}, H))$ is trivial, from which it follows that $\mathcal{P}H \leq T(\mathcal{F}, H)$. As $T(\mathcal{F}, H)$ is perfect, this is in fact an equality, hence establishing (a).

Since the classifying space $B\mathcal{F}$ has the homotopy type of a two-dimensional CW-complex, each homomorphism $\mathcal{F} \rightarrow \pi_1(X)$ is induced by a map $B\mathcal{F} \rightarrow X$.

Hence, by Lemma 4.1, for each $x \in \mathcal{P}\pi_1(X)$ there is a map $h: B\mathcal{F} \rightarrow X$ such that the image of h_* contains x . This implies that $S(B\mathcal{F}, X)$ contains $\mathcal{P}\pi_1(X)$. But using Proposition 2.4 we have that

$$S(B\mathcal{F}, X) \leq T(B\mathcal{F}, X) = T(\mathcal{F}, \pi_1(X)) \leq \mathcal{P}\pi_1(X),$$

from which (b) follows. This, together with part (iii) of Theorem 3.3, proves part (c) as well. \square

It follows from part (b) of this theorem that \mathcal{F} is universal, in that the space $B\mathcal{F}$ is initial in the category \mathbf{SP} defined in Section 2. However, this property does not characterize \mathcal{F} up to group isomorphism, since the category \mathbf{SP} has very few morphisms. For example, if a group \mathcal{F}' is perfect and maps onto \mathcal{F} , then $B\mathcal{F} \cong B\mathcal{F}'$ in \mathbf{SP} and hence, by Proposition 2.3 and part (iii) of Theorem 3.3, the functors $P_{B\mathcal{F}}$ and $P_{B\mathcal{F}'}$ are naturally isomorphic. The same argument shows, more generally, that if G and G' are acyclic groups which sweep each other (that is, $S(G, G') = G'$ and $S(G', G) = G$), then the functors P_{BG} and $P_{BG'}$ are naturally isomorphic.

Thus, the extent to which the group \mathcal{F} is uniquely determined is best discussed by resorting to the appropriate equivalence relation, which we describe in the next section.

5 Characteristic properties

Recall from [12, Proposition 1.4] that, given two groups G and H , the following facts are equivalent:

- (i) The G -reduction $H//G$ is trivial.
- (ii) $T(H, K) \leq T(G, K)$ for every group K .
- (iii) For every group K there is a natural epimorphism $K//H \rightarrow K//G$.

(iv) The class of G -reduced groups is contained in the class of H -reduced groups.

In analogy with [10, § 3], we say that two groups G and G' have the *same nullity* if both $G//G' = \{1\}$ and $G'//G = \{1\}$. This happens if and only if $K//G \cong K//G'$ for all groups K , or equivalently, if the classes of G -reduced and G' -reduced groups coincide. Of course, if G and G' sweep each other, then they have the same nullity. However, the converse need not be true, as illustrated by the example given in (1.1).

Proposition 5.1 *Let W, W' be two connected spaces such that the functors P_W and $P_{W'}$ are naturally isomorphic. Then the groups $\pi_1(W)$ and $\pi_1(W')$ have the same nullity.*

PROOF. Let R be any $\pi_1(W)$ -reduced group. Then $\text{Hom}(\pi_1(W), R)$ is trivial, and hence the classifying space BR is W -null. It follows from our assumption that BR is W' -null, so that R is $\pi_1(W')$ -reduced. The same argument shows that each $\pi_1(W')$ -reduced group is also $\pi_1(W)$ -reduced, from which our assertion follows. \square

The next result implies that there is precisely one nullity class of perfect, locally free groups G such that P_{BG} is naturally isomorphic to the plus-construction.

Proposition 5.2 *Let G and G' be perfect, locally free groups. Then the functors P_{BG} and $P_{BG'}$ are naturally isomorphic if and only if the groups G and G' have the same nullity.*

PROOF. One implication follows of course from Proposition 5.1. To check the converse, we use the fact that BG and BG' are two-dimensional acyclic spaces. By Lemma 3.1, a space X is BG -null if and only if $S(BG, X)$ is trivial, and, by Lemma 1.1, this happens if and only if $\pi_1(X)$ is G -reduced. Hence, the classes of BG -null and BG' -null spaces coincide, from which it follows that P_{BG} and $P_{BG'}$ are naturally isomorphic. \square

Example 5.3 For each sequence $\mathbf{n} = (n_1, n_2, n_3, \dots)$ of positive integers and each $r \geq 1$, let $F_{\mathbf{n},r}$ be the free group freely generated by the following set of $2^r n_1 \cdots n_r$ symbols:

$$\{x_r(\varepsilon_1, \dots, \varepsilon_r; i_1, \dots, i_r) \mid \varepsilon_k \in \{0, 1\}, 1 \leq i_k \leq n_k\}.$$

For $r = 0$, define $F_{\mathbf{n},0}$ to be infinite cyclic with a generator x_0 . Define homomorphisms $\varphi_r: F_{\mathbf{n},r} \rightarrow F_{\mathbf{n},r+1}$ for $r \geq 0$ by sending $x_r(\varepsilon_1, \dots, \varepsilon_r; i_1, \dots, i_r)$ to the following product of commutators:

$$\prod_{i_{r+1}=1}^{n_{r+1}} [x_{r+1}(\varepsilon_1, \dots, \varepsilon_r, 0; i_1, \dots, i_r, i_{r+1}), x_{r+1}(\varepsilon_1, \dots, \varepsilon_r, 1; i_1, \dots, i_r, i_{r+1})].$$

Let $F_{\mathbf{n}}$ be the direct limit of the direct system $(F_{\mathbf{n},r}, \varphi_r)$. Thus, for each sequence \mathbf{n} of positive integers, the group $F_{\mathbf{n}}$ is countable, locally free, and perfect (hence acyclic). Now let \mathcal{F}' be the free product of the groups $F_{\mathbf{n}}$, where \mathbf{n} ranges over all increasing sequences of positive integers. If we denote by a the cardinality of the set of naturals and by c the cardinality of the set of reals, then the disjoint union U of all the groups $F_{\mathbf{n}}$ has cardinality $c \cdot a = c$, and the set S of sequences of elements of U has cardinality $c^a = c$. The group \mathcal{F}' contains the set U and embeds into S ; therefore the cardinality of \mathcal{F}' equals c .

For every perfect group P and each element $x \in P$, one can write x as a product of commutators. As a consequence of this fact, there is a sequence of integers \mathbf{n} (which may be chosen to be increasing) and a homomorphism $\psi: F_{\mathbf{n}} \rightarrow P$ whose image contains x . We omit the details of this claim; cf. [18, Lemma 5.7]. It follows that the group \mathcal{F}' has the same nullity as our former group \mathcal{F} . Therefore, by Proposition 5.2, the functor $P_{B\mathcal{F}'}$ is naturally isomorphic to the plus-construction as well.

One may speculate as to whether it is possible to replace \mathcal{F}' by a locally free, perfect group known to be countable. We thank C. F. Miller III for encouraging us in such speculations, although no conclusion has as yet been forthcoming.

It is worth emphasizing that the $B\mathcal{F}$ -nullification (plus-construction) of any given space X can be carried out by means of one single push-out. Choose a set $\{x_\alpha\}$ whose normal closure in $\pi_1(X)$ is the largest perfect subgroup of $\pi_1(X)$. For every index α , choose a map $g_\alpha: B\mathcal{F} \rightarrow X$ such that x_α belongs to the image of the homomorphism $(g_\alpha)_*$. Let W be a wedge of copies of $B\mathcal{F}$, one for each α , and define a map $g: W \rightarrow X$ using the family $\{g_\alpha\}$. Then the homotopy cofibre of g is homotopy equivalent under X to the space X^+ . However, as X varies, so the cardinality of the set $\{x_\alpha\}$ may increase without bound; thus no one space W allows a single push-out for all spaces by this method.

The story is a little different when we look at the motivating example for the plus-construction, namely the classifying space $BGLR$ of the general linear group of a ring R . Here, as shown in [5], $\mathcal{P}GLR$ is normally generated by any single non-identity finite permutation matrix, regarded as in GLR through the canonical homomorphism from $GL\mathbf{Z}$ to GLR . As a consequence of this fact, there are many spaces $W = BG$ for which a single push-out suffices to yield the plus-construction for all rings R . Specifically,

Proposition 5.4 *Let G be any acyclic group which has a nontrivial finite-dimensional integral representation $\rho: G \rightarrow GL\mathbf{Z}$. Then, for all associative rings R with unit, the homotopy cofibre of the induced map $f: BG \rightarrow BGLR$ is homotopy equivalent to $BGLR^+$. \square*

Any acyclic group G with a finite, nontrivial homomorphic image fulfills the hypothesis made in Proposition 5.4. However, for instance, the acyclic groups in Example 3.5 are excluded.

We shall address one final question. We know that if W is a space for which P_W is naturally isomorphic to the plus-construction on all spaces, then W is acyclic, since in particular $W^+ \simeq P_W W \simeq \text{pt}$. As we next show, in order to infer that W be acyclic, it suffices to assume that P_W coincides with the plus-construction on classifying spaces of discrete groups.

Lemma 5.5 *Let W be a space such that $P_W BGLC \simeq BGLC^+$, where \mathbf{C} is the field of complex numbers and GLC has the discrete topology. Then $H_1(W) = 0$.*

PROOF. If the abelian group $H_1(W)$ is nonzero then it admits a nontrivial homomorphism ψ to the group $GL_1\mathbf{C}$, which is divisible and contains elements of all finite orders. Now the fibre $\mathcal{A}BGLC$ of the plus-construction $q : BGLC \rightarrow BGLC^+$ is acyclic. Thus because every map from W to $BGLC$ factors through $\mathcal{A}BGLC$ it must be trivial on homology. In particular, the composite map

$$W \rightarrow B\pi_1(W) \rightarrow BH_1(W) \xrightarrow{B\psi} BGL_1\mathbf{C} \rightarrow BGLC$$

is homologically trivial. However, on the first homology group it coincides with ψ via the determinant isomorphism $H_1(BGLC) = K_1(\mathbf{C}) \cong GL_1\mathbf{C}$. This gives the desired contradiction. \square

Theorem 5.6 *Let W be a space such that $P_W BG \simeq BG^+$ for every discrete group G . Then W is acyclic.*

PROOF. According to Kan–Thurston [19], there is a discrete group G with a perfect normal subgroup N such that $W \simeq BG_N^+$. By Lemma 5.5, the group $\pi_1(W) \cong G/N$ is perfect. This implies that G is also perfect, and hence

$$W^+ \simeq BG^+ \simeq P_W BG.$$

This shows that W^+ is W -null, and hence the Quillen map $q : W \rightarrow W^+$ is trivial. Since q is a homology isomorphism, W is acyclic. \square

In view of the historical association of the plus-construction with classifying spaces of the form $BGLR$, we ask whether Theorem 5.6 can be improved as follows.

Question 5.7 *Suppose that W is a space with the property that*

$$P_W BGLR \simeq BGLR^+$$

for every associative ring R with unit. Is W then necessarily acyclic?

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