Localization methods in the study of the homology of virtually nilpotent groups

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0 Introduction

In a series of works ([13], [14], [15]), Hilton introduced the terminology *relative group* to denote a group epimorphism $\varepsilon\colon G\twoheadrightarrow Q$, and *relative space* to denote a map $f\colon E\to B$ between connected spaces inducing an epimorphism of fundamental groups. He pointed out the desirability of relativizing the theory of P-localization of nilpotent groups and spaces developed in [17], and carried out the algebraic part of this project in [14], [16]. The homotopy-theoretical part was settled by Llerena in [18], [19].

On the other hand, a certain amount of work has been done in the last few years to develop P-localization methods in the category of all groups ([4], [11], [20], [21], [22], [23], [24]), starting from earlier approaches on radicability in groups ([2]). Given a set of primes P, a group G is called *P-local* ([17]) if the map $x\mapsto x^n$ is bijective in G for every integer $n\geq 1$ whose prime divisors lie outside P. The inclusion of the subcategory of P-local groups in the category of groups has a left adjoint (see Proposition 5 in [22] or Example 3.3 in [7]), which is denoted by $(\quad)_P$ and called *P-localization*. This functor turns out to be *initial* among all possible extensions of P-localization of nilpotent groups to the category of all groups, in the sense that there is a unique natural transformation from it to any other such extension; see Section 2 of [7].

P-localization of groups can also be relativized in the sense of Hilton. In Section 1 we give a simple abstract proof of this fact using the methods of [7], hence enlarging the list of idempotent functors encountered in practice whose construction may be viewed as a particular case of a very general procedure. The existence argument given by Ribenboim in the absolute case ([22]) is recovered by identifying each group G with the corresponding epimorphism $G \twoheadrightarrow \{1\}$. Our method is also guided by the attempt to develop the analog in homotopy theory, i.e., to generalize the results of [19] to arbitrary relative spaces, using the new ideas of [5].

In Sections 2 and 3 we use P-localization of relative groups as a tool to obtain a description of the P-localization of groups having a nilpotent subgroup of finite index (called *virtually nilpotent* or *nilpotent-by-finite*). We prove the following: Given a group extension

$$N \overset{\iota}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$$

in which N is nilpotent of class at most c and Q is nilpotent and torsion, then the sequence

$$\Gamma^c_K(N_P) \rightarrowtail N_P \overset{\iota_P}{\to} G_P \overset{\varepsilon_P}{\twoheadrightarrow} Q_P \tag{0.1}$$

is exact for every set of primes P; here K is the preimage under ε of the P'-torsion subgroup of Q (P' denotes the complement of P), and $\Gamma^c_K(N_P)$ is the cth term of the lower central series ([12]) of the action of K on N_P induced by the conjugation action of G on N. The assumption that Q be nilpotent can be weakened; cf. Theorem 2.1.

This result illustrates the lack of exactness of the P-localization functor in general, and allows us to prove, among other things, that the P-localization of a virtually nilpotent group is again virtually nilpotent (Theorem 3.3). Thus P-localization restricts to the category of virtually nilpotent groups.

Our result (0.1) enables us then to analyze the behaviour of integral homology under P-localization in the class of virtually nilpotent groups. In particular, we are able to show examples of finitely generated nonnilpotent groups G for which $H_k(G_P) \cong \mathbf{Z}_P \otimes H_k(G)$ for all $k \geq 1$ and every set of primes P, where $\mathbf{Z}_P$ denotes the integers localized at P. These examples include the fundamental group of the Klein bottle and the infinite dihedral group. It is remarkable that no (nonnilpotent) examples with that property can be found in the class of finite groups, because, as shown in [4], if G is finite and $H_k(G_p) \cong \mathbf{Z}_p \otimes H_k(G)$ for all $k \geq 1$ and all single primes p, then G has to be nilpotent.

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1 Localization of relative groups

We shall use the terminology and results of [6], [7], which we briefly recall. A morphism $f\colon A \to B$ and an object X in a category $\mathcal{C}$ are called *orthogonal* ([9]) if for every morphism $g\colon A \to X$ there is exactly one $g'\colon B \to X$ such that $g'f = g$; that is, if $f^*\colon \mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$ is bijective. For a class of morphisms $\mathcal{S}$, the class of objects orthogonal to all $f \in \mathcal{S}$ is denoted by $\mathcal{S}^\perp$, and analogously for a class of objects $\mathcal{D}$. An *orthogonal pair* $(\mathcal{S}, \mathcal{D})$ in $\mathcal{C}$ consists of a class of morphisms $\mathcal{S}$ and a class of objects $\mathcal{D}$ such that $\mathcal{S}^\perp = \mathcal{D}$ and $\mathcal{D}^\perp = \mathcal{S}$. If $(\mathcal{S}, \mathcal{D})$ is an orthogonal pair, then the classes $\mathcal{S}$, $\mathcal{D}$ are *saturated*, meaning that $\mathcal{S}^{\perp\perp} = \mathcal{S}$, $\mathcal{D}^{\perp\perp} = \mathcal{D}$.

Orthogonal pairs are useful tools to study idempotent functors, because, as explained in [1, §2], every idempotent functor E gives rise to an orthogonal pair $(\mathcal{S}, \mathcal{D})$, and it is often possible to recover E from information contained in $(\mathcal{S}, \mathcal{D})$. Specifically, Theorem 1.1 below will be useful for our purposes. An object X of a cocomplete category $\mathcal{C}$ is called *presentable* ([10]) if, for a sufficiently large ordinal α, every morphism from X to the colimit of a direct system $F\colon \alpha \to \mathcal{C}$ factors through $F(i)$ for some $i < \alpha$.

Theorem 1.1 ([7]) *Let $\mathcal{C}$ be a cocomplete category and $(\mathcal{S}, \mathcal{D})$ an orthogonal pair in $\mathcal{C}$. Assume that there exists a set $\mathcal{S}_0 \subseteq \mathcal{S}$ such that $(\mathcal{S}_0)^\perp = \mathcal{D}$, and that the domains and codomains of morphisms in $\mathcal{S}_0$ are presentable. Then the inclusion of the full subcategory of objects in $\mathcal{D}$ in the category $\mathcal{C}$ has a left adjoint $E\colon \mathcal{C} \to \mathcal{D}$.* $\square$

The next example is illuminating and relevant for the sequel. In the category $\mathcal{G}$ of groups one may consider, for a set of primes P, the set of maps

$$\mathcal{T}_P = \{\rho_n\colon C \to C,\ \rho_n(t) = t^n,\ n \in P'\},$$

where $C = \langle t \rangle$ is infinite cyclic (we use multiplicative notation), and $n \in P'$ means that n is a positive integer all of whose prime divisors lie outside P. A group G belongs to $\mathcal{D}_P = (\mathcal{T}_P)^\perp$ if and only if every $x \in G$ has a unique nth root for each $n \in P'$, that is, if and only if G is P-*local* as defined in the introduction. We call P-*equivalences* of groups the homomorphisms in the class $\mathcal{S}_P = (\mathcal{D}_P)^\perp$. Observe that $(\mathcal{S}_P)^\perp = (\mathcal{D}_P)^{\perp\perp} = (\mathcal{T}_P)^{\perp\perp\perp} = (\mathcal{T}_P)^\perp = \mathcal{D}_P$, so that $(\mathcal{S}_P, \mathcal{D}_P)$ is an orthogonal pair in $\mathcal{G}$ and, by Theorem 1.1, the inclusion of $\mathcal{D}_P$ in $\mathcal{G}$ has a left adjoint $\mathcal{G} \to \mathcal{D}_P$, which we call P-*localization*. The unit of the adjunction is denoted by l. Thus a group homomorphism $l\colon G \to G_P$ P-*localizes* if and only if $l \in \mathcal{S}_P$ and $G_P \in \mathcal{D}_P$ (or, equivalently, if l is initial among all homomorphisms from G to groups in $\mathcal{D}_P$). If G is nilpotent, then $l\colon G \to G_P$ is the same map as the one described in Chapter I of [17]; in particular, if G is commutative, then $G_P \cong \mathbf{Z}_P \otimes G$.

The aim of this section is to relativize the functor described above. Let $\mathcal{EG}$ denote the category of relative groups in the sense of Hilton (cf. [14], from which our notation is taken). Thus, objects in $\mathcal{EG}$ are group epimorphisms $\varepsilon\colon G \twoheadrightarrow Q$, and morphisms are commutative diagrams

$$\begin{array}{ccc} G & \overset{\varepsilon}{\twoheadrightarrow} & Q \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ G' & \overset{\varepsilon'}{\twoheadrightarrow} & Q'. \end{array} \tag{1.1}$$

A relative group ε is called *P-local* if $\ker\varepsilon$ is a P-local group. This property can be translated into categorical terms as follows.

Proposition 1.2 *A relative group $\varepsilon\colon G \twoheadrightarrow Q$ is P-local if and only if it is orthogonal to*

$$\begin{array}{ccc} C & \twoheadrightarrow & \{1\} \\ {\scriptstyle \rho_n}\downarrow & & \downarrow \\ C & \twoheadrightarrow & \{1\} \end{array} \tag{1.2}$$

for all $n \in P'$, where $C = \langle t\rangle$ is infinite cyclic, and $\rho_n(t) = t^n$. $\square$

In other words, if we denote by $\mathcal{ED}_P$ the class of P-local relative groups and by $\mathcal{ET}_P$ the set of morphisms (1.2) with $n \in P'$, then $\mathcal{ED}_P = (\mathcal{ET}_P)^{\perp}$. It follows, as in the absolute case, that $(\mathcal{ED}_P)^{\perp\perp} = (\mathcal{ET}_P)^{\perp\perp\perp} = (\mathcal{ET}_P)^{\perp} = \mathcal{ED}_P$, i.e., the class $\mathcal{ED}_P$ is saturated. Thus, if we set $\mathcal{ES}_P = (\mathcal{ED}_P)^{\perp}$, we obtain an orthogonal pair $(\mathcal{ES}_P, \mathcal{ED}_P)$ in the category $\mathcal{EG}$. We call morphisms in $\mathcal{ES}_P$ *P-equivalences* of relative groups.

Proposition 1.3 *If the morphism*

$$\begin{array}{ccc} G & \overset{\varepsilon}{\twoheadrightarrow} & Q \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ G' & \overset{\varepsilon'}{\twoheadrightarrow} & Q' \end{array} \tag{1.3}$$

is a P-equivalence of relative groups, then f is a P-equivalence of groups and $\bar{f}$ is an isomorphism.

PROOF. Given a P-local group L, the relative group $L \twoheadrightarrow \{1\}$ is P-local and therefore orthogonal to (1.3). This provides a bijection

$$f^*\colon \mathrm{Hom}(G', L) \cong \mathrm{Hom}(G, L)$$

and hence proves the first assertion. Similarly, a homomorphism $g\colon Q' \to Q$ such that $g\bar{f} = \mathrm{id}$ is obtained using the fact that (1.3) is orthogonal to the relative group $Q \overset{=}{\twoheadrightarrow} Q$, and the equality $\bar{f}g = \mathrm{id}$ is checked by using the relative group $Q' \overset{=}{\twoheadrightarrow} Q'$ in the same way. $\square$

In fact, it is possible to state a necessary and sufficient condition for a morphism between relative groups to be a P-equivalence, but the proof requires additional material; see Corollary 1.7.

Theorem 1.4 *The inclusion of the full subcategory of P-local relative groups in the category $\mathcal{EG}$ of relative groups has a left adjoint.*

PROOF. We want to show that Theorem 1.1 applies here. A diagram of relative groups $F\colon \Lambda \to \mathcal{EG}$ gives rise to two diagrams F_1, F_2 in the category of groups, together with a natural transformation $F_1 \to F_2$. Since the category of groups is cocomplete, there is a relative group $\mathrm{colim}\, F_1 \twoheadrightarrow \mathrm{colim}\, F_2$, which is easily seen to be a colimit of F. Hence, the category $\mathcal{EG}$ is cocomplete. The set $\mathcal{ET}_P$ defined in Proposition 1.2 satisfies $(\mathcal{ET}_P)^\perp = \mathcal{ED}_P$, and the relative group $C \twoheadrightarrow \{1\}$, C infinite cyclic, is presentable in $\mathcal{EG}$. Now our claim follows from Theorem 1.1. $\square$

We shall use the term *P-localization* of relative groups to denote the left adjoint given by Theorem 1.4. For a given object ε in $\mathcal{EG}$, the P-localizing morphism (i.e., the morphism induced by the unit of the adjunction) will be denoted by

$$\begin{array}{ccc} G & \overset{\varepsilon}{\twoheadrightarrow} & Q \\ \lambda \downarrow & & \downarrow = \\ G_{(P)} & \overset{\varepsilon_{(P)}}{\twoheadrightarrow} & Q. \end{array} \tag{1.4}$$

Thus the morphism $\varepsilon \to \varepsilon_{(P)}$ is initial among all morphisms from ε to P-local relative groups, and terminal among all P-equivalences going out of ε. Note that, by Proposition 1.3, we can always choose $\varepsilon_{(P)}$ within its isomorphism class so that the right-hand vertical homomorphism in (1.4) is precisely the identity.

The notation $G_{(P)} \twoheadrightarrow Q$ in (1.4) is taken from [14], where it was first used to denote the P-localization of $G \twoheadrightarrow Q$ as a relative group, in a special case. The group $G_{(P)}$ should not be confused with G_P, the (absolute) P-localization of G. By Corollary 1.6 below, one might think of $G_{(P)}$ as obtained "by localizing $\ker \varepsilon$ inside G". In particular, the absolute P-localization of G can be recovered as the relative P-localization of $G \twoheadrightarrow \{1\}$.

We devote the rest of this section to showing that the functor given by Theorem 1.4 extends, indeed, the functor constructed by Hilton in [14]. That is, if the kernel of $\varepsilon\colon G \twoheadrightarrow Q$ is nilpotent, then the two constructions agree.

Proposition 1.5 *The restriction $\hat{\lambda}\colon \ker \varepsilon \to \ker \varepsilon_{(P)}$ in (1.4) is a P-equivalence of groups.*

Proof. Our argument requires a careful inspection of the procedure used in the construction of $\varepsilon_{(P)}$, along the lines of the proof of Theorem 1.1 (as given e.g. in [7, Theorem 1.4]). The relative group $\varepsilon_{(P)}$ is obtained as a direct limit

$$\varepsilon_{(P)} = \varinjlim \varepsilon_i$$

of a direct system of relative groups $\varepsilon_i\colon G_i \twoheadrightarrow Q$, with $\varepsilon_0 = \varepsilon$, and G_{i+1} has the form

$$G_{i+1} = (G_i * F_i)/N_i, \tag{1.5}$$

where F_i and N_i are defined as follows: F_i is a free group with one generator $y(x)$ for every $x \in \ker \varepsilon_i$ without an nth root for some $n \in P'$, and one generator $z(u,v)$ for every pair of elements $u, v \in \ker \varepsilon_i$ such that $u^n = v^n$ for some $n \in P'$; N_i is the normal subgroup of $G_i * F_i$ generated by the words $y(x)^n x^{-1}$, $z(u,v)u^{-1}$,

$z(u,v)v^{-1}$. The homomorphism $\varepsilon_{i+1}\colon G_{i+1} \twoheadrightarrow Q$ is defined by $\varepsilon_{i+1}(xN_i) = \varepsilon_i(x)$ if $x \in G_i$, and $\varepsilon_{i+1}(xN_i) = 1$ if $x \in F_i$. The morphism

$$\begin{array}{ccc} G_i & \stackrel{\varepsilon_i}{\twoheadrightarrow} & Q \\ {\scriptstyle s_i}\downarrow & & \downarrow {\scriptstyle =} \\ G_{i+1} & \stackrel{\varepsilon_{i+1}}{\twoheadrightarrow} & Q \end{array} \tag{1.6}$$

is defined by $s_i(x) = xN_i$. Now it suffices to show that

$$\hat{s}_i\colon \ker \varepsilon_i \to \ker \varepsilon_{i+1}$$

is a P-equivalence of groups for every $i \geq 0$. Fix such an i. Consider the epimorphism

$$G_i * F_i \stackrel{\kappa}{\twoheadrightarrow} Q \tag{1.7}$$

given by $\kappa(x) = \varepsilon_i(x)$ if $x \in G_i$, and $\kappa(F_i) = \{1\}$. Then $N_i \subseteq \ker \kappa$ and

$$(\ker \kappa)/N_i = \ker \varepsilon_{i+1}. \tag{1.8}$$

Using the methods of Bass-Serre (see [25, §5, Théorème 14]), the kernel of κ is seen to be a free product

$$\ker \kappa = (\ker \varepsilon_i) * F', \tag{1.9}$$

where F' is the free group on the set $\{gy(x)g^{-1},\ gz(u,v)g^{-1}\}$, where $y(x)$, $z(u,v)$ are the generators of F_i, and g ranges within a set of representatives of cosets of G_i mod $\ker \varepsilon_i$.

Now assume there is given a homomorphism $\psi\colon \ker \varepsilon_i \to L$ with L P-local. Then, by (1.9), ψ admits an extension $\tilde{\psi}\colon \ker \kappa \to L$, which is unique if we impose the condition $\tilde{\psi}(N_i) = \{1\}$. Therefore, ψ factors uniquely through a homomorphism $\psi'\colon \ker \varepsilon_{i+1} \to L$ by (1.8), and this shows that

$$(\hat{s}_i)^*\colon \mathrm{Hom}(\ker \varepsilon_{i+1}, L) \cong \mathrm{Hom}(\ker \varepsilon_i, L),$$

as desired. $\square$

Corollary 1.6 *A morphism*

$$\begin{array}{ccc} G & \stackrel{\varepsilon}{\twoheadrightarrow} & Q \\ {}^{f}\downarrow & & \downarrow{}^{=} \\ G' & \stackrel{\varepsilon'}{\twoheadrightarrow} & Q \end{array} \tag{1.10}$$

is a P-localization of relative groups if and only if the induced homomorphism $\hat{f}\colon \ker\varepsilon \to \ker\varepsilon'$ is a P-localization of groups.

Proof. If (1.10) P-localizes, then $\hat{f}$ P-localizes by Proposition 1.5. Conversely, assume that $\hat{f}$ P-localizes. Then (1.4) and (1.10) give rise to a commutative diagram

$$\begin{array}{ccccc} \ker\varepsilon_{(P)} & \rightarrowtail & G_{(P)} & \stackrel{\varepsilon_{(P)}}{\twoheadrightarrow} & Q \\ {}^{\hat{g}}\downarrow & & {}^{g}\downarrow & & {}^{=}\downarrow \\ \ker\varepsilon' & \rightarrowtail & G' & \stackrel{\varepsilon'}{\twoheadrightarrow} & Q \end{array} \tag{1.11}$$

with $g\lambda = f$ and $\hat{g}\hat{\lambda} = \hat{f}$. But $\hat{\lambda}$ and $\hat{f}$ are both P-equivalences, so that $\hat{g}$ is also a P-equivalence and hence an isomorphism. It follows that g is also an isomorphism. □

Corollary 1.7 *The morphism* (1.3) *is a P-equivalence of relative groups if and only if $\bar{f}$ is an isomorphism and the restriction $\hat{f} : \ker\varepsilon \to \ker\varepsilon'$ is a P-equivalence of groups.* □

Given a relative group $\varepsilon\colon G \twoheadrightarrow Q$, the conjugation action of G on $\ker\varepsilon$ induces a unique action on $(\ker\varepsilon)_P$ such that $l\colon \ker\varepsilon \to (\ker\varepsilon)_P$ is a G-homomorphism. Consider the semidirect product $(\ker\varepsilon)_P \rtimes G$ with respect to this action, and call H the subgroup generated by the elements of the form $(l(x), x^{-1})$, $x \in \ker\varepsilon$. Then the following diagram is commutative and its rows are exact (cf. [3, page 117] or [14, Proposition 1.1])

$$\begin{array}{ccccc} \ker\varepsilon & \rightarrowtail & G & \stackrel{\varepsilon}{\twoheadrightarrow} & Q \\ {}^{l}\downarrow & & \downarrow & & {}^{=}\downarrow \\ (\ker\varepsilon)_P & \rightarrowtail & ((\ker\varepsilon)_P \rtimes G)/H & \stackrel{\varepsilon'}{\twoheadrightarrow} & Q. \end{array} \tag{1.12}$$

It follows from Corollary 1.6 that $\varepsilon' \cong \varepsilon_{(P)}$, so that we can compute $G_{(P)}$ in practice provided we have a sufficiently good description of $(\ker\varepsilon)_P$. The diagram (1.12) also shows that our functor coincides with the one defined in [14] when $\ker\varepsilon$ is nilpotent.

2 On the lack of exactness of P-localization

It is well-known that P-localization is *not* an exact functor in the category of all groups. An easy counterexample is given by the extension

$$\mathbf{Z}/3 \rightarrowtail \Sigma_3 \twoheadrightarrow \mathbf{Z}/2, \tag{2.1}$$

where Σ_3 is the symmetric group on three elements, since localization of (2.1) at $P = \{3\}$ gives the sequence $\mathbf{Z}/3 \to \{1\} \to \{1\}$. However, P-localization always preserves epimorphisms ([22, Proposition 10]). Moreover, for every extension $N \rightarrowtail G \twoheadrightarrow Q$ there is a commutative diagram with exact rows

$$\begin{array}{ccccc} N & \overset{\iota}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & Q \\ {\scriptstyle\varphi}\downarrow & & {\scriptstyle l}\downarrow & & {\scriptstyle l}\downarrow \\ \ker(\varepsilon_P) & \rightarrowtail & G_P & \overset{\varepsilon_P}{\twoheadrightarrow} & Q_P, \end{array} \tag{2.2}$$

where φ is the composite of the following homomorphisms:

$$N \xrightarrow{l} N_P \overset{\iota_P}{\twoheadrightarrow} \operatorname{im}(\iota_P) \hookrightarrow \ker(\varepsilon_P). \tag{2.3}$$

If G is nilpotent, then $\ker(\varepsilon_P) \cong N_P$ and φ P-localizes ([17, Theorem I.2.4]). In general, $\ker(\varepsilon_P)$ is the smallest P'-ideal ([2]) of G_P containing $l(\iota(N))$, cf. [11, Teorema 1.3.4]. An interesting abstract description of $\ker(\varepsilon_P)$ was given in [20] in the case when the extension $N \rightarrowtail G \twoheadrightarrow Q$ splits. Other characterizations of $\ker(\varepsilon_P)$ in special cases are contained in [21] and [24].

Our main theorem below precisely identifies $\ker(\varepsilon_P)$ in (2.2) when N is nilpotent and Q is torsion (a group is said to be *torsion* if all its elements have finite order).

Theorem 2.1 *Let $N \overset{\iota}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$ be a group extension in which Q is torsion and N is nilpotent of class at most c. Let P be a set of primes. Denote by S the kernel of $l\colon Q \to Q_P$, and set $K = \varepsilon^{-1}(S)$. Then:*

(a) *There is a group extension*

$$(K/\Gamma^c K)_P \rightarrowtail G_P \overset{\varepsilon_P}{\twoheadrightarrow} Q_P, \tag{2.4}$$

where $\Gamma^k K$, $k \geq 0$, denotes the lower central series of the group K.

(b) *If S is nilpotent, then there is a commutative diagram with exact rows*

$$\begin{array}{ccccc} N & \overset{\iota}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & Q \\ {\scriptstyle l\pi}\downarrow & & {\scriptstyle l}\downarrow & & {\scriptstyle l}\downarrow \\ (N/\Gamma_K^c N)_P & \rightarrowtail & G_P & \overset{\varepsilon_P}{\twoheadrightarrow} & Q_P, \end{array} \tag{2.5}$$

in which $\Gamma_K^k N$, $k \geq 0$, denotes the lower central series of the conjugation action of K on N, and π is the projection $N \twoheadrightarrow N/\Gamma_K^c N$. Moreover, the following sequence is exact:

$$\Gamma_K^c(N_P) \rightarrowtail N_P \overset{\iota_P}{\to} G_P \overset{\varepsilon_P}{\twoheadrightarrow} Q_P. \tag{2.6}$$

Before proving this theorem, we need a few observations:

Lemma 2.2 *Let $N \rightarrowtail G \overset{\varepsilon}{\twoheadrightarrow} Q$ be a group extension. Then ε is a P-equivalence if and only if every homomorphism $\varphi\colon G \to L$ with L P-local satisfies $\varphi(N) = \{1\}$.*

Proof. This follows directly from the definition. □

Lemma 2.3 *A torsion group is P-local if and only if it is P-torsion. If G is a torsion group, then the P-localization homomorphism $l\colon G \to G_P$ is surjective, and* $\ker l$ *is generated by the P'-torsion elements of G.*

Proof. A P-local group cannot contain nontrivial P'-torsion, because $x^n = 1$ with $n \in P'$ forces $x = 1$. Conversely, if G is P-torsion, then it is automatically

P-local ([24, Corollary 6.20]). Now, if G is torsion and S is the subgroup of G generated by its P'-torsion elements, then G/S is P-torsion and, by Lemma 2.2, the projection $G \twoheadrightarrow G/S$ is a P-equivalence. Hence, $G_P \cong G/S$. □

Lemma 2.4 (cf. [2, Theorem 11.5]) *Let $N \rightarrowtail G \twoheadrightarrow Q$ be a group extension in which N is P-local and Q is P-torsion. Then G is P-local.* □

Corollary 2.5 *Given a group extension $N \overset{\iota}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$ in which Q is P-torsion, the sequence $\{1\} \to N_P \xrightarrow{\iota_P} G_P \xrightarrow{\varepsilon_P} Q_P \to \{1\}$ is exact.*

Proof. This was first pointed out in [11, Teorema 1.4.5]. Our own argument is as follows. Consider the commutative diagram with exact rows

$$\begin{array}{ccccc} N & \overset{\iota}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & Q \\ {\scriptstyle l}\downarrow & & {\scriptstyle \lambda}\downarrow & & {\scriptstyle =}\downarrow \\ \ker \varepsilon_{(P)} & \rightarrowtail & G_{(P)} & \overset{\varepsilon_{(P)}}{\twoheadrightarrow} & Q \end{array} \tag{2.7}$$

given by (1.4). Thus $\varepsilon_{(P)}\colon G_{(P)} \twoheadrightarrow Q$ is the P-localization of $\varepsilon\colon G \twoheadrightarrow Q$ as a relative group. By Corollary 1.6, $\ker \varepsilon_{(P)} \cong N_P$. Since Q is P-torsion, Lemma 2.3 tells us that $Q = Q_P$. By Lemma 2.4, $G_{(P)}$ is P-local, and, by Proposition 1.3, λ is a P-equivalence, ensuring that $G_{(P)} \cong G_P$. Finally, the commutativity of (2.7), together with the universal property of P-localization, tell us that the bottom maps in (2.7) are ι_P and ε_P, up to isomorphism. □

Lemma 2.6 *Let $N \rightarrowtail G \overset{\varepsilon}{\twoheadrightarrow} Q$ be a group extension in which*

(i) *N is nilpotent of class at most c, and*

(ii) *Q is generated by P'-torsion elements.*

Then each homomorphism $\varphi\colon G \to L$ with L P-local satisfies $\varphi(\Gamma^c G) = \{1\}$. Furthermore, G_P is nilpotent of class at most c.

Proof. Let I be the P'-isolator ([22]) of $\varphi(N)$ in the group L, i.e., the smallest P-local subgroup of L containing $\varphi(N)$. By [2, Theorem 15.1], I is nilpotent of class at most c. We next prove that

$$\varphi(G) \subseteq I. \tag{2.8}$$

Given $y \in G$, write $\varepsilon(y) = \varepsilon(x_1) \cdots \varepsilon(x_r)$, $x_i \in G$, where each $\varepsilon(x_i)$ is a P'-torsion element in Q. Then we can choose an integer $m \in P'$ such that $(x_i)^m \in N$ for all $i = 1, \ldots, r$. Now

$$\varphi(y) = \varphi(yx_r^{-1} \cdots x_1^{-1})\varphi(x_1) \cdots \varphi(x_r)$$

belongs to I, because $y(x_1 \cdots x_r)^{-1} \in N$ and $\varphi(x_i)^m \in \varphi(N)$ for all i. This proves (2.8). Next, observe that $\varphi(\Gamma^c G) \subseteq \Gamma^c I = \{1\}$, which proves our first assertion.

Now choose φ to be the P-localization homomorphism $l\colon G \to G_P$. By (2.8), $l(G)$ is nilpotent of class at most c. But the P'-isolator of $l(G)$ in G_P is G_P itself ([22, Proposition 6]), and hence the group G_P is also nilpotent of class at most c. $\square$

Proof of Theorem 2.1. The assumption that Q is torsion ensures that $l\colon Q \to Q_P$ is surjective (Lemma 2.3). Now observe that K is precisely the kernel of the composition $G \overset{\varepsilon}{\twoheadrightarrow} Q \overset{l}{\twoheadrightarrow} Q_P$. Moreover, K contains all elements $x \in G$ such that $\varepsilon(x) = 1$. Thus we can consider the group extensions

$$N \rightarrowtail K \overset{\varepsilon}{\twoheadrightarrow} S, \tag{2.9}$$

$$K \rightarrowtail G \overset{l\varepsilon}{\twoheadrightarrow} Q_P. \tag{2.10}$$

We look first at (2.10). Since Q_P is P-torsion, from Corollary 2.5 we obtain a commutative diagram with exact rows

$$\begin{array}{ccccc} K & \rightarrowtail & G & \overset{l\varepsilon}{\twoheadrightarrow} & Q_P \\ {\scriptstyle l}\downarrow & & {\scriptstyle l}\downarrow & & {\scriptstyle =}\downarrow \\ K_P & \rightarrowtail & G_P & \overset{\varepsilon_P}{\twoheadrightarrow} & Q_P. \end{array} \tag{2.11}$$

Now we work with (2.9). By Lemma 2.3, S is generated by P'-torsion elements. Therefore, by Lemma 2.6, each homomorphism $\varphi\colon K \to L$ with L P-local satisfies $\varphi(\Gamma^c K) = \{1\}$, and hence, by Lemma 2.2, the projection

$K \twoheadrightarrow K/\Gamma^c K$ is a P-equivalence. In other words, $K_P \cong (K/\Gamma^c K)_P$, which proves part (a).

Next, assume that S is nilpotent. Then, since $\Gamma^c_K N \subseteq \Gamma^c_K K = \Gamma^c K$, Lemma 2.6 tells us that every homomorphism $\varphi : K \to L$ with L P-local satisfies $\varphi(\Gamma^c_K N) = \{1\}$. Hence, by Lemma 2.2, the projection ν in the diagram

$$\begin{array}{ccccc} N & \rightarrowtail & K & \overset{\varepsilon}{\twoheadrightarrow} & S \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \nu}\downarrow & & {\scriptstyle =}\downarrow \\ N/\Gamma^c_K N & \rightarrowtail & K/\Gamma^c_K N & \twoheadrightarrow & S \end{array} \tag{2.12}$$

is a P-equivalence. Now the conjugation action of $K/\Gamma^c_K N$ on $N/\Gamma^c_K N$ is nilpotent. Since S is assumed to be nilpotent, $K/\Gamma^c_K N$ is also nilpotent ([15, Theorem 1.1g]), and P-localization preserves exactness in the bottom row of (2.12). Since $S_P = \{1\}$, we obtain a commutative diagram

$$\begin{array}{ccc} N & \rightarrowtail & K \\ {\scriptstyle l\pi}\downarrow & & \downarrow{\scriptstyle l\nu} \\ (N/\Gamma^c_K N)_P & \overset{\cong}{\to} & (K/\Gamma^c_K N)_P \end{array} \tag{2.13}$$

in which the composition $l\nu$ P-localizes because ν is a P-equivalence. Putting diagrams (2.13) and (2.11) together, we obtain diagram (2.5). Finally, since the homomorphism $N/\Gamma^c_K N \to N_P/\Gamma^c_K(N_P)$ P-localizes ([12, Theorem 2.8]), the exactness of (2.6) follows from the exactness of the bottom row in (2.5). □

The assumption that S be nilpotent cannot be removed from part (b) of Theorem 2.1, as the following counterexample shows.

Example 2.7 Take $G = SL_2(5)$, $Q = PSL_2(5)$, and $P = \{2\}$. Then $N \cong \mathbf{Z}/2$ is central in G. Hence, $\Gamma^1_K N = \{0\}$ and $(N/\Gamma^1_K N)_P \cong \mathbf{Z}/2$. But G is generated by P'-torsion elements, so that $G_P = \{1\}$.

On the other hand, the hypothesis on Q in part (a) can be weakened. To carry out the proof of part (a), we only need, for a fixed set P, that Q_P be a torsion group and that S be generated by P'-torsion elements. This happens, of course, if Q is torsion, but also in other interesting cases, for example when Q itself is generated by P'-torsion elements.

3 Applications to the study of virtually nilpotent groups

A group G is *virtually nilpotent* if it contains a nilpotent subgroup H of finite index. If this is the case, then H contains a subgroup N which is normal in G and still of finite index in G. It follows that G has a *Fitting subgroup*, i.e., a unique normal nilpotent subgroup $F(G)$ such that any other normal nilpotent subgroup is contained in it. We may thus associate to G, in a canonical way, a group extension

$$F(G) \rightarrowtail G \stackrel{\varepsilon}{\twoheadrightarrow} Q(G) \tag{3.1}$$

in which $F(G)$ is nilpotent and $Q(G)$ is finite. Let us call (3.1) the *Fitting extension* associated to G, and $Q(G)$ the *Fitting quotient*. Given a set of primes P, we denote by $S(G,P)$ the kernel of $l\colon Q(G) \to Q(G)_P$, and by $K(G,P)$ its preimage in G.

We devote this section to pointing out several consequences of Theorem 2.1 applied to the extension (3.1).

Proposition 3.1 *Let G be a virtually nilpotent group, and P a set of primes for which $S(G,P)$ is nilpotent. Then the kernel of $l\colon G \to G_P$ is the set of elements $x \in G$ such that $x^m \in \Gamma^c_K F(G)$ for some $m \in P'$, where c is the nilpotency class of $F(G)$, and $K = K(G,P)$.*

Proof. By part (b) of Theorem 2.1, we have $l(\Gamma^c_K F(G)) = \{1\}$. If $x^m \in \Gamma^c_K F(G)$ with $m \in P'$, then $l(x)^m = 1$ and hence $l(x) = 1$ because G_P is P-local. Conversely, if $l(x) = 1$, then $\varepsilon(x)$ belongs to $S(G,P)$, which is nilpotent and generated by P'-torsion elements, and hence P'-torsion itself. Thus $x^n \in F(G)$ for some $n \in P'$. But x^n is sent to 1 by the composition $F(G) \twoheadrightarrow F(G)/\Gamma^c_K F(G) \to (F(G)/\Gamma^c_K F(G))_P$. Therefore the class of x^n in $F(G)/\Gamma^c_K F(G)$ has P'-order, and this implies that $x^m \in \Gamma^c_K F(G)$ for some $m \in P'$. $\square$

Proposition 3.2 *Let G be a virtually nilpotent group, and P a set of primes for which $S(G,P)$ is nilpotent. Then the homomorphisms s_i in the tower*

$$\cdots \to F(G)/\Gamma_K^{i+1}F(G) \xrightarrow{s_i} F(G)/\Gamma_K^{i}F(G) \to \cdots \xrightarrow{s_0} \{1\}, \tag{3.2}$$

where $K = K(G,P)$, are P-equivalences for i greater than or equal to the nilpotency class of $F(G)$.

Proof. By Theorem 2.1, $(F(G)/\Gamma_K^i F(G))_P$ is isomorphic to $\ker(\varepsilon_P)$ for all i greater than or equal to the nilpotency class of $F(G)$. □

Theorem 3.3 *If G is virtually nilpotent, then, for every set of primes P, its P-localization G_P is also virtually nilpotent.*

Proof. Consider the Fitting extension (3.1) associated to G, and let c be the nilpotency class of $F(G)$. By part (a) of Theorem 2.1, there is a group extension $N \rightarrowtail G_P \twoheadrightarrow Q(G)_P$ in which N is nilpotent of class at most c. Since $Q(G)_P$ is finite (by Lemma 2.3), the group G_P is virtually nilpotent. □

Corollary 3.4 *If G is a virtually nilpotent subgroup of a P-local group L, then the P'-isolator I of G in L is also virtually nilpotent.*

Proof. Let $\iota\colon G \hookrightarrow L$ be the inclusion. Then $\iota_P(G_P)$ is a P-local subgroup of L containing G, and hence I is contained in $\iota_P(G_P)$. But the class of virtually nilpotent groups is closed under taking subgroups and epimorphic images, so that our claim follows from Theorem 3.3. □

Corollary 3.4 says precisely that the property of being virtually nilpotent is *closure-preserved* in the sense of [8].

Example 3.5 Consider the fundamental group of the Klein bottle

$$\pi = \langle x, y \mid yxy^{-1} = x^{-1} \rangle.$$

There is a (non-split) extension

$$\mathbf{Z} \oplus \mathbf{Z} \overset{\iota}{\rightarrowtail} \pi \overset{\varepsilon}{\twoheadrightarrow} \mathbf{Z}/2, \tag{3.3}$$

where $\iota(1,0) = x$, $\iota(0,1) = y^2$, $\varepsilon(x) = 1$, $\varepsilon(y) = \xi$, by writing $\mathbf{Z}/2 = \langle \xi \mid \xi^2 = 1 \rangle$. Hence, π is virtually nilpotent and (3.3) is its associated Fitting extension. If $2 \in P$, then Theorem 2.1 gives a commutative diagram of extensions

$$\begin{array}{ccccc} \mathbf{Z} \oplus \mathbf{Z} & \rightarrowtail & \pi & \twoheadrightarrow & \mathbf{Z}/2 \\ {\scriptstyle l} \downarrow & & {\scriptstyle l} \downarrow & & {\scriptstyle =} \downarrow \\ \mathbf{Z}_P \oplus \mathbf{Z}_P & \rightarrowtail & \pi_P & \twoheadrightarrow & \mathbf{Z}/2. \end{array} \tag{3.4}$$

In particular, $l\colon \pi \to \pi_P$ is a monomorphism if $2 \in P$. On the other hand, if $2 \notin P$, then $K(\pi, P) = \pi$ and the projection $\pi \twoheadrightarrow \pi/\Gamma^1\pi$ is a P-equivalence. Thus

$$\pi_P \cong \mathbf{Z}_P, \tag{3.5}$$

and the kernel of $l\colon \pi \to \pi_P$ is the subgroup generated by x. Observe also that, if $2 \notin P$, then the tower (3.2) takes the form:

$$\cdots \to (\mathbf{Z}/8) \oplus \mathbf{Z} \to (\mathbf{Z}/4) \oplus \mathbf{Z} \to (\mathbf{Z}/2) \oplus \mathbf{Z} \to 0.$$

Example 3.6 Consider the infinite dihedral group $D_\infty = \langle u, v \mid u^2 = v^2 = 1 \rangle$. One obtains, as in Example 3.5, that

$$(D_\infty)_P \cong \begin{cases} \mathbf{Z}_P \rtimes (\mathbf{Z}/2) & \text{if } 2 \in P \\ \{1\} & \text{if } 2 \notin P. \end{cases} \tag{3.6}$$

It is well-known that, if G is nilpotent and $l\colon G \to G_P$ is its P-localization, then the homomorphisms

$$l_*\colon H_k(G) \to H_k(G_P) \tag{3.7}$$

P-localize for all $k \geq 1$ and each set of primes P. This is false, in general, for nonnilpotent groups. We wish to obtain information on (3.7) when the group G is virtually nilpotent. The next observation is an immediate consequence of Corollary 2.5.

Theorem 3.7 *The class of groups for which the homomorphism $l\colon G \to G_P$ induces P-localization on integral homology is closed under extensions by finite P-groups.*

Proof. Given a group extension $N \rightarrowtail G \twoheadrightarrow Q$ in which Q is a finite P-group, by Corollary 2.5 we have a commutative diagram with exact rows

$$\begin{array}{ccccc} N & \rightarrowtail & G & \twoheadrightarrow & Q \\ {\scriptstyle l}\downarrow & & {\scriptstyle l}\downarrow & & {\scriptstyle =}\downarrow \\ N_P & \rightarrowtail & G_P & \twoheadrightarrow & Q. \end{array}$$

Now, if we assume that $l_*\colon H_k(N) \to H_k(N_P)$ P-localizes for all $k \geq 1$, then we can use the induced morphism at the corresponding Lyndon-Hochschild-Serre spectral sequences with $\mathbf{Z}_P$ coefficients to obtain an isomorphism

$$l_*\colon H_k(G; \mathbf{Z}_P) \cong H_k(G_P; \mathbf{Z}_P)$$

for all $k \geq 1$. But, since both N_P and Q have P-local integral homology groups, so does G_P. Hence, $l_*\colon H_k(G) \to H_k(G_P)$ P-localizes for all $k \geq 1$. □

Corollary 3.8 *If G is virtually nilpotent and the set P contains all prime divisors of the order of its Fitting quotient $Q(G)$, then $l : G \to G_P$ induces P-localization on integral homology.* □

Example 3.9 Let G be either the fundamental group π of the Klein bottle or the infinite dihedral group D_∞ (Examples 3.5 and 3.6). Corollary 3.8 applies if $2 \in P$. But, in the case $2 \notin P$, direct computation shows that $l\colon G \to G_P$ *also* induces P-localization on integral homology; to carry out this computation one may use Theorem 3.10 below.

The groups in Example 3.9 are somehow exceptional. If G is an arbitrary virtually nilpotent group, no good homological behaviour is to be expected at the primes not dividing the order of the Fitting quotient. For example, the Fitting extension of the symmetric group Σ_3 is

$$\mathbf{Z}/3 \rightarrowtail \Sigma_3 \twoheadrightarrow \mathbf{Z}/2, \tag{3.8}$$

and for $P = \{3\}$ one has $(\Sigma_3)_P = \{1\}$, while $H_*(\Sigma_3)$ contains nontrivial 3-torsion. In fact, the following holds.

Theorem 3.10 *Let $A \rightarrowtail G \twoheadrightarrow S$ be a group extension in which A is commutative and S is a finite P'-torsion nilpotent group. Then, for each given $k \geq 1$, the homomorpism $l_*\colon H_k(G) \to H_k(G_P)$ P-localizes if and only if the natural homomorphism*

$$H_0(S; H_k(A)) \to H_k(H_0(S; A))$$

is a P-equivalence.

(Note that, since A is commutative, the action of S on $H_*(A)$ is induced by an action of S on A. We are thus measuring the difference between dividing out the action of S before and after applying the homology functors. For example, in the extension (3.8) we have $H_0(S; A) = 0$, while S acts trivially on $H_k(A) \cong \mathbf{Z}/3$ for $k = 3, 7, 11, \ldots$)

Proof. By part (b) of Theorem 2.1, there is a commutative diagram of extensions

$$\begin{array}{ccccc} A & \rightarrowtail & G & \twoheadrightarrow & S \\ {\scriptstyle l\pi}\downarrow & & {\scriptstyle l}\downarrow & & \downarrow \\ H_0(S; A)_P & \rightarrowtail & G_P & \twoheadrightarrow & \{1\}. \end{array} \tag{3.9}$$

Consider the induced morphism at the corresponding Lyndon-Hochschild-Serre spectral sequences with $\mathbf{Z}_P$ coefficients. Since S is a finite P'-torsion group, both sequences collapse at the E^2-term and give a commutative diagram

$$\begin{array}{ccc} H_0(S; H_k(A; \mathbf{Z}_P)) & \stackrel{\cong}{\to} & H_k(G; \mathbf{Z}_P) \\ \downarrow & & \downarrow {\scriptstyle l_*} \\ H_k(H_0(S; A)_P; \mathbf{Z}_P) & \stackrel{\cong}{\to} & H_k(G_P; \mathbf{Z}_P) \end{array} \tag{3.10}$$

for all $k \geq 1$. Now, since G_P is commutative, $l_*\colon H_k(G) \to H_k(G_P)$ P-localizes if and only if the right-hand arrow in (3.10) is an isomorphism, and this is equivalent to the condition stated. □

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