EXTENDED GENUS OF TORSION-FREE FINITELY GENERATED NILPOTENT GROUPS OF CLASS TWO

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Abstract

The extended genus of a nilpotent group N is the set of isomorphism classes of nilpotent groups M, *not necessarily finitely generated*, such that the p-localizations M_p, N_p are isomorphic for all primes p. In this article, for any torsion-free finitely generated nilpotent group N of nilpotency class 2, the extended genus of N is analyzed by assigning to each of its members a sequence of triads of matrices with rational entries, generalizing the sequential representation which has been exploited elsewhere in the case when N is abelian. This approach allows, among other things, to obtain examples of groups in the ordinary (Mislin) genus of N.

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0 Introduction

Torsion-free finitely generated nilpotent groups of class 2 have been extensively studied; see e.g. [4], [5], [7], [10], [12]. Every such group N may be written as a central extension $\mathbf{Z}^m \rightarrowtail N \twoheadrightarrow \mathbf{Z}^n$, where the commutator operation in N defines an alternating bilinear map $\mathbf{Z}^n \times \mathbf{Z}^n \to \mathbf{Z}^m$. The analysis of this map gives a big deal of information about N; indeed, it has been used to classify torsion-free finitely generated nilpotent groups of class 2 up to Hirsch length 6. This has been achieved in [6], [7], where the problem of deciding whether two such groups are in the same Pickel genus (see [13]) has also been considered.

We warn the reader that, in the present paper, we will be dealing with genus in the sense of Mislin ([11]), instead of Pickel. The Mislin genus of a finitely generated nilpotent group N is the set of isomorphism classes of finitely generated nilpotent groups M such that the p-localizations N_p, M_p (as defined e.g. in [8]) are isomorphic for all primes p. It is well-known that, if N, M are in the same Mislin genus, then they are also in the same Pickel genus. Thus, both Pickel genera and Mislin genera are finite sets.

It is possible to dispense with the restriction that groups be finitely generated if one is willing to deal with possibly infinite (even uncountable) genus sets. Thus, if N, M are nilpotent groups, not necessarily finitely generated, such that $N_p \cong M_p$ for all primes p, then we say, as in [2], that N and M are in the same *extended genus*. In Section 2 we carry out a very natural generalization of several parts of [2], where suitable tools were given to study groups in the extended genus of $\mathbf{Z}^k$, by associating to every such group a sequence of matrices with rational entries, one for each prime p. A similar representation is described in the present paper for torsion-free finitely generated nilpotent groups of class 2 (in fact, we show that the first steps in the construction are valid for every torsion-free nilpotent group). Given such a group N, we associate to each group M in its extended genus a sequence of triads of matrices $\{A_p, B_p, C_p\}$ with rational entries, which is uniquely determined up to a certain equivalence relation. If N is a central extension of $\mathbf{Z}^m$ by $\mathbf{Z}^n$, then the sequence $\{A_p\}$ represents a group in the extended genus of $\mathbf{Z}^n$ and the sequence $\{C_p\}$ represents a group in the extended genus of $\mathbf{Z}^m$. A key result

(Theorem 3.4 below) states that, if the group represented by $\{A_p\}$ is finitely generated —which, as explained in [2], can be detected by merely inspecting the entries of each A_p— then M is also finitely generated. Hence, our approach allows to recognize groups in the Mislin genus of N within the extended genus of N by analyzing their associated matrices. Unfortunately, even though we can easily decide if a given sequence of matrices $\{A_p, B_p, C_p\}$ is realized by some group in the extended genus of N, we know of no effective procedure to decide whether two given sequences represent isomorphic groups or not. This is of course a serious difficulty if one aims to computing orders of Mislin genera. On the other hand, we emphasize that the applicability of our approach is not limited by the Hirsch number of N.

More powerful methods for the analysis of genera are available if the Hirsch number of N is less than or equal to 6, or also if the commutator subgroup of N is cyclic. Under any of these restrictions, Mislin genera turn out to be trivial except when $n = 4$, $m = 2$; see [6], [7], or [14, ch. 11]. In the case $n = 4$, $m = 2$, a complete characterization of Mislin genera by means of a finite set of arithmetical invariants has recently been described in [3].

1 The groups $G_\phi(R)$

If N is a torsion-free nilpotent group of class 2, then by considering the isolator of the commutator subgroup

$$I = I([N, N]) = \{x \in N \mid x^k \in [N, N] \text{ for some integer } k\},$$

one obtains a central extension $I \rightarrowtail N \twoheadrightarrow N/I$ of torsion-free nilpotent groups. If one assumes, in addition, that N is finitely generated, then $I \cong \mathbf{Z}^m$ and $N/I \cong \mathbf{Z}^n$ for some positive integers n, m. In this situation, we say, as in [7] or [14, ch. 11], that the group N belongs to the class $T(n, m)$. One can associate with every group N in $T(n, m)$ the map $(N/I) \times (N/I) \to I$ sending (xI, yI) to the commutator $[x, y] = x^{-1}y^{-1}xy$. This map is *full*, in the sense that its image generates a subgroup of rank m (or, equivalently, of finite index) in I. Thus, after choosing bases in I and N/I, one obtains a full alternating bilinear map $\phi\colon \mathbf{Z}^n \times \mathbf{Z}^n \to \mathbf{Z}^m$. Any other choice of bases will give rise to another map ψ, fitting into a commutative diagram

$$\begin{array}{ccc} \mathbf{Z}^n \times \mathbf{Z}^n & \xrightarrow{f\times f} & \mathbf{Z}^n \times \mathbf{Z}^n \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathbf{Z}^m & \xrightarrow{h} & \mathbf{Z}^m \end{array} \tag{1.1}$$

for some $f \in GL_n(\mathbf{Z})$ and $h \in GL_m(\mathbf{Z})$. In fact, the assignment $N \mapsto \phi$ sets up a bijective correspondence between the set of all isomorphism classes of groups in $T(n,m)$ and the set of all equivalence classes (in the sense of (1.1)) of full alternating bilinear maps $\phi\colon \mathbf{Z}^n \times \mathbf{Z}^n \to \mathbf{Z}^m$; see [7, Theorem 2].

We next describe a broader class of torsion-free nilpotent groups, not necessarily finitely generated, which contains the above as a special case. Let R be a subring of $\mathbf{Q}$ containing 1 and $\phi\colon R^n \times R^n \to R^m$ be any full alternating bilinear map. Note that, since the matrix of ϕ is skew-symmetric, the assumption that ϕ be full forces the numbers m, n to satisfy the inequality $m \le \frac{1}{2}n(n-1)$. We denote by $G_\phi(R)$ the group whose underlying set is the Cartesian product $R^n \times R^m$, equipped with the multiplication

$$(x,y)(x',y') = (x+x',\, y+y'+\bar{\phi}(x,x')), \tag{1.2}$$

where $\bar{\phi}\colon R^n \times R^n \to R^m$ is the bilinear map obtained by replacing the entries below the diagonal in the matrix of ϕ with zeros (so that $\phi(x,y) = \bar{\phi}(x,y) - \bar{\phi}(y,x)$ for all x, y). By construction, $G_\phi(R)$ fits into a central extension

$$R^m \rightarrowtail G_\phi(R) \twoheadrightarrow R^n, \tag{1.3}$$

showing that $G_\phi(R)$ is torsion-free nilpotent of class 2. Moreover, the kernel of (1.3) is equal to the isolator $I\left([G_\phi(R), G_\phi(R)]\right)$. The bilinear map $\bar{\phi}$ may be viewed as a 2-cocycle and, as such, it determines a cohomology class $[\bar{\phi}] \in H^2(R^n; R^m)$ attached to the extension (1.3). In this situation, the next result follows by standard arguments; cf. [1], [15].

Theorem 1.1 *Let R, S be subrings of $\mathbf{Q}$ with $1 \in R \subseteq S$. Assume given full alternating bilinear maps $\phi\colon R^n \times R^n \to R^m$, $\psi\colon S^n \times S^n \to S^m$. Then every homomorphism $F\colon G_\phi(R) \to G_\psi(S)$ factors to a homomorphism $f\colon R^n \to S^n$. On the other hand, a homomorphism $f\colon R^n \to S^n$ can be lifted to* *a homomorphism* $F\colon G_\phi(R) \to G_\psi(S)$ *if and only if there is a homomorphism* $h\colon R^m \to S^m$ *such that* $h\phi = \psi(f\times f)$. *If one lifting exists, then* h *is uniquely determined by* f*, and the set of all liftings is in one-to-one correspondence with the set of homomorphisms* $g\colon R^n \to S^m$. $\square$

More specifically, any homomorphism $F\colon G_\phi(R) \to G_\psi(S)$ has the form

$$F(x,y) = (f(x), h(y) + \theta(x)), \tag{1.4}$$

where θ is some function satisfying $\partial\theta = h\bar{\phi} - \bar{\psi}(f\times f)$ and $\theta(0) = 0$; any two choices θ, θ' must differ in a 1-cocycle, i.e., a homomorphism. Note also that a bilinear map $R^n \times R^n \to S^m$ is a coboundary if and only if it is symmetric.

It will be useful to introduce some terminology. In the hypotheses of Theorem 1.1, a homomorphism $f: R^n \to S^n$ will be called *liftable* if there is a (uniquely determined) homomorphism $h\colon R^m \to S^m$ such that $h\phi = \psi(f\times f)$. In other words, f is liftable if it is the factorization of some homomorphism $F\colon G_\phi(R) \to G_\psi(S)$. A more practical way to describe liftable homomorphisms is given in the next lemma.

Lemma 1.2 *A homomorphism* $f\colon R^n \to S^n$ *is liftable if and only if, for every vanishing linear combination of the form* $r_1\phi(x_1,y_1) + \cdots + r_k\phi(x_k,y_k) = 0$ *with* $r_1,\ldots,r_k$ *in* R*, the following equality also holds:*

$$r_1\psi(f(x_1), f(y_1)) + \cdots + r_k\psi(f(x_k), f(y_k)) = 0. \qquad \square$$

In particular, if f satisfies $\phi(x,y) = \psi(f(x), f(y))$ for all x, y, then f is liftable and any lifting F restricts to the canonical inclusion of R^m into S^m. Observe also that, in order to test if a given map $f: R^n \to S^n$ is liftable, it suffices to check the condition of Lemma 1.2 for the vectors $\{e_1,\ldots,e_n\}$ of a fixed basis of R^n. We also record the following remark for further use:

Lemma 1.3 *Given an arbitary basis* $\{e_1,\ldots,e_n\}$ *of* R^n *and a liftable homomorphism* $f\colon R^n \to S^n$*, there is a unique lifting* $F\colon G_\phi(R) \to G_\psi(S)$ *satisfying* $F(e_i,0) = (f(e_i),0)$ *for* $i = 1,\ldots,n$.

Proof. If $h\colon R^m \to S^m$ is the homomorphism determined by f, then we set $\varphi = h\bar{\phi} - \bar{\psi}(f \times f)$ and $\theta(x) = -\frac{1}{2}[\varphi(x,x) - x \cdot \operatorname{diag}\varphi]$, where $\operatorname{diag}\varphi$ denotes the vector whose components are $\varphi(e_1, e_1), \ldots, \varphi(e_n, e_n)$. It follows that $\partial\theta = \varphi$ and $\theta(e_i) = 0$ for $i = 1, \ldots, n$. $\square$

Given a full alternating bilinear map $\phi\colon \mathbf{Z}^n \times \mathbf{Z}^n \to \mathbf{Z}^m$ and a set of primes P, we can consider the unique extension of ϕ to a map $\mathbf{Z}_P^n \times \mathbf{Z}_P^n \to \mathbf{Z}_P^m$, which we denote by the same letter ϕ (here, and in all of what follows, $\mathbf{Z}_P$ denotes the ring of integers localized at P). Then, as a special case of Theorem 1.1, we have a commutative diagram of group extensions

$$\begin{array}{ccccc} \mathbf{Z}^m & \rightarrowtail & G_\phi(\mathbf{Z}) & \twoheadrightarrow & \mathbf{Z}^n \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Z}_P^m & \rightarrowtail & G_\phi(\mathbf{Z}_P) & \twoheadrightarrow & \mathbf{Z}_P^n, \end{array} \tag{1.5}$$

where the vertical arrows are canonical embeddings. Since the first and third vertical arrows in (1.5) are P-localizations, so is also the middle arrow (see [9, Corollary I.2.6]). In other words,

$$G_\phi(\mathbf{Z}_P) \cong G_\phi(\mathbf{Z})_P. \tag{1.6}$$

If a group $N = G_\phi(\mathbf{Z})$ is in the class $T(n,m)$, and M is in the Mislin genus of N, then M also belongs to the class $T(n,m)$. Furthermore, (1.6) and Theorem 1.1 give the following.

Theorem 1.4 *Two groups $G_\phi(\mathbf{Z})$, $G_\psi(\mathbf{Z})$ in $T(n,m)$ are in the same Mislin genus if and only if, for every prime p, there exist $f_p \in \operatorname{Aut}(\mathbf{Z}_p^n)$ and $h_p \in \operatorname{Aut}(\mathbf{Z}_p^m)$ such that $h_p\phi = \psi(f_p \times f_p)$.* $\square$

2 On the extended genus

For a fixed nilpotent group N, the extended genus $EG(N)$ is the set of isomorphism classes of (not necessarily finitely generated) nilpotent groups M such that $N_p \cong M_p$ for all primes p. In this broader sense, extended genera need not be finite, and turn out to be relevant even for abelian groups (for which Mislin genera are always trivial). For example, $EG(\mathbf{Z})$ contains uncountably many non-isomorphic groups.

We recall from [2] that, if M and N are in the same extended genus, then they have the same nilpotency class and their torsion subgroups TM, TN are isomorphic; moreover, M/TM and N/TN still belong to the same extended genus. Thus the essential part of the study is concerned with extended genera of torsion-free nilpotent groups.

If A is finitely generated abelian and torsion-free, then the extended genus of A can be studied by representing it into a certain set of matrix sequences with rational entries (see [2]). In this section, we generalize these methods to torsion-free finitely generated nilpotent groups of class 2.

Before particularizing to this class of groups, however, we point out that some basic results of [2] may be generalized to arbitrary torsion-free nilpotent groups. Thus let N be torsion-free nilpotent; then, for every $M \in EG(N)$ we can fix, for each prime p, an isomorphism $g_p\colon M_p \cong N_p$. We can also choose an isomorphism of the rationalizations $g_0\colon M_0 \cong N_0$ (not necessarily compatible with any of the g_p's). Now, since M is torsion-free, M_p embeds into M_0. Thus we may consider the homomorphism $F_p = g_0(g_p)^{-1}$, which is an injection of N_p into N_0. Note that F_p extends in a unique way to an automorphism of N_0; hence we may view F_p either as a monomorphism from N_p to N_0 or as an automorphism of N_0, depending on the context.

Of course, a different choice of isomorphisms $\{g'_p\}$, g'_0 gives rise to a different family $\{F'_p\}$. But the two families $\{F_p\}$ and $\{F'_p\}$ are then related by $F'_p = \left(g'_0 g_0^{-1}\right) F_p \left(g_p (g'_p)^{-1}\right)$, and hence belong to the same coset in

$$\mathrm{Aut}(N_0) \setminus \prod_p \mathrm{Aut}(N_0) \,/\, \prod_p \mathrm{Aut}(N_p). \tag{2.1}$$

Since two isomorphic groups $M \cong M'$ are obviously represented by the same element in (2.1), we have defined a function

$$\Phi\colon EG(N) \longrightarrow \mathrm{Aut}(N_0) \setminus \prod_p \mathrm{Aut}(N_0) \,/\, \prod_p \mathrm{Aut}(N_p). \tag{2.2}$$

Furthermore, any group $M \in EG(N)$ can be reconstructed inside N_0 from a representing sequence $\{F_p\}$, since

$$\bigcap_p \mathrm{im}\, F_p = g_0(M) \cong M. \tag{2.3}$$

The proof is the same as in [2, Theorem 2.1], for the argument only requires that N be torsion-free. Observe that (2.3) implies that the function Φ is injective. However, Φ is far from being surjective in general. We say that a sequence of monomorphisms $F_p\colon N_p \to N_0$, one for each prime p, is *realizable* if its equivalence class is in the image of Φ; that is, if there exists a group M together with isomorphisms $g_p\colon M_p \cong N_p$ for all primes p and $g_0\colon M_0 \cong N_0$, such that $F_p = g_0(g_p)^{-1}$ for all p. By (2.3), if $\{F_p\}$ is realizable, then such a group M is determined up to isomorphism. We call it the *realization* of $\{F_p\}$. Realizable sequences can be characterized as follows.

Theorem 2.1 *Let N be a torsion-free nilpotent group. Assume given a monomorphism $F_p\colon N_p \to N_0$ for each prime p. Consider $H = \cap_p \mathrm{im}\, F_p$, and identify H_0 with the isolator of H in N_0. Then the following statements are equivalent:*

(i) $H_0 = N_0$.

(ii) $H \in EG(N)$ *and* $\Phi(H) = [\{F_p\}]$.

(iii) $\{F_p\}$ *is realizable.*

Proof. Assume that (i) holds. We first show that $H_p = \mathrm{im}\, F_p$ for all p. Thus fix a prime p, and observe that $\mathrm{im}\, F_p$ is isomorphic to N_p and hence p-local. Therefore, we only have to prove that the inclusion of H in $\mathrm{im}\, F_p$ is p-surjective; cf. [9]. Assume given $x \in \mathrm{im}\, F_p$. Since we are assuming that $H_0 = N_0$, we may pick an integer n such that $x^n \in H$. Write $n = p^k m$ with $(m,p) = 1$, and $y = x^m$, which obviously belongs to $\mathrm{im}\, F_p$. Let q be any prime different from p. Since $\mathrm{im}\, F_q$ is q-local and contains H, it also contains the q'-isolator of H in N_0 (where q' denotes the set of all primes $p \neq q$). But y satisfies $y^{p^k} \in H$, so that $y \in \mathrm{im}\, F_q$ for every $q \neq p$ as well. This tells us that $y \in H$, as desired. We conclude that $H_p = \mathrm{im}\, F_p \cong N_p$, so that $H \in EG(N)$. Moreover, if we define isomorphisms $\{g_p\}$, g_0 by setting $g_p = F_p^{-1}$ for all p, and $g_0 = \mathrm{id}$, then we find that $\{F_p\}$ precisely represents the group H. This shows that (i)$\Rightarrow$(ii).

The implication (ii)$\Rightarrow$(iii) is trivial. We finally prove that (iii)$\Rightarrow$(i). Assume that $\{F_p\}$ realizes a certain group M, and let $g_0\colon M_0 \cong N_0$ be the associated isomorphism of the rationalizations. Then $g_0(M) = H$ by (2.3). Since the restriction $g_0 \colon M \to N_0$ is 0-surjective, it follows that every element $x \in N_0$ has a power in H, showing that $H_0 = N_0$. $\square$

Corollary 2.2 *With the same hypotheses as in Theorem* 2.1*, if we assume in addition that the group N is finitely generated, then $\{F_p\}$ is realizable if and only if $H_0 \cong N_0$.*

Proof. Recall that any monomorphism of the rationalization of a finitely generated nilpotent group into itself must be an isomorphism (this is shown by arguing by induction on the nilpotency class). It follows that, if $H_0 \cong N_0$, then the inclusion of H_0 in N_0 has to be an equality. Then we can use Theorem 2.1 to complete the argument. $\square$

If we omit the hypothesis that N be finitely generated, then it is not enough to assume that $H_0 \cong N_0$ in order to conclude that $\{F_p\}$ is realizable; not even in the abelian case. For example, let N be an infinite direct sum of copies of $\mathbf{Z}$, indexed by the natural numbers, and let F_p be the canonical embedding of N_p into N_0, followed by multiplication by p on the first coordinate. Then $H = \cap_p \mathrm{im}\, F_p$ is isomorphic to N, so that $H_0 \cong N_0$, yet H_0 is properly contained in N_0 and hence $\{F_p\}$ is not realizable.

3 Extended genus of groups in $T(n,m)$

From now on we specialize to the case $N = G_\phi(\mathbf{Z})$, for a certain full alternating bilinear map $\phi \colon \mathbf{Z}^n \times \mathbf{Z}^n \to \mathbf{Z}^m$. In all of what follows, coordinates are referred to the canonical bases of $\mathbf{Q}^n$ and $\mathbf{Q}^m$. By (1.6) and Theorem 1.1, if P is any set of primes, then an automorphism of N_P is determined by three matrices $A \in GL_n(\mathbf{Z}_P)$, $B \in M_{m\times n}(\mathbf{Z}_P)$, $C \in GL_m(\mathbf{Z}_P)$ satisfying $C\phi = A^{\mathrm{t}}\phi A$, where we denote the matrix of ϕ by the same letter. Hence, the representation (2.2) gives us a chance to study the extended genus of groups in $T(n,m)$ by working with matrices with rational entries. The main difficulty is that —already in the abelian case— given two representatives of cosets in (2.1), it may be very hard to decide in practice whether they belong to the same coset or not.

However, we will be able to describe a practical procedure for constructing realizable sequences $\{F_p\}$. Furthermore, it will be possible to decide when the realization of such a sequence is finitely generated.

Thus, assume given an arbitrary family of monomorphisms

$$F_p\colon G_\phi(\mathbf{Z}_p) \to G_\phi(\mathbf{Q}),$$

one for each prime p. Each F_p fits into a commutative diagram

$$\begin{array}{ccccc} \mathbf{Z}_p^m & \rightarrowtail & G_\phi(\mathbf{Z}_p) & \twoheadrightarrow & \mathbf{Z}_p^n \\ \downarrow{\scriptstyle h_p} & & \downarrow{\scriptstyle F_p} & & \downarrow{\scriptstyle f_p} \\ \mathbf{Q}^m & \rightarrowtail & G_\phi(\mathbf{Q}) & \overset{\pi}{\twoheadrightarrow} & \mathbf{Q}^n \end{array} \tag{3.1}$$

where h_p is injective because F_p is injective, and f_p is injective because h_0 and F_0, and hence also f_0, are isomorphisms. As in (1.4), we can write

$$F_p(x, y) = (f_p(x),\, h_p(y) + \theta_p(x)) \tag{3.2}$$

for a certain function $\theta_p\colon \mathbf{Z}_p^n \to \mathbf{Q}^m$ satisfying $\theta_p(0) = 0$.

As we next explain, if $\{F_p\}$ is realizable then both $\{f_p\}$ and $\{h_p\}$ are realizable. However, the converse turns out to be false. If we consider the subgroup $H = \cap_p \mathrm{im}\, F_p$ of $G_\phi(\mathbf{Q})$, then we have a commutative diagram with exact rows,

$$\begin{array}{ccccc} \mathbf{Q}^m \cap H & \rightarrowtail & H & \twoheadrightarrow & \pi(H) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Q}^m & \rightarrowtail & G_\phi(\mathbf{Q}) & \overset{\pi}{\twoheadrightarrow} & \mathbf{Q}^n, \end{array} \tag{3.3}$$

where the vertical arrows are inclusions.

Lemma 3.1 *With the above notation, we have $\mathbf{Q}^m \cap H = \cap_p \mathrm{im}\, h_p$ and $\pi(H) \subseteq \cap_p \mathrm{im}\, f_p$.*

Proof. If $y \in \cap_p \mathrm{im}\, h_p$, then for every p we have $y = h_p(t_p)$ for some $t_p \in \mathbf{Z}_p^m$, and hence $F_p(0, t_p) = (0, h_p(t_p)) = (0, y)$ for all p, so that $(0, y) \in H$. Conversely, if $(0, y) \in H$, then we can write $(0, y) = F_p(z_p, t_p)$ for each p. It follows that $f_p(z_p) = 0$ for all p, which forces $z_p = 0$ for all p. Therefore $(0, y) = F_p(0, t_p) = (0, h_p(t_p))$ for all p, which implies $y \in \cap_p \mathrm{im}\, h_p$. This concludes the proof of the first equality. Now, if $(x, y) \in H$, then $(x, y) =$

$F_p(z_p, t_p) = (f_p(z_p), h_p(t_p) + \theta_p(z_p))$ for all p. Hence, $x = f_p(z_p)$ for all p, which implies that $x \in \cap_p \mathrm{im}\, f_p$. $\square$

We note that the inclusion $\pi(H) \subseteq \cap_p \mathrm{im}\, f_p$ need not be an equality in general (see remark 3 below).

Theorem 3.2 *Assume given monomorphisms* $F_p \colon G_\phi(\mathbf{Z}_p) \to G_\phi(\mathbf{Q})$, *one for each prime* p. *Let* $\{f_p\}$ *and* $\{h_p\}$ *be given by* (3.1), *and let* $H = \cap_p \mathrm{im}\, F_p$. *Then the following statements are equivalent.*

(i) $\{F_p\}$ *is realizable.*

(ii) $\{h_p\}$ *is realizable and* $\pi(H)$ *has rank* n.

(iii) $\{f_p\}$ *is realizable and* $\pi(H)$ *has rank* n.

PROOF. If $\{F_p\}$ is realizable, then it follows from Theorem 2.1 that the middle vertical arrow in (3.1) becomes an equality after rationalization. Hence, $\pi(H)_0 = \mathbf{Q}^n$ and $\pi(H)$ has exactly rank n. By Lemma 3.1, $\cap_p \mathrm{im}\, f_p$ has also rank n, which implies that the sequence $\{f_p\}$ is realizable. Moreover, $(\mathbf{Q}^m \cap H)_0 = \mathbf{Q}^m$, which tells us that $\cap_p \mathrm{im}\, h_p$ has rank m, so that $\{h_p\}$ is realizable. We have proved that (i)⇒(ii) and (i)⇒(iii). Now (ii)⇒(i) is obtained by rationalizing (3.1) and invoking Corollary 2.2. Finally, the implication (iii)⇒(ii) follows from part (a) of Theorem 3.3 below. $\square$

In what follows, we no longer assume that a sequence $\{F_p\}$ has been given, but only a sequence of monomorphisms $f_p \colon \mathbf{Z}_p^n \to \mathbf{Q}^n$ which are liftable, in the sense of Section 1. Recall that, in this situation, all liftings F_p of f_p restrict to the same homomorphism $h_p \colon \mathbf{Z}_p^m \to \mathbf{Q}^m$.

Theorem 3.3 *Given a full alternating bilinear map* $\phi \colon \mathbf{Z}_p^n \times \mathbf{Z}_p^n \to \mathbf{Z}_p^m$ *and a realizable sequence of liftable monomorphisms* $f_p \colon \mathbf{Z}_p^n \to \mathbf{Q}^n$, *then:*

(a) *The associated homomorphisms* $h_p : \mathbf{Z}_p^m \to \mathbf{Q}^m$ *are monomorphisms, and the sequence* $\{h_p\}$ *is realizable.*

(b) *Every lifting* F_p *of* f_p *is a monomorphism.*

(c) *There is at least one lifting F_p for each p such that the sequence $\{F_p\}$ is realizable.*

Proof. Since $\{f_p\}$ is realizable, we can pick a basis $\{x_1, \ldots, x_n\}$ of $\mathbf{Q}^n$ such that $x_i \in \cap_p \mathrm{im}\, f_p$ for all i. Since ϕ is full, $\mathbf{Q}^m$ is generated as a $\mathbf{Q}$-vector space by the elements $\phi(x_i, x_j)$, $i = 1, \ldots, n$, $j = 1, \ldots, n$. Now, in order to prove that $\cap_p \mathrm{im}\, h_p$ has rank m, it suffices to see that $\phi(x_i, x_j) \in \cap_p \mathrm{im}\, h_p$ for all i, j. But for each fixed pair i, j we may write $(x_i, x_j) = (f_p(z_p), f_p(t_p))$ for all p, so that $\phi(x_i, x_j) = \phi(f_p(z_p), f_p(t_p)) = h_p\phi(z_p, t_p)$ for all p, as desired. The fact that $\cap_p \mathrm{im}\, h_p$ has rank m implies that every h_p is injective, and also shows that the sequence $\{h_p\}$ is realizable. Now any lifting F_p has the form (3.2) and hence it is injective.

We finally prove (c). Choose again a basis $\{x_1, \ldots, x_n\}$ of $\mathbf{Q}^n$ which is contained in $\cap_p \mathrm{im}\, f_p$. Set $x_i = f_p(z_p^i)$ for each i and each p. For every fixed p, the family $\{z_p^1, \ldots, z_p^n\}$ is a basis of $\mathbf{Z}_p^n$. Then, by Lemma 1.3, there is precisely one lifting F_p satisfying $F_p(z_p^i, 0) = (x_i, 0)$ for $i = 1, \ldots, n$. This tells us that $(x_i, 0) \in \cap_p \mathrm{im}\, F_p$ for all i. Hence $x_i \in \pi(H)$ for all i, so that $\pi(H)$ has rank n. It follows that $\{F_p\}$ is realizable, by Theorem 3.2. □

Remark 1. If $\cap_p \mathrm{im}\, f_p$ is finitely generated, then, in the last paragraph of the above proof, we can choose $\{x_1, \ldots, x_n\}$ to be a basis of $\cap_p \mathrm{im}\, f_p$ as a $\mathbf{Z}$-module. If we do so, then the family $\{F_p\}$ of liftings satisfying $F_p\left(f_p^{-1}(x_i),\, 0\right) = (x_i, 0)$ for every i, p, is realizable and has the property that the inclusion $\pi(H) \subseteq \cap_p \mathrm{im}\, f_p$ is an equality.

Remark 2. Suppose that we have in fact $h_p\bar{\phi} = \bar{\phi}(f_p \times f_p)$ for all p. Then we may choose $\theta_p = 0$ in (3.2) and hence, for every p, the map $F_p(x, y) = (f_p(x), h_p(y))$ is a homomorphism. For this sequence of liftings, we have $\pi(H) = \cap_p \mathrm{im}\, f_p$, and therefore $\{F_p\}$ is realizable.

Remark 3. It is not true that every lifting of a realizable sequence $\{f_p\}$ is realizable. For example, set $m = 1$, $n = 2$, let f_p and h_p be the canonical inclusions, and define for each p a homomorphism $g_p\colon \mathbf{Z}_p^2 \to \mathbf{Q}$ by $g_p(e_1) = 0$, $g_p(e_2) = 1/p$. Then $F_p(x, y) = (x,\, y + g_p(x))$ is a homomorphism lifting f_p. However, if $(z, t) \in \mathrm{im}\, F_p$ for all p, then $g_p(z) \in \mathbf{Z}_p$ for almost all p. This forces that, if $z = z_1e_1 + z_2e_2$, then $z_2 = 0$. Thus we find that $\pi(H)$ has rank 1, and hence, by Theorem 3.2, $\{F_p\}$ is not realizable.

In conclusion, we are able to construct groups in the extended genus of $G_\phi(\mathbf{Z})$ by exhibiting realizable sequences $\{f_p\}$ of liftable maps in $GL_n(\mathbf{Q})$. In practice, we can search for liftable maps by using Lemma 1.2. Also, in order to decide whether a given sequence $\{f_p\}$ is realizable, the techniques of [2] are available. Specifically, we recall from [2, Theorem 2.5] that $\{f_p\}$ is realizable if and only if the matrix of f_p^{-1} has entries in $\mathbf{Z}_p$ for almost all primes p. If this is the case, then the realization of $\{f_p\}$ is finitely generated if and only if the matrix of f_p has entries in $\mathbf{Z}_p$ for almost all p; see [2, Theorem 4.1].

Surprisingly, it turns out that if $\cap_p \mathrm{im}\, f_p$ is finitely generated, so is $\cap_p \mathrm{im}\, F_p$ for any sequence of liftings $\{F_p\}$. Hence, among such sequences $\{F_p\}$, the ones which are realizable give us groups which are in the Mislin genus of $G_\phi(\mathbf{Z})$. We next prove this claim.

Theorem 3.4 *In the hypotheses of Theorem* 3.3*, assume that the realization of* $\{f_p\}$ *is finitely generated. Then the realization of* $\{h_p\}$ *is also finitely generated, and so is* $H = \cap_p \mathrm{im}\, F_p$ *for every sequence* $\{F_p\}$ *lifting* $\{f_p\}$*.*

Proof. Let $\{F_p\}$ be any sequence lifting $\{f_p\}$. We know from Lemma 3.1 that $\pi(H) \subseteq \cap_p \mathrm{im}\, f_p$, and hence $\pi(H)$ is finitely generated. Thus, in order to prove that H is finitely generated, it suffices to see that $\mathbf{Q}^m \cap H$ —that is, the realization of $\{h_p\}$— is finitely generated. We next show that the matrix of h_p has entries in $\mathbf{Z}_p$ for almost all primes p. Fix any prime p such that

(i) the matrix of f_p has entries in $\mathbf{Z}_p$, and

(ii) p does not divide the index of the subgroup generated by $\mathrm{im}\, \phi$ in $\mathbf{Z}_p^m$.

Then we can ensure that, for such a prime p, the image of ϕ generates $\mathbf{Z}_p^m$ as a $\mathbf{Z}$-module. Thus, for each fixed i, the vector e_i of the canonical basis of $\mathbf{Z}_p^m$ is a sum of elements belonging to the image of ϕ, and hence the ith column of the matrix of h_p has the form

$$h_p(e_i) = \sum_j h_p\phi(z_j, t_j) = \sum_j \phi(f_p(z_j), f_p(t_j)),$$

which has entries in $\mathbf{Z}_p$. Since all primes except a finite number satisfy (i) and (ii), the proof is complete. $\square$

Example 1. Let $G_\phi(\mathbf{Z})$ be any group in $T(n,m)$. In order to study its extended genus, we start by looking for realizable sequences of liftable monomorphisms $f_p\colon \mathbf{Z}_p^n \to \mathbf{Q}^n$. One obvious choice is $f_p = \alpha_p\,\mathrm{id}$, where each α_p is an arbitrary nonzero rational number. By [2, Theorem 2.5], the sequence $\{f_p\}$ will be realizable if and only if $1/\alpha_p$ belongs to $\mathbf{Z}_p$ for almost all p. The associated maps $h_p\colon \mathbf{Z}_p^m \to \mathbf{Q}^m$ can be computed using the relation $h_p\phi = \phi(f_p \times f_p)$. The result is $h_p = (\alpha_p)^2\,\mathrm{id}$. In fact, under our choices, we have $h_p\bar{\phi} = \bar{\phi}(f_p \times f_p)$ for every p. Hence, by remark 2 above, the homomorphisms $F_p\colon G_\phi(\mathbf{Z}_p) \to G_\phi(\mathbf{Q})$ given by $F_p(x,y) = (f_p(x), h_p(y))$ form a realizable sequence. Therefore, if $H = \cap_p \mathrm{im}\, F_p$, then $H \in EG(G_\phi(\mathbf{Z}))$.

In this way we obtain an uncountable family of groups in the extended genus of $G_\phi(\mathbf{Z})$. Namely, if we enumerate the primes as $p_1, p_2, \ldots, p_i, \ldots$ and pick an arbitrary sequence of non-negative integers $k_1, k_2, \ldots, k_i, \ldots$, then by choosing $\alpha_{p_i} = 1/p_i^{k_i}$, the resulting group H is the subgroup of $G_\phi(\mathbf{Q})$ generated by

$$\left\{ \frac{1}{p_i^{k_i}} e_1, \ldots, \frac{1}{p_i^{k_i}} e_n,\ \frac{1}{p_i^{2k_i}} e_1', \ldots, \frac{1}{p_i^{2k_i}} e_m' \ \middle|\ i = 1,2,3,\ldots \right\};$$

cf. [2, Theorem 3.3], where $\{e_1, \ldots, e_n\}$ and $\{e_1', \ldots, e_m'\}$ denote the canonical bases of $\mathbf{Q}^n$ and $\mathbf{Q}^m$, respectively. Two such groups cannot be isomorphic unless the corresponding sequences $\{k_i\}$ are almost equal.

Example 2. It was shown in [7] that any group in $T(4,2)$ admits a presentation of the form

$$\begin{aligned} G(\delta,\lambda; a,b,c) = \langle x_1, x_2, x_3, x_4, u_1, u_2 \mid{}& u_1, u_2 \text{ central}, \\ & [x_1,x_2] = [x_3,x_4] = 1,\ [x_1,x_3] = {u_2}^{\delta\lambda}, \\ & [x_1,x_4] = {u_1}^{\delta},\ [x_2,x_3] = {u_1}^{a\delta}{u_2}^{b\delta\lambda},\ [x_2,x_4] = {u_2}^{-c\delta\lambda} \rangle, \end{aligned}$$

where δ and λ are positive integers and a, b, c are viewed as the coefficients of a binary integral quadratic form $ax^2 + bxy + cy^2$. Using our notation, this group is isomorphic to $G_\phi(\mathbf{Z})$ where $\phi\colon \mathbf{Z}^4 \times \mathbf{Z}^4 \to \mathbf{Z}^2$ is the alternating bilinear map whose components are given by the following matrices:

$$\phi_1 = \begin{pmatrix} 0 & 0 & 0 & \delta \\ 0 & 0 & \delta a & 0 \\ 0 & -\delta a & 0 & 0 \\ -\delta & 0 & 0 & 0 \end{pmatrix} \qquad \phi_2 = \begin{pmatrix} 0 & 0 & \delta\lambda & 0 \\ 0 & 0 & \delta\lambda b & -\delta\lambda c \\ -\delta\lambda & -\delta\lambda b & 0 & 0 \\ 0 & \delta\lambda c & 0 & 0 \end{pmatrix}. \tag{3.4}$$

Then, by Lemma 1.2, a map $f_p\colon \mathbf{Z}_p^4 \to \mathbf{Q}^4$ will be liftable to a homomorphism $F_p\colon G_\phi(\mathbf{Z}_p) \to G_\phi(\mathbf{Q})$ if and only if

$$\begin{cases} \phi(f_p(e_1), f_p(e_2)) = (0,0) \\ \phi(f_p(e_3), f_p(e_4)) = (0,0) \\ \phi(f_p(e_2), f_p(e_3)) = b\,\phi(f_p(e_1), f_p(e_3)) + a\,\phi(f_p(e_1), f_p(e_4)) \\ \phi(f_p(e_2), f_p(e_4)) = -c\,\phi(f_p(e_1), f_p(e_3)). \end{cases} \tag{3.5}$$

One family of solutions of (3.5) is

$$f_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha_p + b\beta_p & -c\beta_p \\ 0 & 0 & a\beta_p & \alpha_p \end{pmatrix}, \tag{3.6}$$

where α_p, β_p are arbitrary rational numbers such that $\det f_p \neq 0$. As explained in [2], any choice of α_p, β_p belonging to $\mathbf{Z}_p$ for almost all p, and such that f_p^{-1} has entries in $\mathbf{Z}_p$ for almost all p, will provide a finitely generated group in the Mislin genus of $G_\phi(\mathbf{Z})$.

For instance, by specializing to $G_\phi(\mathbf{Z}) = G(1,1;1,0,14)$, we know from [3] or [7] that the group $G_\psi(\mathbf{Z}) = G(1,1;2,0,7)$ is in the same Mislin genus as $G_\phi(\mathbf{Z})$. Indeed, we can represent $G_\psi(\mathbf{Z})$ by choosing $f_p = \mathrm{id}$ if $p \neq 2$, $\alpha_2 = 0$, $\beta_2 = 1/7$ in (3.6). From (2.3) we find that $G_\psi(\mathbf{Z})$ is isomorphic to the subgroup of $G_\phi(\mathbf{Z})$ generated by $\{e_1,\, e_2,\, 2e_3,\, e_4,\, e_1',\, 2e_2'\}$.

Of course, the roles of $G_\phi(\mathbf{Z})$ and $G_\psi(\mathbf{Z})$ can be exchanged, so as to obtain a certain embedding of $G_\phi(\mathbf{Z})$ into $G_\psi(\mathbf{Z})$. The phenomenon of groups in the same Mislin genus embedding into each other turns out to be much more general, far beyond the class $T(4,2)$. We plan to address this question in a forthcoming paper.

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