On genus and embeddings of torsion-free nilpotent groups of class two

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Abstract

We study embeddings between torsion-free nilpotent groups having isomorphic localizations. Firstly, we show that for finitely generated torsion-free nilpotent groups of nilpotency class 2, the property of having isomorphic P-localizations (where P denotes any set of primes) is equivalent to the existence of mutual embeddings of finite index not divisible by any prime in P. We then focus on a certain family $\mathbf{\Gamma}$ of nilpotent groups whose Mislin genera can be identified with quotient sets of ideal class groups in quadratic fields. We show that the multiplication of equivalence classes of groups in $\mathbf{\Gamma}$ induced by the ideal class group structure can be described by means of certain pull-back diagrams reflecting the existence of enough embeddings between members of each Mislin genus. In this sense, the family $\mathbf{\Gamma}$ resembles the family $\mathbf{N}_0$ of infinite, finitely generated nilpotent groups with finite commutator subgroup. We also show that, in further analogy with $\mathbf{N}_0$, two groups in $\mathbf{\Gamma}$ with isomorphic localizations at every prime have isomorphic localizations at every finite set of primes. We supply counterexamples showing that this is not true in general, neither for finitely generated torsion-free nilpotent groups of class 2 nor for torsion-free abelian groups of finite rank.

[*]Supported by DGICYT grant PB94–0725

0 Introduction

For a finitely generated nilpotent group N, the Mislin genus $\mathcal{G}(N)$ is the set of isomorphism classes of finitely generated nilpotent groups M such that the p-localizations N_p and M_p are isomorphic for all primes p. The same notion is defined for nilpotent spaces of finite type; see [17]. We also recall that, for any set of primes P, a homomorphism $\varphi\colon N \to M$ of nilpotent groups is called a P-equivalence if the induced homomorphism of P-localizations, $\varphi_P\colon N_P \to M_P$, is an isomorphism.

If a group N belongs to the family $\mathbf{N}_0$ of infinite, finitely generated, nilpotent groups with finite commutator subgroup, then the Mislin genus $\mathcal{G}(N)$ admits a finite abelian group structure, which was first discussed in [16]. Moreover, if two groups N and M belong to $\mathbf{N}_0$, then the following properties are equivalent:

(i) $N_p \cong M_p$ for all primes p;

(ii) $N_P \cong M_P$ for every finite set of primes P;

(iii) there exist injective P-equivalences $N \to M$ and $M \to N$ for every finite set of primes P.

These properties need not be equivalent beyond $\mathbf{N}_0$. Indeed, it has long been known that if two finitely generated nilpotent groups N and M satisfy $N_P \cong M_P$ for some set P, then it does not follow in general that there is a P-equivalence between N and M in either direction; see Roitberg's article [31] and the refinement made recently in [32]. We note, however, that no counterexample has been given so far in which N and M are in the same Mislin genus. It is remarkable that such counterexamples do exist in homotopy theory, as one can find nilpotent spaces of finite type in the same Mislin genus admitting no essential maps between them in either direction; this was reported by Møller in [27, Example 3.11] and by McGibbon and Møller in [25, Example 4.1].

We shall examine to what extent properties (i)–(iii) are related to each other in the family of torsion-free finitely generated nilpotent groups of nilpotency class 2, which will be called $\mathcal{T}_2$-groups for brevity. In Section 1 we show that for any two $\mathcal{T}_2$-groups N and M, the property $N_P \cong M_P$ (where P is arbitrary) implies that N and M embed into each other via P-equivalences. This yields an alternative approach to a result of Smith [34], stating that $\mathcal{T}_2$-groups are compressible.

In Section 3 we focus on a family $\mathbf{\Gamma}$ of $\mathcal{T}_2$-groups for which all three properties (i)–(iii) are equivalent, and which shares other interesting similarities with $\mathbf{N}_0$. It consists of all nilpotent groups of the form

$$\Gamma(R,I) = \begin{pmatrix} 1 & R & I \\ 0 & 1 & I \\ 0 & 0 & 1 \end{pmatrix},$$

where R is any order in a quadratic field and I is any invertible ideal of R. Groups belonging to $\mathbf{\Gamma}$, which have been studied notably by Grunewald and Scharlau in [10] (see also [11], [29], or [33]), possess a very rich genus theory which is best handled by means of the classical theory of binary quadratic forms or the theory of ideals in orders of quadratic fields. In particular, the equivalence between properties (i) and (ii) for groups in $\mathbf{\Gamma}$ may be viewed as a consequence of the fact that binary integral quadratic forms which are equivalent over $\mathbf{Z}_p$ for all primes p are also equivalent over $\mathbf{Z}_P$ for every finite set of primes P; see [8, Theorem 9.7.1] and [9, Theorem 1.1]. This is false for forms of higher order, as shown in Example 2.2 below, and also if the usual equivalence of forms is replaced by λ-equivalence in the sense of [11]; cf. Example 3.6 below. Thus, property (i) fails to imply (ii) for $\mathcal{T}_2$-groups in general. We shall refer to (ii) by saying that N and M are in the same *strong genus*. It is interesting that the notion of strong genus is distinct from ordinary genus already in the family of abelian groups, as we can exhibit in Section 2 an example of two abelian groups of finite rank whose localizations are isomorphic at every prime but not at a set consisting of three primes.

Mislin genera of groups in $\mathbf{\Gamma}$ can be identified with quotient sets of ideal class groups, as we next explain. For a fixed ring R, we define $\Gamma(R,I)$ and $\Gamma(R,J)$ to be equivalent if I and J are isomorphic as R-modules, that is, if $rI = sJ$ for some nonzero elements $r, s \in R$. The set of equivalence classes, which we denote by $\mathcal{C}_R$, admits a commutative group structure induced by the multiplication of ideals —hence, $\mathcal{C}_R$ is isomorphic to the ideal class group of R. On the other hand, it follows from results in [9] and [33] that groups in $\mathbf{\Gamma}$ with a common R belong to the same Mislin genus and, in fact, their isomorphism classes constitute a whole genus. Thus, for each invertible ideal I of R there is a surjection

$$\mathcal{C}_R \twoheadrightarrow \mathcal{G}(\Gamma(R,I)),$$

since equivalent groups in $\mathbf{\Gamma}$ are indeed isomorphic. Due to the fact that $\Gamma(R,I)$ is also isomorphic to $\Gamma(R,J)$ if I and J lie in conjugate ideal classes, the above surjection is not one-to-one in general, but $\mathcal{G}(\Gamma(R,I))$ corresponds bijectively with the set of orbits of the involution $x \mapsto x^{-1}$ in $\mathcal{C}_R$. Contrary to what happens in [16, Theorem 1.4], the group structure in $\mathcal{C}_R$ cannot be transported to the genus set in general. However, if the ideal class group of R has exponent 2, then the sets $\mathcal{C}_R$ and $\mathcal{G}(\Gamma(R,I))$ coincide and hence, in this special case, the Mislin genus of $\Gamma(R,I)$ does acquire a commutative group structure.

We have observed, moreover, that the multiplication in $\mathcal{C}_R$ is analogous to the standard multiplication in Mislin genera within the family $\mathbf{N}_0$, in the sense that both multiplications can be defined by means of a certain pull-back construction reflecting the existence of enough P-equivalences between members of each genus for any finite set P of primes.

Acknowledgements We are indebted to P. Hilton, C. A. McGibbon, P. Ribenboim, J. Roitberg, L. Vaserstein, and W. C. Waterhouse, as well as to the editor, R. Oliver, for their interest and help. We also thank the Centre de Recerca Matemàtica of Barcelona for gathering us together during the preparation of this article.

1 Embeddings of nilpotent groups of class two

Throughout the article, torsion-free finitely generated nilpotent groups will be called $\mathcal{T}$-groups, and we shall refer to $\mathcal{T}$-groups of nilpotency class 2 as $\mathcal{T}_2$-groups.

Let P be any set of primes and P' denote its complement. Recall from [17, Ch. I, § 1] that a homomorphism $\varphi : N \to M$ of nilpotent groups is called P-injective if $\ker\varphi$ is P'-torsion, P-surjective if for every element $x \in M$ there is a P'-number m such that $x^m \in \operatorname{im}\varphi$, and P-bijective or a P-equivalence if it is both P-injective and P-surjective. The latter condition holds if and only if the induced homomorphism $\varphi_P \colon N_P \to M_P$ is an isomorphism. We emphasize that P-equivalences between torsion-free nilpotent groups are necessarily injective. An embedding $M \hookrightarrow N$ of $\mathcal{T}$-groups is a P-equivalence if and only if the index $[N : M]$ is finite and it is a P'-number (this follows, as in [12, Satz 5.81], from the fact that each subgroup of a nilpotent group is subnormal).

Let M be any $\mathcal{T}_2$-group. Then the isolator of the commutator subgroup,

$$I = \{x \in M \mid x^n \in [M,M] \text{ for some } n > 0\},$$

is a free abelian subgroup in the center of M. We can choose a Mal'cev basis

$\{v_1, \ldots, v_r, w_1, \ldots, w_s\}$ in M (see [22, Ch. 6]), where $w_1, \ldots, w_s$ form a basis of I. The group M is then said to be in the family $\mathcal{T}(r, s)$.

This allows to assign coordinates $(X, Y) = (x_1, \ldots, x_r, y_1, \ldots, y_s)$ to each element of M, and the multiplication in M can be written as

$$(X, Y)(X', Y') = (X + X',\ Y + Y' + \zeta(X, X')), \tag{1.1}$$

where $\zeta\colon \mathbf{Z}^r \times \mathbf{Z}^r \to \mathbf{Z}^s$ is a bilinear map which is determined by the coordinates of the commutators $[v_j, v_i]$, with $j > i$, in the chosen Mal'cev basis. The associated alternating bilinear map

$$\phi(X, X') = \zeta(X, X') - \zeta(X', X) \tag{1.2}$$

is the expression in coordinates of the commutator map $(M/I) \times (M/I) \to I$ sending $(vI, v'I)$ to $[v, v']$.

Thus, groups in $\mathcal{T}(r, s)$ are precisely central extensions

$$\mathbf{Z}^s \rightarrowtail M \twoheadrightarrow \mathbf{Z}^r$$

in which the commutator subgroup $[M, M]$ has rank s. If we view the bilinear map ζ as a 2-cocycle, then its cohomology class in $H^2(\mathbf{Z}^r; \mathbf{Z}^s)$ characterizes the group M in the usual way [4].

It is important to observe that every $\mathcal{T}_2$-group M embeds into itself properly via the function $h_n\colon M \to M$ defined in coordinates by

$$h_n(X, Y) = (nX,\ n^2 Y), \tag{1.3}$$

which is indeed a monomorphism for every integer $n \neq 0$.

Theorem 1.1 *Let M and N be torsion-free nilpotent groups of class at most two. Assume that M is finitely generated. If $M_P \cong N_P$ for a certain set of prime numbers P, then there exists a P-equivalence $h\colon M \to N$.*

Proof. By [17, Theorem I.3.3], there is a finitely generated nilpotent group K together with P-equivalences $K \to N$ and $K \to M$. On the other hand, the existence of the homomorphisms h_n defined in (1.3) ensures that $\mathcal{T}_2$-groups are P-universal for each set of primes P, in the sense of Kahn–Scheerer [20], [21], Lemaire [23], or Mimura–Toda [26]. Hence, there is a P-equivalence $M \to K$ yielding the desired result, since the composition of two P-equivalences is again a P-equivalence. □

In fact, under the assumptions of Theorem 1.1, it is not difficult to exhibit an explicit P-equivalence $M \to N$. It can be done as follows. Let $f\colon M_P \to N_P$ be an isomorphism. View both M and N as subgroups of N_P via the injections

$$l\colon N \hookrightarrow N_P \qquad \text{and} \qquad fl\colon M \hookrightarrow N_P,$$

where l denotes the P-localization homomorphism. Let $\{v_1, \ldots, v_r, w_1, \ldots, w_s\}$ be a Mal'cev basis of M, chosen as explained above (we do not exclude the possibility that M be abelian, in which case $s = 0$ and ζ is identically zero). Since M is contained in N_P, there are P'-numbers $n_1, \ldots, n_r, m_1, \ldots, m_s$ such that $v_i^{n_i} \in N$ for $i = 1, \ldots, r$ and $w_j^{m_j} \in N$ for $j = 1, \ldots, s$. Take

$$n = \operatorname{lcm}(n_1, \ldots, n_r, m_1, \ldots, m_s),$$

which is also a P'-number, and consider the homomorphism $h_n\colon M \to M$ defined in (1.3), where coordinates are of course referred to $\{v_1, \ldots, v_r, w_1, \ldots, w_s\}$. Then $h_n(M) \subset N$, since all the coordinates of an arbitrary element of $h_n(M)$ are integer multiples of n. Thus, we may view h_n as a homomorphism from M to N. Moreover, as such, it is a P-equivalence, since the smallest P-local subgroup of N_P containing $h_n(M)$ is N_P itself.

Corollary 1.2 *Let M and N be $\mathcal{T}_2$-groups. Then, for any set P of prime numbers, the following statements are equivalent:*

(i) $M_P \cong N_P$.

(ii) *N embeds into M as a subgroup of finite index prime to P.*

(iii) *M embeds into N as a subgroup of finite index prime to P.* $\square$

This yields an alternative proof of Proposition 2 in [34], as it implies that $\mathcal{T}_2$-groups are compressible. (A group G is compressible if whenever H is a subgroup of finite index in G, there exists a subgroup K of finite index in H which is isomorphic to G.)

Corollary 1.2 does not hold in general for arbitrary $\mathcal{T}$-groups, since, as observed in [34, Proposition 4], there exist $\mathcal{T}$-groups of nilpotency class 6 which are not compressible. Similar examples were displayed by Roitberg in [31] and [32].

2 Strong genus

Motivated by Example 2 in [5], we have been interested in deciding if any two nonisomorphic $\mathcal{T}$-groups M, N in the same Mislin genus necessarily embed into each other with relatively prime indices. We still do not know the answer to this question. Note that, in view of Corollary 1.2, the answer would be affirmative for $\mathcal{T}_2$-groups if it were true that $M_P \cong N_P$ for every finite set of primes P whenever M and N are in the same Mislin genus. Unfortunately, Example 2.2 below will show that this is not the case in general. This led us to introduce the following terminology.

Definition 2.1 We say that two nilpotent groups M, N are in the same *strong genus* if $M_P \cong N_P$ for all finite sets of primes P.

Of course, two nilpotent groups in the same strong genus are also in the same genus. However, as we next show, the converse is not true (see also Example 2.3 and Example 3.6).

Example 2.2 Consider the groups

$$N = \langle v_1, \ldots, v_8, w_1, w_2 \mid w_1 \text{ and } w_2 \text{ central}, \ [v_2, v_1] = [v_4, v_3] = w_2, \\ [v_6, v_5] = w_1, \ [v_8, v_7] = w_1^6 w_2, \ [v_j, v_i] = 1 \text{ otherwise, for } j > i\rangle,$$

$$M = \langle v_1, \ldots, v_8, w_1, w_2 \mid w_1 \text{ and } w_2 \text{ central}, \ [v_2, v_1] = [v_4, v_3] = w_2, \\ [v_6, v_5] = w_1^3 w_2, \ [v_8, v_7] = w_1^2 w_2, \ [v_j, v_i] = 1 \text{ otherwise, for } j > i\rangle.$$

These are groups in $\mathcal{T}(8, 2)$. In the given Mal'cev bases, the respective commutator maps ϕ, ψ, defined as in (1.2), have Pfaffians

$$\sqrt{\det(x\phi_1 + y\phi_2)} = y^2 x(6x + y), \qquad \sqrt{\det(x\psi_1 + y\psi_2)} = y^2(3x + y)(2x + y).$$

These two forms are equivalent at each prime p, in the sense that there is a linear change of variables over $\mathbf{Z}_p$ transforming the first form into the second; namely, $x = x' + \frac{1}{3}y'$, $y = y'$ if $p \neq 3$, and $x = x' + \frac{1}{2}y'$, $y = -y'$ for $p = 3$. We use these transformations, together with [5, Theorem 1.4], to show that the groups N and M are in the same Mislin genus. Indeed, if $p \neq 3$, then one isomorphism $N_p \to M_p$ is defined by sending $v_6 \mapsto \frac{1}{3}v_6$, $v_8 \mapsto 3v_8$, $w_1 \mapsto w_1 + \frac{1}{3}w_2$; for $p = 3$, take instead $v_2 \mapsto -v_2$, $v_4 \mapsto -v_4$, $v_5 \mapsto v_7$, $v_6 \mapsto \frac{1}{2}v_8$, $v_7 \mapsto v_5$, $v_8 \mapsto 2v_6$, $w_1 \mapsto w_1 + \frac{1}{2}w_2$, $w_2 \mapsto -w_2$.

However, the two Pfaffians above are not equivalent at the set $P = \{2,3\}$, since no other linear change of variables takes one form to the other (up to scalar factors). It follows that the groups N_P and M_P fail to be isomorphic if $P = \{2,3\}$; see [11, § 3] or [33, Ch. 11].

The two forms used in this example were supplied by W. C. Waterhouse, whom we here thank; in spirit, this goes back to his example in [2, Lemma 2.2].

Of course, counterexamples are more abundant if one considers localization of not necessarily nilpotent groups. For instance, the strong genus of any finite group G consists of the isomorphism class of G only, while there exist many isomorphism classes of (nonnilpotent) finite groups in the genus of the trivial group [3].

Likewise, if we omit the assumption that the groups involved be finitely generated, then we can exhibit examples of abelian groups A and B which are in the same genus but not in the same strong genus.

Example 2.3 Let x, y be any two linearly independent elements in $\mathbf{Q}^2$. Consider the subgroups of $\mathbf{Q}^2$ given by

$$A = \langle\, \mathbf{Z}[\tfrac{1}{2}]\, x,\ \mathbf{Z}[\tfrac{1}{3}]\, y,\ \tfrac{1}{5}(x+y)\,\rangle, \qquad B = \mathbf{Z}[\tfrac{1}{2}]\, x \,\oplus\, \mathbf{Z}[\tfrac{1}{3}]\, y.$$

Then, according to [1, Example 2.2], the group A is indecomposable, i.e., it is not a direct sum of groups of rank 1. However, it is almost completely decomposable, in the sense that

$$5A \subseteq B \subseteq A. \tag{2.1}$$

It follows from (2.1) that $A_p \cong B_p$ if $p \neq 5$. Moreover, $A_5 = \mathbf{Z}_5\, x \oplus \mathbf{Z}_5\, z$, where $z = \frac{1}{5}(x+y)$, so that A and B are in the same genus. Now, if we choose $P = \{2,3,5\}$, then

$$A_P = \langle\, \mathbf{Z}_{\{3,5\}}\, x,\ \mathbf{Z}_{\{2,5\}}\, y,\ \tfrac{1}{5}(x+y)\,\rangle,$$

which is indecomposable. Therefore, A and B are not in the same strong genus.

On the positive side, we offer the following result. A nilpotent group N is called *fg-like* if it is in the genus of some finitely generated nilpotent group.

Theorem 2.4 *Let A and B be abelian fg-like groups. Then A and B are in the same strong genus if and only if they are in the same genus.*

Proof. From the discussion in [6, § 1] it follows that it is sufficient to prove the statement for A and B torsion-free. Thus, we assume that A and B are in the genus of $\mathbf{Z}^k$ for some common $k \geq 1$, and we aim to proving that $A_P \cong B_P$ for any finite set of primes P.

Pick any set $\{x_1, \ldots, x_k\}$ of linearly independent elements of A, and write $L = \langle x_1, \ldots, x_k \rangle$. An element $a \in A$ is said to have p-height n if $p^n a$ belongs to L but $p^{n-1}a$ does not. By [7, Theorem 1.4], for each prime p there is a positive number $\nu(p)$ such that all elements of A have p-height less than or equal to $\nu(p)$. Now write $P = \{p_1, \ldots, p_r\}$ and choose $\lambda = p_1^{\nu(p_1)} \cdots p_r^{\nu(p_r)}$. Then $A_P \subseteq (\lambda^{-1}L)_P \cong \mathbf{Z}_P^k$. Due to the fact that $\mathbf{Z}_P$ is a principal ideal domain, A_P is itself a free $\mathbf{Z}_P$-module of rank k. Since the same argument applies to B, the proof is complete. □

It was observed by Hilton and Militello [13, Theorem 3.3] that, if a nilpotent group G is fg-like and the commutator subgroup $[G, G]$ is finite, then every subgroup H of G is fg-like. However, if the assumption on $[G, G]$ is removed, then this need not be true; it is even possible to construct a counterexample where G is torsion-free and has nilpotency class 2; see [14, § 4]. Still, one could expect that each subgroup H of finite index in an fg-like group G could be fg-like. We prove this for nilpotent groups of class at most 2.

Theorem 2.5 *Let G be a torsion-free fg-like nilpotent group of class at most two. Then the following hold:*

(a) *There is a finitely generated subgroup $K \leq G$ in the genus of G.*

(b) *If H is a subgroup of G of finite index, then H is fg-like.*

Proof. We shall obtain (a) in the course of our argument to prove (b). Since all subgroups of a nilpotent group are subnormal [22, Theorem 16.2.2], there is no restriction in assuming that the index $[G : H]$ is a power of a prime p, so that the inclusion $H \hookrightarrow G$ is p'-surjective. Let K be a finitely generated group in the genus of G. Then $K_p \cong G_p$ and from Theorem 1.1 it follows that there is a p-surjective embedding $K \hookrightarrow G$. Then $H \cap K$ is finitely generated and belongs to the genus of H, as we next show. Firstly, the inclusion $H \cap K \hookrightarrow H$ is p-surjective and hence $(H \cap K)_p \cong H_p$. Secondly, if $q \neq p$, then the inclusion $H \cap K \hookrightarrow K$ is q-surjective, so that $(H \cap K)_q \cong K_q \cong G_q \cong H_q$ as well. □

Theorem 2.5 remains true if the assumption about G is weakened by imposing only that G be in the genus of a $\mathcal{T}$-group which is p-universal for all primes p. In fact, as explained in [21, Theorem 1], it is sufficient to have p-universality for one prime p in order to have P-universality for all sets of primes P.

3 Ideal class groups and Mislin genera

We now consider a family $\mathbf{\Gamma}$ of $\mathcal{T}_2$-groups, whose Mislin genera can be arbitrarily large, while remaining well understood thanks to the methods of Grunewald–Segal–Sterling [11].

Let R be an order (i.e., a subring with additive group $\mathbf{Z} \oplus \mathbf{Z}$) in the field $\mathbf{Q}(\sqrt{D})$, for some square-free integer $D \neq 0, 1$, and let I be any invertible ideal of R. Let $\Gamma(R, I)$ be the group of matrices of the form

$$\begin{pmatrix} 1 & r & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{with } r \in R,\ s \in I,\ t \in I. \tag{3.1}$$

Thus, $\Gamma(R, I)$ is a subgroup of the group $\mathrm{UT}_3(R)$ of upper-triangular matrices in $\mathrm{GL}_3(R)$ with all diagonal entries equal to 1, which is a torsion-free finitely generated nilpotent group of class 2; in fact, every such group $\Gamma(R, I)$ belongs to $\mathcal{T}(4, 2)$. The family of groups $\Gamma(R, I)$, for all choices of R and I as above, will hereafter be denoted by $\mathbf{\Gamma}$. The extent to which this family $\mathbf{\Gamma}$ fails to exhaust $\mathcal{T}(4, 2)$ is made precise in the Appendix.

We recall from [33, Theorem 3, p. 275] that two groups $\Gamma(R_1, I_1)$ and $\Gamma(R_2, I_2)$ are isomorphic if and only if $R_1 \cong R_2$ by an isomorphism θ such that $\theta(I_1)$ and I_2 lie in the same or in conjugate ideal classes of R_2.

We declare that two groups $\Gamma(R, I)$ and $\Gamma(R, J)$ are *equivalent* if I and J are in the same ideal class. Then the quotient set $\mathcal{C}_R$ of groups in $\mathbf{\Gamma}$ with a fixed R under this equivalence relation admits a (finite, commutative) group structure, which is naturally induced by the multiplication of ideals; thus, it is isomorphic to the well-known ideal class group of R (see e.g. [28] or [30]). The unit element is the equivalence class containing the group $\Gamma(R, R)$. The following result describes how the Mislin genus of any $\Gamma(R, I)$ can be obtained from the group $\mathcal{C}_R$.

Theorem 3.1 *Let D be any square-free integer, $D \neq 0, 1$. Let R be an order in the field $\mathbf{Q}(\sqrt{D})$, and let I be any invertible ideal of R. Then the Mislin genus* *of* $\Gamma(R,I)$ *is in one-to-one correspondence with the set obtained by identifying each element of the ideal class group of* R *with its multiplicative inverse.*

Proof. If we fix R and let I run over all invertible ideals of R, then all groups of the form $\Gamma(R,I)$ belong to the same Mislin genus. This result is inferred from [9, Theorem 1.1]; see also the argument in [29, Théorème 2.3] in the case when R is the maximal order (i.e., the ring of algebraic integers in $\mathbf{Q}(\sqrt{D})$). Actually, by [33, Theorem 3, p. 275] and [9, Theorem 1.1], the isomorphism classes of such groups constitute a whole genus. It follows that there is a surjection

$$\mathcal{C}_R \twoheadrightarrow \mathcal{G}(\Gamma(R,I)),$$

which is not one-to-one in general, since conjugate ideals I, $\bar{I}$ give isomorphic groups $\Gamma(R,I) \cong \Gamma(R,\bar{I})$, yet they may correspond to distinct elements of $\mathcal{C}_R$. Thus, the genus set $\mathcal{G}(\Gamma(R,I))$ is precisely the set of orbits of the involution $x \mapsto x^{-1}$ in $\mathcal{C}_R$. □

Corollary 3.2 *If the ideal class group of* R *has exponent 2, then the multiplication of ideals induces a commutative group structure on the Mislin genus of* $\Gamma(R,I)$. □

For each invertible ideal I of an order R, the norm $N(I)$ is defined as the index of I in R. Thus, every group $\Gamma(R,I)$ is a subgroup of $\Gamma(R,R)$ with index $N(I)^2$. Since every group in the genus of $\Gamma(R,R)$ can be identified with some $\Gamma(R,I)$ in several ways (in fact, infinitely many), the inclusions

$$\Gamma(R,I) \hookrightarrow \Gamma(R,R)$$

give us plenty of embeddings from any given group in the genus into $\Gamma(R,R)$. In particular, every principal ideal aR, where $a \in R$, provides a self-embedding of the group $\Gamma(R,R)$.

We say that two inclusions $\Gamma(R,I) \hookrightarrow \Gamma(R,R)$ and $\Gamma(R,J) \hookrightarrow \Gamma(R,R)$ form an *exhaustive pair* if

- the ideals I, J are relatively prime, and
- $\gcd(N(I),F) = \gcd(N(J),F) = 1$, where F is the index of R in the maximal order of $\mathbf{Q}(\sqrt{D})$.

(The latter condition is necessary to get a good factorization theory of ideals, whenever R is not the ring of algebraic integers in $\mathbf{Q}(\sqrt{D})$. The number F is called the *conductor* of R; cf. [30, § 16].)

Theorem 3.3 *Let R be an order in $\mathbf{Q}(\sqrt{D})$, where D is any square-free integer, $D \neq 0, 1$. Suppose that I and J are invertible ideals of R such that the inclusions*

$$\phi\colon \Gamma(R, I) \hookrightarrow \Gamma(R, R), \qquad \psi\colon \Gamma(R, J) \hookrightarrow \Gamma(R, R)$$

form an exhaustive pair. Then the diagram

$$\begin{array}{ccc} \Gamma(R, IJ) & \rightarrow & \Gamma(R, J) \\ \downarrow & & \downarrow \psi \\ \Gamma(R, I) & \xrightarrow{\phi} & \Gamma(R, R), \end{array}$$

where all the arrows are inclusions, is both a pull-back and a push-out diagram in the category of nilpotent groups.

Proof. The pull-back of ϕ, ψ over $\Gamma(R, R)$ is isomorphic to $\Gamma(R, I \cap J)$. Since ϕ, ψ form an exhaustive pair, we have $I \cap J = IJ$. This follows from [28, Proposition 1.5] in the case when R is a Dedekind ring, since I, J are relatively prime. If R is not the ring of algebraic integers in $\mathbf{Q}(\sqrt{D})$, then we must restrict further the choice of I, J to ideals whose norms are prime to the conductor F of R; but this is embodied in our definition of an exhaustive pair. It then follows from [15, Corollary 2.2] that the diagram is also a push-out. □

Consequently, the multiplication in the group $\mathcal{C}_R$ can alternatively be described by means of pull-back diagrams, provided that one chooses representatives in $\mathcal{C}_R$ that form exhaustive pairs. Such a choice is always possible, since any class of invertible ideals in an order of a quadratic field contains a representative I whose norm $N(I)$ is prime to any given finite set of prime numbers (this follows from the generalized Dirichlet prime number theorem [28, p. 358]). However, we warn that Theorem 3.3 does not provide us with a well-defined binary operation on isomorphism classes of groups, since $\Gamma(R, I) \cong \Gamma(R, \bar{I})$, yet $\Gamma(R, IJ)$ need not be isomorphic to $\Gamma(R, \bar{I}J)$.

Corollary 3.4 *Let M, N be groups in $\mathbf{\Gamma}$ which are in the same Mislin genus. Then, for any finite set of primes P, there exist embeddings $M \hookrightarrow N$ and $N \hookrightarrow M$ whose indices are finite, relatively prime, and prime to P.*

Proof. Of course, it suffices to prove that, for any finite set of primes P, there exists an embedding $M \hookrightarrow N$ whose index is finite and prime to P. Choose invertible ideals K, L such that $M \cong \Gamma(R, K)$ and $N \cong \Gamma(R, L)$. Then choose an exhaustive pair

$$\phi\colon \Gamma(R, I) \hookrightarrow \Gamma(R, R), \qquad \psi\colon \Gamma(R, J) \hookrightarrow \Gamma(R, R)$$

where I is in the same ideal class as KL^{-1}, J is in the same ideal class as L, and the norm $N(I)$ is prime to P. By Theorem 3.3, the group $\Gamma(R, IJ)$ is the pull-back of ϕ, ψ over $\Gamma(R, R)$ and, by our choice of I, we have

$$[\Gamma(R, R) : \Gamma(R, I)] = N(I)^2,$$

which is prime to P. It then follows from [15, Proposition 1.7] that the index $[N : M] = [\Gamma(R, J) : \Gamma(R, IJ)]$ is also prime to P. □

Corollary 3.5 *If M, N are two groups in the family $\mathbf{\Gamma}$, then M and N are in the same strong genus if and only if they are in the same Mislin genus.* □

It is natural to ask if this result can be extended to all groups in $\mathcal{T}(4, 2)$. Our next example shows that it is not the case, as we can display two groups N and M in $\mathcal{T}(4, 2)$ which are in the same Mislin genus but not in the same strong genus.

Example 3.6 Using the notation described in the Appendix, we consider the groups $N = G(\delta, \lambda, f)$ and $M = G(\delta, \lambda, f')$, where $\delta = 1$, $\lambda = 6$, $f(x, y) = 6x^2 + xy$ and $f'(x, y) = 6x^2 + 5xy + y^2$ (these choices are inspired by Example 2.2). According to [9, Theorem 1.1], the groups N and M are in the same Mislin genus. However, the forms f and f' are not λ-equivalent over $\mathbf{Z}_P$ if $P = \{2, 3\}$, from which it follows that the groups N_P and M_P are not isomorphic.

The conclusions of Corollaries 3.4 and 3.5 hold in the family $\mathbf{N}_0$ as well; see [16, Corollary 2.2]. However, there are also important differences between the properties of the genus sets in $\mathbf{\Gamma}$ and in $\mathbf{N}_0$. For example, in $\mathbf{N}_0$, a group M belongs to $\mathcal{G}(N)$ if and only if

$$M \times \mathbf{Z} \cong N \times \mathbf{Z}$$

(see [35]); however, this is not true in $\mathbf{\Gamma}$. We recall that the family $\mathbf{\Gamma}$ consists of $\mathcal{T}_2$-groups and hence the condition $M \times \mathbf{Z} \cong N \times \mathbf{Z}$ necessarily implies that

$M \cong N$, as shown by Hirshon in [19]. Hence, groups in the same genus in $\mathbf{\Gamma}$ do not provide examples of non-cancellation.

Moreover, if we denote by N^k the cartesian product of k copies of N, then the growth of $\mathcal{G}(N^k)$, as a function of k, also behaves quite differently. In $\mathbf{N}_0$, we have $|\mathcal{G}(N^k)| \leq |\mathcal{G}(N)|$ for every positive integer k; see [16]. On the contrary, the cardinality $|\mathcal{G}(N^k)|$ increases with k in the family $\mathbf{\Gamma}$. If, say, $|\mathcal{G}(N)| = n$ for a group $N \in \mathbf{\Gamma}$ (where n can be as big as one wants), then

$$|\mathcal{G}(N^k)| \geq \frac{(n+k-1)!}{k!\,(n-1)!} \geq (k+1)^{n-1}.$$

Actually, if $M_1, \ldots, M_k$, $N_1, \ldots, N_k$ belong to $\mathcal{G}(N)$ for a group $N \in \mathbf{\Gamma}$, then both $M_1 \times \cdots \times M_k$ and $N_1 \times \cdots \times N_k$ belong to $\mathcal{G}(N^k)$, and

$$M_1 \times \cdots \times M_k \cong N_1 \times \cdots \times N_k$$

if and only if $M_i \cong N_i$, $i = 1, \ldots, k$, up to reordering; cf. [18], [24].

4 Appendix

For the convenience of the reader, we recall the basic ingredients in the classification of groups in $\mathcal{T}(4,2)$, as described in [11] or [33, Ch. 11].

For every group in $\mathcal{T}(r,s)$ one can choose a Mal'cev basis, which determines an alternating bilinear map $\phi\colon \mathbf{Z}^r \times \mathbf{Z}^r \to \mathbf{Z}^s$, as in (1.2). If r is even, then the Pfaffian of the skew-symmetric matrix $x_1\phi_1 + \cdots + x_s\phi_s$ (i.e., a square root of the determinant) is an integral form of degree $\frac{1}{2}r$ in the variables $x_1, \ldots, x_s$. If two groups in $\mathcal{T}(r,s)$ have isomorphic localizations at a set P, where r is even, then the corresponding Pfaffians are equivalent over $\mathbf{Z}_P$; cf. [11, § 3]. However, we emphasize that the converse is not true in general.

To any binary integral quadratic form $f(x,y) = ax^2 + bxy + cy^2$ and positive integers δ, λ, one can associate a group

$$G(\delta,\lambda,f) = \langle v_1, v_2, v_3, v_4, w_1, w_2 \mid w_1 \text{ and } w_2 \text{ central},\ [v_2,v_1] = [v_4,v_3] = 1,$$
$$[v_3,v_1] = w_2^{\delta\lambda},\ [v_4,v_1] = w_1^{\delta},\ [v_3,v_2] = w_1^{a\delta} w_2^{b\delta\lambda},\ [v_4,v_2] = w_2^{-c\delta\lambda}\rangle.$$

Any such group $G(\delta,\lambda,f)$ is in $\mathcal{T}(4,2)$ and the torsion subgroup of its abelianization is isomorphic to $(\mathbf{Z}/\delta\mathbf{Z}) \oplus (\mathbf{Z}/\delta\lambda\mathbf{Z})$. The Pfaffian of the alternating bilinear map associated with the given basis is equal to

$$\delta^2(a\,x^2 + b\lambda\,xy + c\lambda^2\,y^2).$$

According to [11, Theorem 1], every group G in $\mathcal{T}(4,2)$ is isomorphic to $G(\delta,\lambda,f)$ for an appropriate choice of δ, λ, and f. The integers δ and λ are fully determined by G, but the quadratic form f is not. In fact, two groups $G(\delta,\lambda,f)$ and $G(\delta,\lambda,f')$ are isomorphic if and only if f and f' are λ-equivalent over $\mathbf{Z}$, in the following sense. Two binary quadratic forms f and f' are λ-equivalent over a ring R if there exists a unit $u \in R$ and a matrix

$$\begin{pmatrix} r_1 & r_2 \\ \lambda r_3 & r_4 \end{pmatrix} \in \mathrm{GL}_2(R), \qquad r_i \in R,$$

such that $f'(x,y) = u\, f(r_1x + r_2y,\ \lambda r_3x + r_4y)$. Furthermore, the localizations of two groups $G(\delta,\lambda,f)$ and $G(\delta',\lambda',f')$ at a set of primes P are isomorphic if and only if $\delta' = u\delta$, $\lambda' = v\lambda$, where u, v are units in $\mathbf{Z}_P$, and the forms f and f' are λ-equivalent over $\mathbf{Z}_P$; see [11, § 3].

We conclude by recalling how one uses ideals of orders in quadratic fields to deal with most groups in $\mathcal{T}(4,2)$ with $\lambda = \delta = 1$. The *content* $C(f)$ of a form $f(x,y) = ax^2 + bxy + cy^2$ is defined to be the greatest common divisor of a, b, c, and it is set to be zero if $f = 0$. The discriminant $D(f)$ is the number $b^2 - 4ac$. Both the content and the discriminant are invariant under λ-equivalence over $\mathbf{Z}$, for any λ. Thus, given a group G in $\mathcal{T}(4,2)$, we may speak without ambiguity of the invariants $C(G)$, $D(G)$, $\lambda(G)$ and $\delta(G)$. If $C(G) = 0$, then the center of G has rank 3 and the group G cannot be represented inside $\mathrm{UT}_3(R)$ for any ring R. If $C(G) \neq 0$, then the center of G has rank 2 and, as shown in [11, § 7], the group G is isomorphic to the matrix group

$$\Gamma(R,I,M) = \begin{pmatrix} 1 & R & M \\ 0 & 1 & I \\ 0 & 0 & 1 \end{pmatrix}$$

for a suitable choice of a ring R, an ideal I of R, and an additive subgroup M of R containing I. Moreover, the ring R is determined up to isomorphism by the group G, by [33, p. 273] and [11, Lemma 7]. If the discriminant $D(G)$ is a square, then the ring R necessarily contains zero-divisors. In the remaining cases, i.e., when $D(G)$ is not a square, one has $G \cong \Gamma(R,I,M)$ for some order R in the quadratic field $\mathbf{Q}(\sqrt{D})$, where $D = D(G)$. The index F of R in the maximal order is determined as follows. If $D(G) = k^2 d$, where d is square-free and k is a positive integer, then $F = k$ if $d \equiv 1 \bmod 4$, or else k is necessarily even and $F = \frac{1}{2}k$ if $d \not\equiv 1 \bmod 4$; cf. [11, Lemma 10].

The abelianization of $\Gamma(R, I, M)$ has a torsion subgroup which is isomorphic to M/I. Hence, $M = I$ if and only if the abelianization of G is torsion-free (i.e., $\lambda = \delta = 1$). In other words, the family $\mathbf{\Gamma}$ which we discussed in Section 3 consists precisely of those groups in $\mathcal{T}(4, 2)$ whose abelianization is torsion-free and whose discriminant is not a square.

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