Relative group completions

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Dedicated to Peter Hilton on his 80th birthday

Abstract

We extend arbitrary group completions to the category of pairs (G, N) where G is a group and N is a normal subgroup of G. Relative localizations are special cases. Our construction is a group-theoretical analogue of fibrewise completion and fibrewise localization in homotopy theory, and generalizes earlier work of Hilton and others on relative localization at primes. We use our approach to find conditions under which factoring out group radicals preserves exactness. This has implications in the study of the effect of plus-constructions on homotopy fibre sequences.

Introduction

Relative completions and relative localizations of groups are group-theoretical analogues of fibrewise completions and fibrewise localizations in homotopy theory. The latter were first discussed in the work of Sullivan [25] and Bousfield–Kan [7], and since then by many other authors, primarily focusing on localization at primes; see e.g. [15], [16], [19]. More recently, Bousfield [6, § 4] and Dror Farjoun [13, Ch. 1] have defined a fibrewise version of homotopical localization of spaces with respect to any map. The aim of this paper is to provide a similar tool for groups, and give applications within group theory and to homotopy theory.

Thus, our initial goal was to state and prove that localization of groups with respect to any group homomorphism (see [8]) also admits a relative version, that is, can be suitably defined on pairs of groups. However, we will present this result in much greater generality, by proving that every pointed endofunctor which is compatible with conjugation (as defined

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in Section 1) admits a relative version. We will then prove that completions are compatible with conjugation. Our notion of “completion” is explained in Section 2 and intends to be as general as possible, certainly applicable to many well-known constructions such as profinite completion and pronilpotent completion. By a “localization” we mean an idempotent completion.

Theorem 2.2 below, together with results in Section 1, imply the existence of fibrewise R-completion of groups for a commutative ring R, which was sketched by Bousfield and Kan in [7, IV.5.6]. Our approach also generalizes, very naturally, Hilton's construction in [14] and the one described by Descheemaeker and Malfait in [12].

Although localizations are special cases of completions, the relevance of their universal property justifies a distinguished treatment. Thus, if L is any localization in the category of groups, then the relative L-localization of a pair (G, N) is a pair (E, LN) with $E/LN \cong G/N$, together with a morphism $(G, N) \to (E, LN)$ which is initial among morphisms from (G, N) to pairs where the normal subgroup is L-local. We may describe this transformation by saying that the effect of L on N is uniformly extended over all cosets of G relative to N.

In the last two sections, we restrict ourselves to a special kind of group localizations, namely the ones consisting of dividing out radicals. These are called epireflections. Using their relative version, we analyze the effect of epireflections on group extensions. By doing so, we extend results of Berrick [2] about hypoabelianization of groups and preservation of homotopy fibre sequences under Quillen's plus-construction. Our results apply to the case of plus-construction with respect to homology with mod n coefficients, among other examples.

A preliminary version of the present article was useful to Rodríguez and Scevenels in [23, Theorem 6], in order to relate exactness of certain localizations with closure properties of the corresponding image classes. Some of their results are improved in Section 4 of this article.

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1 Relativizing pointed endofunctors

Let $\mathcal{C}$ be any category. A functor $L\colon \mathcal{C} \to \mathcal{C}$ together with a natural transformation $l\colon \mathrm{Id} \to L$ is called a *pointed endofunctor* or a *coaugmented functor* in the literature.

If (L, l) is a pointed endofunctor in a category $\mathcal{C}$, and G is a group, then (L, l) defines, in a canonical way, a pointed endofunctor in the category of G-objects (that is, objects of $\mathcal{C}$ equipped with an action of G) with G-maps as morphisms. Indeed, for every G-object N

and every element $g \in G$, there is a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{g} & N \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle l} \\ LN & \xrightarrow{Lg} & LN \end{array} \tag{1.1}$$

yielding a G-action on LN such that l is a G-map.

From now on, we will exclusively work in the categories of groups and relative groups. A *relative group* is a pair (G, N) where G is a group and N is a normal subgroup of G. It is usually more convenient to regard a relative group as a group extension $N \rightarrowtail G \twoheadrightarrow Q$. A morphism of relative groups is a commutative diagram

$$\begin{array}{ccccc} N & \rightarrowtail & G & \twoheadrightarrow & Q \\ {\scriptstyle l}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow \\ N' & \rightarrowtail & G' & \twoheadrightarrow & Q' \end{array}$$

of group extensions. Note that such a diagram yields an action of G on N' via f, and l is automatically a G-map, that is,

$$l(gng^{-1}) = f(g)\, l(n)\, f(g)^{-1}$$

for all $n \in N$ and $g \in G$.

Now let (L, l) be any pointed endofunctor in the category of groups. As in [5] or [14], we look for suitable conditions ensuring that the diagram

$$\begin{array}{ccccc} N & \rightarrowtail & G & \twoheadrightarrow & Q \\ {\scriptstyle l}\downarrow & & & & \\ LN & & & & \end{array}$$

can be embedded into a morphism of relative groups

$$\begin{array}{ccccc} N & \rightarrowtail & G & \twoheadrightarrow & Q \\ {\scriptstyle l}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\ LN & \overset{j}{\rightarrowtail} & E & \overset{p}{\twoheadrightarrow} & Q, \end{array} \tag{1.2}$$

specializing to the standard homomorphism

$$l_*\colon H^2(Q; N) \to H^2(Q; LN)$$

whenever N and LN are abelian, and admitting a similar description in terms of cohomology classes of nonabelian 2-cocycles otherwise.

This is the precise group-theoretical analogue of fibrewise functorial transformations in homotopy theory, where "continuity" of the functor is a necessary assumption to ensure that it can be coherently extended over the fibres. Similarly, we have to require that (L, l) be compatible with conjugation, in the sense that we now make precise. Our aim is to guarantee that the action of G on LN via f in (1.2) agrees with the action defined by the functoriality of L as depicted in (1.1).

For a group N, we denote by $\mathrm{Aut}(N)$ the group of automorphisms of N, and by $\mathrm{Out}(N)$ the quotient of $\mathrm{Aut}(N)$ by the inner automorphisms. That is, $\mathrm{Out}(N)$ is the cokernel of the homomorphism $\tau\colon N \to \mathrm{Aut}(N)$ given by $\tau_x(y) = xyx^{-1}$. We also need to consider the homomorphism $\mathrm{Aut}(N) \to \mathrm{Aut}(LN)$ sending each φ to $L\varphi$, which we do not label.

Definition 1.1 A pointed endofunctor (L, l) in the category of groups is *compatible with conjugation* if the diagram

$$\begin{array}{ccc} N & \xrightarrow{l} & LN \\ {\scriptstyle\tau}\downarrow & & {\scriptstyle\tau}\downarrow \\ \mathrm{Aut}(N) & \longrightarrow & \mathrm{Aut}(LN) \end{array} \qquad (1.3)$$

is commutative for every group N.

This property is not a tautology. For instance, it fails if L is the identity functor and $l(n) = 1$ for every group N and all $n \in N$. On the other hand, as we prove in Section 2, group completions are compatible with conjugation.

Note that, if (L, l) is compatible with conjugation, then L induces a group homomorphism

$$\mathrm{Out}(N) \to \mathrm{Out}(LN),$$

but the converse need not be true (as the same counterexample given above shows). Observe also that the diagram (1.3) trivially commutes if N and LN are abelian.

We next prove that compatibility with conjugation is sufficient to define a relative version of (L, l). We will give two different, equivalent constructions. The first one is based on earlier work of Hilton [14] and Bokor–Hilton [5]. The second one generalizes the method of Descheemaeker–Malfait published in [12].

First construction. Suppose given a group extension $N \rightarrowtail G \twoheadrightarrow Q$. Using the technique described in [5], [7, IV.5.6] or [14], we may embed the given extension into a commutative diagram with exact rows

$$\begin{array}{ccccc} N & \rightarrowtail & G & \twoheadrightarrow & Q \\ {\scriptstyle l}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\ LN & \xrightarrow{j} & E & \overset{p}{\twoheadrightarrow} & Q, \end{array} \qquad (1.4)$$

where we emphasize that the homomorphism j need not be injective if we omit the condition that (L, l) be compatible with conjugation. The group E and the maps in (1.4) are defined as follows; cf. [14]. Let $LN \rtimes G$ denote the semi-direct product with respect to the G-action on LN induced by functoriality from the conjugation action of G on N. Let S be the normal subgroup of $LN \rtimes G$ generated by all the elements of the form $(l(n)^{-1}, n)$ with $n \in N$. For $x \in LN$ and $g \in G$, we denote by $(x, g)S$ the class of (x, g) in the quotient

$$E = (LN \rtimes G)/S.$$

The homomorphism f is defined as $f(g) = (1, g)S$, and the homomorphisms j and p are given respectively by $j(x) = (x, 1)S$ and $p((x, g)S) = gN$. Then the diagram (1.4) commutes, and the kernel of p coincides with the image of j, since, for all $n \in N$ and $x \in LN$, we may write

$$(x, n) = (x\, l(n), 1)(l(n)^{-1}, n).$$

Now, the assumption that (L, l) is compatible with conjugation implies that j is injective. To see this, it is sufficient to prove that the set of elements of the form $(l(n)^{-1}, n)$ with $n \in N$ is closed under conjugation inside $LN \rtimes G$, and this follows precisely from the commutativity of (1.3) and the fact that $l\colon N \to LN$ is a G-map by (1.1).

Note that if L is the identity and $l(n) = 1$ for every N and all $n \in N$, then E is isomorphic to $(N/[N, N]) \rtimes Q$, so in this example j is not injective, unless N is abelian.

Second construction. Recall that every extension $N \rightarrowtail G \twoheadrightarrow Q$ of groups determines a group homomorphism $\psi\colon Q \to \mathrm{Out}(N)$, which is called an abstract kernel. An extension $N \rightarrowtail G \twoheadrightarrow Q$ is defined to be equivalent to another extension $N \rightarrowtail G' \twoheadrightarrow Q$ if there is an isomorphism $G \cong G'$ inducing the identity on both N and Q. As explained e.g. in [17], there is a one-to-one correspondence between the set $\mathrm{Ext}_\psi(Q, N)$ of equivalence classes of extensions $N \rightarrowtail G \twoheadrightarrow Q$ inducing an abstract kernel $\psi\colon Q \to \mathrm{Out}(N)$ and the set $H^2_\psi(Q; N)$ of cohomology classes of pairs (φ, c), where $\varphi\colon Q \to \mathrm{Aut}(N)$ is a function lifting ψ, and $c\colon Q \times Q \to N$ is a normalized 2-cocycle, that is,

$$\varphi(x) \circ \varphi(y) = c(x, y)\, \varphi(xy)\, c(x, y)^{-1}$$
$$\varphi(x)(c(y, z))\, c(x, yz) = c(x, y)\, c(xy, z),$$

for all $x, y, z \in Q$, and $\varphi(1) = \mathrm{id}$ and $c(x, 1) = c(1, x) = 1$ for all $x \in Q$.

Given a 2-cocycle (φ, c), the associated extension of N by Q can be explicitly described as the set $N \times Q$ equipped with the multiplication

$$(n, x)(m, y) = (n\ (\varphi(x)(m))\ c(x, y),\ xy)\,. \tag{1.5}$$

Now suppose given an extension $N \rightarrowtail G \twoheadrightarrow Q$ and a pointed endofunctor (L, l). Choose a normalized set-theoretical section $\sigma\colon Q \to G$ and let (φ, c) be the corresponding 2-cocycle, classifying the given extension. Specifically, φ is the composite of σ with the conjugation homomorphism $G \to \mathrm{Aut}(N)$, and c is defined as $c(x,y) = \sigma(x)\sigma(y)\sigma(xy)^{-1}$. Then the assumption that (L, l) is compatible with conjugation ensures that $(L \circ \varphi, l \circ c)$ is also a normalized 2-cocycle. Indeed, in order to verify that

$$L(\varphi(x)) \circ L(\varphi(y)) = l(c(x,y))\, L(\varphi(xy))\, l(c(x,y))^{-1},$$

one needs the functoriality of L and the assumption that (1.3) commutes. The second condition

$$L(\varphi(x))(l(c(y,z))) = l(c(x,y))\, l(c(xy,z))\, l(c(x,yz))^{-1}$$

only requires that l be a natural transformation.

The 2-cocycle $(L \circ \varphi, l \circ c)$ yields a group extension $LN \rightarrowtail E \twoheadrightarrow Q$, which is independent of the choices made, up to equivalence. Moreover, the homomorphism $f\colon G \to E$ sending (n, x) to $(l(n), x)$ renders commutative the diagram

$$\begin{array}{ccccc} N & \rightarrowtail & G & \twoheadrightarrow & Q \\ {\scriptstyle l}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\ LN & \rightarrowtail & E & \twoheadrightarrow & Q, \end{array}$$

in such a way that the action of G on LN defined via f agrees with the action given by conjugation on N and functoriality of L.

We conclude by checking that the bottom extension in this diagram agrees, up to equivalence, with the extension obtained with the first construction. For this, define a section $\sigma'\colon Q \to (LN \rtimes G)/S$ by $\sigma'(q) = (1, \sigma(q))S$. Then the corresponding 2-cocycle is

$$c'(x,y) = \sigma'(x)\sigma'(y)\sigma'(xy)^{-1} = (1, c(x,y))S = (l(c(x,y)), 1)S.$$

Hence $c' = l \circ c$, as needed.

2 Relative completions and localizations

Our main goal in this section is to show that, if the pointed endofunctor (L, l) is a completion, then it is compatible with conjugation.

We first make precise what we mean by a completion functor. Let $\mathcal{D}$ be any class of objects in a category $\mathcal{C}$ and let $K\colon \mathcal{D} \to \mathcal{C}$ denote the embedding, where $\mathcal{D}$ is now viewed as a full subcategory of $\mathcal{C}$. For any object X in $\mathcal{C}$, the objects of the comma category $(X \downarrow K)$ are the morphisms $X \to D$ with D in $\mathcal{D}$, and morphisms in $(X \downarrow K)$ are commutative triangles.

The *$\mathcal{D}$-completion* of an object X is the inverse limit $\widehat{X}$ (if it exists) of the functor from $(X \downarrow K)$ to the category $\mathcal{C}$ sending $X \to D$ to D. In other words, $\mathcal{D}$-completion is the codensity monad (or codensity triple) of the full embedding $K: \mathcal{D} \to \mathcal{C}$. That is, the right Kan extension of K along itself; see [7, IV.2.2], after Artin–Mazur, and [18, p. 246].

Thus, if $\mathcal{D}$-completion exists for an object X, then there is a natural morphism $l: X \to \widehat{X}$, which may be viewed as "the closest approximation to X by an inverse limit of objects in $\mathcal{D}$". Well-known examples in the category of groups include profinite completion (where $\mathcal{D}$ is the class of finite groups) and pronilpotent completion (where $\mathcal{D}$ is the class of nilpotent groups). Each of these admits local versions at primes.

If $\mathcal{D}$-completion exists for all objects, then it is a pointed endofunctor. The existence of $\mathcal{D}$-completion is guaranteed for all objects if (small) inverse limits exist in $\mathcal{C}$ and the class $\mathcal{D}$ is a set, but also in other cases, for instance when $\mathcal{D}$ is reflective. If $\widehat{X}$ belongs itself to $\mathcal{D}$, then $\mathcal{D}$-completion is idempotent on X. It is in fact idempotent on all objects in many cases, but not always (see [7, IV.2 and IV.5.4]).

Recall that a pointed endofunctor (L, l) is called *idempotent* if $Ll: L \to LL$ is an isomorphism and $Ll = lL$; see [8] for a recent survey about this concept. Idempotent pointed endofunctors will be called *localizations*. Such functors L are characterized by the following universal property. For all X, Y and every morphism $\varphi: X \to LY$, there is a unique morphism $\psi: LX \to LY$ such that $\psi \circ l = \varphi$.

If L is a localization in any category, then *L-local* objects are those isomorphic to LX for some X, and *L-equivalences* are morphisms f such that Lf is an isomorphism. Thus, for every X, the morphism $l: X \to LX$ is an L-equivalence, and it is the unique L-equivalence from X to an L-local object, up to isomorphism. Furthermore, local objects X and equivalences $f: A \to B$ are *orthogonal*, meaning that the induced function

$$f^*: \mathcal{C}(B, X) \to \mathcal{C}(A, X)$$

is a bijection, where $\mathcal{C}(B, X)$ denotes the set of morphisms $B \to X$ in $\mathcal{C}$.

Some examples of localizations in the category of groups are abelianization, hypoabelianization (i.e., dividing out the perfect radical), and localization at primes.

Localizations are special cases of completions. Specifically,

Proposition 2.1 *Every localization (L, l) is isomorphic to $\mathcal{D}$-completion where $\mathcal{D}$ is the class of L-local groups.*

PROOF. For each object X, the morphism $l\colon X \to LX$ is an initial object in the comma category $(X \downarrow K)$, where $K\colon \mathcal{D} \to \mathcal{C}$ is the full embedding. $\square$

We now prove that completions are compatible with conjugation, as defined in Section 1.

Theorem 2.2 *Let $\mathcal{D}$ be a class of groups such that $\mathcal{D}$-completion exists for all groups. Let N be any group and let $l\colon N \to \widehat{N}$ denote its $\mathcal{D}$-completion. Then the action of N on $\widehat{N}$ induced by conjugation and functoriality satisfies*

$$n \cdot x = l(n)\, x\, l(n)^{-1} \qquad \text{for all } n \in N \text{ and } x \in \widehat{N}.$$

PROOF. First we need to describe the action of N on $\widehat{N}$ in explicit terms, using the fact that $\widehat{N}$ is an inverse limit. For any element $n \in N$, let $\tau_n\colon N \to N$ denote conjugation by n. For each homomorphism $f_D\colon N \to D$ with D in $\mathcal{D}$, we consider the composite

$$N \xrightarrow{\tau_n} N \xrightarrow{f_D} D$$

and denote by $g_{n,D}$ the corresponding homomorphism $\widehat{N} \to D$ in the limiting cone, satisfying $g_{n,D} \circ l = f_D \circ \tau_n$ (abbreviated to g_D when $n = 1$). Then the automorphism $\gamma_n\colon \widehat{N} \to \widehat{N}$ induced by τ_n is uniquely characterized by the property that $g_D \circ \gamma_n = g_{n,D}$ for every $f_D\colon N \to D$.

Now fix any $n \in N$. Observe that, for each $f_D\colon N \to D$ with D in $\mathcal{D}$, we have

$$\tau_{f_D(n)} \circ f_D = f_D \circ \tau_n,$$

and this implies that $\tau_{f_D(n)} \circ g_D = g_{n,D}$. Thus, we write $\nu(x) = l(n)\, x\, l(n)^{-1}$ for $x \in \widehat{N}$ and check that ν satisfies the property that characterizes the automorphism $\gamma_n\colon \widehat{N} \to \widehat{N}$, namely

$$g_D(\nu(x)) = f_D(n)\, g_D(x)\, f_D(n)^{-1} = \tau_{f_D(n)}(g_D(x)) = g_{n,D}(x)$$

for all $x \in \widehat{N}$, as needed. $\square$

As we saw in Section 1, this implies that completion functors admit relative versions. This enlightens and generalizes the construction made by Bousfield and Kan in [7, IV.5.6], dealing with completion with respect to a ring.

Since localizations are special cases of completions, Theorem 2.2 applies to localizations as well. However, the proof of Theorem 2.2 is much simpler in the case of localizations, since we may then use their universal property to infer the result from the equation

$$n \cdot l(m) = l(n)\, l(m)\, l(n)^{-1}$$

for $n, m \in N$, as done in earlier articles about localization at primes.

3 Universal property of relative localizations

In this section we make explicit the universal property of relative group localizations, hence generalizing substantial parts of [5], [9], [12], and [14].

Given a localization L in the category of groups, we say that a relative group $N \rightarrowtail G \twoheadrightarrow Q$ is *L-local* if N is an L-local group. A morphism of relative groups is an *L-equivalence* if it is orthogonal to all L-local relative groups.

Proposition 3.1 *If a morphism of relative groups*

$$\begin{array}{ccccc} N & \rightarrowtail & G & \twoheadrightarrow & Q \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow \\ N' & \rightarrowtail & G' & \twoheadrightarrow & Q' \end{array}$$

is an L-equivalence, then β is an L-equivalence of groups and γ is an isomorphism.

PROOF. This is proved precisely as in [9, Proposition 1.3]. First use that (α, β, γ) is orthogonal to $D \rightarrowtail D \twoheadrightarrow 1$, where D is any L-local group, and then choose $1 \rightarrowtail E \twoheadrightarrow E$, where E is any group. $\square$

Our main result in this section is the following. It includes the statement that the middle map in a relative L-localization is an L-equivalence of groups, which is of great importance for the applications. Note the analogy with fibrewise localization in homotopy theory, by comparing it with [13, Theorem 1.F.3].

Theorem 3.2 *Let $N \rightarrowtail G \twoheadrightarrow Q$ be any extension of groups, and let (L, l) be any localization in the category of groups. Then there is a commutative diagram*

$$\begin{array}{ccccc} N & \rightarrowtail & G & \twoheadrightarrow & Q \\ {\scriptstyle l}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\ LN & \rightarrowtail & E & \twoheadrightarrow & Q \end{array} \qquad (3.1)$$

which is an L-equivalence of relative groups, and where f is an L-equivalence of groups.

PROOF. Theorem 2.2 tells us that localization functors are compatible with conjugation, and this suffices to obtain (3.1) with the property that the action of G on LN given via f agrees with the action given by functoriality of L, as shown in Section 1. Now we want to prove that the morphism (l, f, id) is an L-equivalence of relative groups. Suppose given a group extension $N' \rightarrowtail G' \twoheadrightarrow Q'$ where N' is L-local, and a morphism (α, β, γ) from $N \rightarrowtail G \twoheadrightarrow Q$ to $N' \rightarrowtail G' \twoheadrightarrow Q'$. Since $l\colon N \to LN$ is an L-localization of groups, there is a unique $\alpha'\colon LN \to N'$ such that $\alpha' \circ l = \alpha$.

We first view E as $(LN \rtimes G)/S$, using the notation of Section 1, and define

$$\beta' \colon E \to G', \qquad \beta'((n,g)S) = \alpha'(n)\,\beta(g).$$

Then β' is a well-defined group homomorphism. In order to check this, the following equality is needed, as in [14, Theorem 1.2]:

$$\alpha'(g \cdot x) = \beta(g)\,\alpha'(x)\,\beta(g)^{-1} \qquad \text{for all } g \in G \text{ and all } x \in LN. \tag{3.2}$$

This is proved by picking first $x \in \operatorname{im} l$ and then using the universal property of L.

Alternatively, we may view E as an extension of LN by Q classified by $(L \circ \varphi, l \circ c)$, where (φ, c) is a 2-cocycle classifying $N \rightarrowtail G \twoheadrightarrow Q$. Then we define a function $\beta' \colon E \to G'$ as follows, where E is written as a twisted product $LN \times Q$ with the multiplication described in (1.5):

$$\beta'(x,q) = (\alpha'(x), 1)\,\beta(1,q).$$

In order to show that β' is a group homomorphism, one needs that

$$\alpha'(((L \circ \varphi)(q))(x)) = \beta(1,q)\,(\alpha'(x),1)\,\beta(1,q)^{-1},$$

and this follows again from the universal property of L, by checking it first for $x \in \operatorname{im} l$.

In either case, we find that $(\alpha', \beta', \gamma)$ is a morphism of relative groups, and it satisfies

$$(\alpha', \beta', \gamma) \circ (l, f, \mathrm{id}) = (\alpha, \beta, \gamma).$$

There only remains to prove the uniqueness of $(\alpha', \beta', \gamma)$. Suppose that $(\alpha'', \beta'', \gamma'')$ satisfies

$$(\alpha'', \beta'', \gamma'') \circ (l, f, \mathrm{id}) = (\alpha, \beta, \gamma).$$

Then $\gamma'' = \gamma$, and we also have $\alpha'' = \alpha'$ by the universal property of L. Finally,

$$\beta''((n,g)S) = \beta''((n,1)S)\,\beta''((1,g)S) = \alpha''(n)\,\beta''(f(g)) = \alpha''(n)\,\beta(g),$$

so $\beta'' = \beta'$. This proves that (l, f, id) is an L-equivalence of relative groups. The fact that f is then an L-equivalence of groups follows from Proposition 3.1. □

Corollary 3.3 *A morphism (α, β, γ) of relative groups is an L-equivalence if and only if α is an L-equivalence of groups and γ is an isomorphism.*

Proof. This is seen by applying relative L-localization to both the domain and the codomain of the morphism (α, β, γ). □

Note that the converse of Proposition 3.1 is not true in general (although it is true in some special cases, such as in [14, Theorem 2.5]). For example, if we consider the diagram

$$\begin{array}{ccccc} \mathbf{Z}/3 & \rightarrowtail & \Sigma_3 & \twoheadrightarrow & \mathbf{Z}/2 \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\mathrm{id}}\downarrow \\ 1 & \rightarrowtail & \mathbf{Z}/2 & \twoheadrightarrow & \mathbf{Z}/2, \end{array}$$

where Σ_3 denotes the symmetric group on three elements, and choose L to be localization at the prime 3, then β is an L-equivalence of groups, yet the diagram is not an L-equivalence of relative groups.

4 Dividing out group radicals

In this section we analyze the extent to which dividing out radicals preserves exactness or half-exactness of group extensions. Relative localization is used at a few key places.

We recall from [11] that a *radical* R in the category of groups is a subfunctor of the identity (i.e., a functor assigning to each group G a subgroup RG in such a way that every homomorphism $G \to K$ induces a homomorphism $RG \to RK$ by restriction), with the property that RG is normal in G and $R(G/RG) = 1$. The functoriality of R implies that RG is a characteristic subgroup of G for every group G; therefore, if G embeds as a normal subgroup into some group K, then RG is also normal in K.

A radical R is *idempotent* if $RRG = RG$ for all groups G. To every radical R one can associate an idempotent radical R^∞ as follows; cf. [11, Proposition 1.2]. For each group G, let $R^\infty G$ be the product of all subgroups H of G such that $RH = H$. Then $RR^\infty G = R^\infty G$, and $R^\infty G$ is maximal with this property. We have $R^\infty G \subseteq RG$ for all groups G. In fact, R^∞ is the largest idempotent radical that is a subfunctor of R. It can alternatively be defined as the inverse limit of the following series of radicals: $R^0 = R$; $R^\alpha = RR^{\alpha-1}$ for every successor ordinal α, and R^α is the intersection of R^β for all $\beta < \alpha$ if α is a limit ordinal. An illuminating example is $RG = [G,G]$, the commutator radical, for which $R^\infty G = \mathcal{P}G$, the perfect radical, which is the intersection of the (transfinite) derived series.

If (L, l) is any localization functor in the category of groups, then we may associate with it a radical, by defining RG to be the kernel of the localizing homomorphism $l\colon G \to LG$. However, such radicals need not be idempotent in general; for instance, $[G,G]$ is the kernel of the abelianization homomorphism.

A localization functor (L, l) is called an *epireflection* if, for every group G, the natural map $l\colon G \to LG$ is an epimorphism. If R is any radical in the category of groups, then $LG = G/RG$ defines an epireflection. Thus, there is a bijective correspondence between radicals and epireflections. An epireflection is called a *reduction* if the corresponding radical is idempotent.

Among localization functors, it is possible to characterize epireflections and reductions in terms of closure properties of the class of L-local groups. The following result, whose proof uses relative localization, generalizes Theorem 6 in [23].

Theorem 4.1 *Let (L, l) be a localization in the category of groups. Then L is an epireflection if and only if the class of L-local groups is closed under taking subgroups, and L is a reduction if and only if the class of L-local groups is closed under taking subgroups and forming extensions.*

PROOF. Suppose that L is an epireflection. Let G be an L-local group, and S a subgroup of G. Then, by assumption, $l\colon S \to LS$ is surjective. Since the inclusion $S \hookrightarrow G$ factors through LS, $l\colon S \to LS$ is also injective, so S is L-local. Conversely, suppose that the class of L-local groups is closed under subgroups. Let G be any group and $l\colon G \to LG$ its localization. We may factor l as $G \twoheadrightarrow \operatorname{im} l \hookrightarrow LG$ and, by assumption, $\operatorname{im} l$ is L-local. Since every homomorphism from G to an L-local group factors uniquely through $\operatorname{im} l$, the arrow $G \twoheadrightarrow \operatorname{im} l$ is an L-equivalence. Hence, the inclusion $\operatorname{im} l \hookrightarrow LG$ is also an L-equivalence and therefore it is an isomorphism. This shows that $l\colon G \to LG$ is surjective.

Now suppose that L is a reduction. This means that $LG = G/RG$ where $RRG = RG$ for all groups G. As this is an epireflection, the class of L-local groups is closed under subgroups. Let $N \rightarrowtail G \twoheadrightarrow Q$ be a group extension where N and Q are L-local. Thus, $RN = 1$ and $RQ = 1$. Since RG maps into RQ, we have $RG \subseteq N$. Then $RG = RRG \subseteq RN$ implies that $RG = 1$, so G is L-local as well. Finally, suppose that the class of L-local groups is closed under subgroups and formation of extensions. Then L is an epireflection and hence $LG = G/RG$ for some radical R and all groups G. Hence, it is enough to show that RRG equals RG for all G. This is the same as proving that $L(RG) = 1$ for all G. Take any group G and consider the relative localization

$$\begin{array}{ccccc} RG & \rightarrowtail & G & \twoheadrightarrow & LG \\ {\scriptstyle l}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\ L(RG) & \rightarrowtail & E & \twoheadrightarrow & LG, \end{array}$$

where, by Theorem 3.2, the map f is an L-equivalence. Since the composite of f with the surjection $E \twoheadrightarrow LG$ is an L-equivalence, we may infer that the latter is also an L-equivalence. Since we are assuming that the class of L-local groups is closed under extensions, the group E is L-local. Therefore, the surjection $E \twoheadrightarrow LG$ is an isomorphism, and this implies that $L(RG) = 1$, as needed. □

Let R be any radical in the category of groups, and consider the epireflection given by $LG = G/RG$ for all G. In this situation, relative L-localization of any group extension $N \rightarrowtail G \twoheadrightarrow Q$ yields a short exact sequence

$$N/RN \rightarrowtail G/RN \twoheadrightarrow Q, \tag{4.1}$$

since RN is normal in G. Examples include

$$N/[N,N] \rightarrowtail G/[N,N] \twoheadrightarrow Q \qquad \text{and} \qquad N/IN \rightarrowtail G/IN \twoheadrightarrow Q,$$

where IN is the *isolator subgroup* of N, that is, the smallest normal subgroup of N such that N/IN is torsion-free.

Furthermore, we can consider the commutative diagram

$$\begin{array}{ccccc} RN & \overset{Ri}{\rightarrowtail} & RG & \overset{Rp}{\to} & RQ \\ \downarrow & & \downarrow & & \downarrow \\ N & \overset{i}{\rightarrowtail} & G & \overset{p}{\twoheadrightarrow} & Q \\ \downarrow & & \downarrow & & \downarrow \\ LN & \overset{Li}{\to} & LG & \overset{Lp}{\twoheadrightarrow} & LQ \end{array}$$

where the upper and lower rows need not be exact in general, although the restriction Ri is necessarily injective, and Lp is necessarily surjective. It is also clear that the composites $Rp \circ Ri$ and $Lp \circ Li$ are trivial, by functoriality.

Proposition 4.2 *Let $N \overset{i}{\rightarrowtail} G \overset{p}{\twoheadrightarrow} Q$ be any group extension. Let R be any radical, and let $LG = G/RG$ be the associated epireflection. Then the following assertions hold.*

(i) $\operatorname{im} Ri = \ker Rp$ *if and only if Li is injective.*

(ii) $\operatorname{im} Li = \ker Lp$ *if and only if Rp is surjective.*

(iii) *The sequence $1 \to RN \to RG \to RQ \to 1$ is exact if and only if the sequence $1 \to LN \to LG \to LQ \to 1$ is exact.*

PROOF. This is a standard diagram-chase argument. In fact, the Nine-Lemma, as stated e.g. in [24, p. 98], holds in the category of groups as well. □

In other words, an epireflection L preserves exactness of a group extension if and only if the associated radical does. Now the following result extends [23, Proposition 5].

Theorem 4.3 *Let $N \rightarrowtail G \twoheadrightarrow Q$ be any group extension and let (L, l) be an epireflection. If Q is L-local, then the sequence $LN \to LG \to Q \to 1$ is exact. Moreover, if L is a reduction, then the sequence $1 \to LN \to LG \to Q \to 1$ is exact.*

Proof. The assumption that Q is L-local tells us that $RQ = 1$, hence the first claim, by Proposition 4.2. To prove the second statement, apply relative L-localization to the extension $N \rightarrowtail G \twoheadrightarrow Q$ and recall from Theorem 4.1 that the class of L-local groups for any reduction is closed under extensions. □

5 Word radicals and plus-constructions

Recall from [21] that, given any set W of elements of a free group F_∞ on a countably infinite set of generators (called *words*), the *variety* defined by W is the class of groups G such that every homomorphism $f: F_\infty \to G$ satisfies $f(w) = 1$ for all $w \in W$. For an arbitrary group G, the *verbal subgroup* with respect to W is the subgroup of G generated by all the images of elements of W through homomorphisms $F_\infty \to G$. Thus, the variety defined by W consists precisely of the groups whose verbal subgroup with respect to W is trivial. Factoring out the verbal subgroup is an epireflection, which is called the *projection* onto the variety. As proved in [21], a class of groups forms a variety if and only if it is closed under taking subgroups and forming quotients and direct products.

Let $\mathcal{V}$ be any variety of groups, and denote by RG the corresponding verbal subgroup of a group G. Then R is a radical. Radicals of this kind will be called *word radicals*. Examples are $RG = [G, G]$ (the commutator subgroup); $RG = \Gamma^i G$ (the ith term of the lower central series); $RG = D^i G$ (the ith term of the derived series); or $RG = G^n$ (the subgroup generated by all nth powers). In fact, as shown in [21], every variety can be defined by a set of words $W = \{x^n, c_1, c_2, c_3, \ldots\}$ where each c_i is an iterated commutator and n is a nonnegative integer, which is called the *exponent* of the variety.

We emphasize that, if R is a word radical and $G \twoheadrightarrow Q$ is any group epimorphism, then the restriction $RG \to RQ$ is necessarily surjective. Therefore, we may use Proposition 4.2 to conclude that, for any word radical R, every group extension $N \rightarrowtail G \twoheadrightarrow Q$ yields a four-term exact sequence

$$1 \to (N \cap RG)/RN \to N/RN \to G/RG \to Q/RQ \to 1.$$

The perfect radical $\mathcal{P}$ is not a word radical. For the perfect radical, a group extension $N \rightarrowtail G \twoheadrightarrow Q$ in which the restriction $\mathcal{P}G \to \mathcal{P}Q$ is surjective was called an "extension preserving perfect radicals" in [2]. Thus, by Proposition 4.2, an extension $N \rightarrowtail G \twoheadrightarrow Q$ preserves perfect radicals if and only if the sequence

$$1 \to (N \cap \mathcal{P}G)/\mathcal{P}N \to N/\mathcal{P}N \to G/\mathcal{P}G \to Q/\mathcal{P}Q \to 1$$

is exact.

As we said in Section 4, the perfect radical may be viewed as R^∞ where R is the commutator radical. The next proposition gives sufficient conditions under which, for a radical R and a group epimorphism $G \twoheadrightarrow Q$, the restriction $R^\infty G \to R^\infty Q$ is surjective. This generalizes Proposition 2.3 of [2].

Proposition 5.1 *Let $N \rightarrowtail G \twoheadrightarrow Q$ be any group extension and R a radical. Then the sequence*

$$1 \to (N \cap R^\infty G)/R^\infty N \to N/R^\infty N \to G/R^\infty G \to Q/R^\infty Q \to 1$$

is exact in each of the following cases:

(i) *The extension $N \rightarrowtail G \twoheadrightarrow Q$ splits.*

(ii) *R is a word radical and $R^m G \subseteq N \cdot R^\infty G$ for some finite m.*

(iii) *R is a word radical and $R^n N \subseteq R^\infty G$ for some finite n.*

Proof. Part (i) and part (ii) follow as in the proof of [2, Proposition 2.3]. In the course of the proof of (ii), one uses the fact that the restriction $R^m G \to R^m Q$ is surjective for all m, since R is a word radical. Part (iii) uses the fact that, if J is the inverse image of $R^\infty Q$ in G, then $R^n J = R^{n+1} J$, where n is the integer given by the assumption made in (iii). The details are as in [2, Lemma 2.1]. □

Certain results about Quillen's plus-construction remain valid if one replaces the commutator radical by any word radical R. Recall that the *plus-construction* associates to each connected topological space X, by attaching cells, another space X^+ with the same integral homology groups as X and such that the fundamental group $\pi_1(X^+)$ is isomorphic to the quotient of $\pi_1(X)$ by its perfect radical $\mathcal{P}\pi_1(X)$. This construction was introduced by Quillen in [22] and plays an important role in algebraic K-theory.

An analogue of the plus-construction with respect to any word radical was defined in [11] as follows. For every word radical R there is a locally free group Φ_R with the property that, given any group G, the subgroup $R^\infty G$ is generated by the images of all homomorphisms $\Phi_R \to G$. As in [3], we say that Φ_R *sweeps* the radical R^∞. The group Φ_R was constructed by refining the technique used in [3, Example 5.3] to obtain a locally free group sweeping the perfect radical.

Now, for each connected space X and each word radical R, the plus-construction X_R^+ of X with respect to R is defined as the f-localization of X, in the sense of Farjoun [13], with respect to the map $f\colon K(\Phi_R, 1) \longrightarrow *$, where $K(\Phi_R, 1)$ is an Eilenberg–Mac Lane space with fundamental grup Φ_R, and $*$ is a point. Thus, there is a natural map $l\colon X \longrightarrow X_R^+$ which is universal in the homotopy category with the property that the space X_R^+ is $K(\Phi_R, 1)$-*null*. Recall from [13] that this means that every map $\Sigma^n K(\Phi_R, 1) \longrightarrow X_R^+$ is nullhomotopic for all $n \geq 0$ (where Σ denotes suspension), or, equivalently, that the pointed mapping space

$$\mathrm{map}_*(K(\Phi_R, 1),\, X_R^+)$$

is weakly contractible (that is, it is path-connected and all its homotopy groups are trivial). Hence, the plus-construction with respect to R assigns to X, in a universal way, a space X_R^+ which is "invisible" from $K(\Phi_R, 1)$.

If R is the commutator radical, then X_R^+ is homotopy equivalent to Quillen's plus-construction, as proved in [3]. If R is any word radical, then the map $l\colon X \longrightarrow X_R^+$ induces an epimorphism $\pi_1(X) \twoheadrightarrow \pi_1(X_R^+)$ factoring out precisely the radical $R^\infty \pi_1(X)$. Moreover, this map l induces an isomorphism in homology with $\mathbf{Z}/n$ coefficients if the variety associated with R has exponent $n > 1$; see [11, Theorem 4.1]. In particular, if R corresponds to the words x^n and $[x, y]$, then X_R^+ is a plus-construction for homology with mod n coefficients, in the sense of [20], also called a partial $\mathbf{Z}/n$-completion in [7, VII.6].

Proposition 5.2 *Let $N \rightarrowtail G \twoheadrightarrow Q$ be any group extension and R any radical. If $R^\infty Q = 1$, then the sequence $1 \to N/R^\infty N \to G/R^\infty G \to Q \to 1$ is exact.*

Proof. This is a direct consequence of Theorem 4.3 and the fact that dividing out R^∞ is a reduction because R^∞ is idempotent. □

The following homotopy-theoretical version of Proposition 5.2 extends [1, Theorem 1.1] to arbitrary word radicals. See also [4].

Theorem 5.3 *Let $F \longrightarrow E \longrightarrow B$ be a homotopy fibre sequence of connected spaces, and let R be any word radical. If $B_R^+ \simeq B$, then $F_R^+ \longrightarrow E_R^+ \longrightarrow B_R^+$ is a homotopy fibre sequence.*

Proof. Apply the plus-construction fibrewise,

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ {\scriptstyle l}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\ F_R^+ & \longrightarrow & X & \longrightarrow & B, \end{array}$$

and recall from [13, Theorem 1.F.3] that f induces a homotopy equivalence $E_R^+ \simeq X_R^+$. Now, if Φ_R is a locally free group sweeping the radical R^∞, we have that B and F_R^+ are both $K(\Phi_R, 1)$-null, so X is also $K(\Phi_R, 1)$-null and this implies that $X_R^+ \simeq X$. □

We end the article by showing that, if the radical R corresponds to a variety of exponent zero, then the condition that $B_R^+ \simeq B$ is equivalent to the condition that $R^\infty \pi_1(B)$ be trivial. For varieties of nonzero exponent, this is not true, as we prove by means of a counterexample. Before displaying it, we need to analyze more closely the effect of plus-construction with respect to any radical R. Recall from [7] that an abelian group A is called Ext-p-complete, where p is a prime, if $\mathrm{Hom}(\mathbf{Z}[1/p], A) = 0$ and $\mathrm{Ext}(\mathbf{Z}[1/p], A) = 0$.

Theorem 5.4 *Let Φ be the direct limit of any sequence of free groups indexed by the first infinite ordinal. Then a connected space X is $K(\Phi, 1)$-null if and only if the following conditions hold:*

(i) $\mathrm{Hom}(\Phi, \pi_1(X))$ *is trivial.*

(ii) $\mathrm{Hom}(H_1(\Phi), \pi_i(X)) = 0$ *and* $\mathrm{Ext}(H_1(\Phi), \pi_i(X)) = 0$ *for* $i \geq 2$.

Proof. Write Φ as the direct limit of

$$F_1 \xrightarrow{\varphi_1} F_2 \xrightarrow{\varphi_2} F_3 \xrightarrow{\varphi_3} \cdots \tag{5.1}$$

where each F_i is free on a set $\{x_i^j\}$ of free generators. Let F_g be the free product of all the groups F_i, and let F_r be another free group with a set of generators r_i^j corresponding bijectively with the generators x_i^j of F_g. Consider the homomorphism $\psi\colon F_r \to F_g$ defined as $\psi(r_i^j) = (x_i^j)^{-1}\varphi_i(x_i^j)$ for all i, j. Thus, the normal closure N of the image of ψ in F_g is the kernel of a free presentation of Φ,

$$N \rightarrowtail F_g \twoheadrightarrow \Phi,$$

yielding, by abelianization, a sequence

$$F_r/[F_r, F_r] \rightarrowtail F_g/[F_g, F_g] \twoheadrightarrow \Phi/[\Phi, \Phi] \tag{5.2}$$

which is a free abelian presentation of the abelianization of Φ, hence exact.

In homotopy-theoretical terms, we may consider a wedge of circles W_i with fundamental group F_i, for every i, and consider the homotopy colimit of the sequence of maps corresponding to (5.1). This space is a 2-dimensional $K(\Phi, 1)$, since it is an ascending union of 2-dimensional $K(F_i, 1)$ spaces. (In the case of sequences indexed by ordinals bigger than the first infinite ordinal, the homotopy colimit could fail to be 2-dimensional.) The cell decomposition of this homotopy colimit is described by a homotopy cofibre sequence

$$W_r \longrightarrow W_g \longrightarrow K(\Phi, 1), \tag{5.3}$$

inducing precisely (5.2) on H_1. The spaces W_r and W_g are wedges of circles with fundamental groups F_r and F_g, respectively.

Now a connected space X is $K(\Phi, 1)$-null if and only if the sets of pointed homotopy classes of maps $[\Sigma^n K(\Phi, 1), X]$ are trivial for all $n \geq 0$. Hence, the long exact sequence

$$\cdots \to [\Sigma K(\Phi, 1), X] \to [\Sigma W_g, X] \to [\Sigma W_r, X] \to [K(\Phi, 1), X] \to [W_g, X] \to [W_r, X]$$

obtained from (5.3) proves the statement, since

$$[\Sigma^n W_g, X] \cong [W_g, \Omega^n X] \cong \mathrm{Hom}(F_g, \pi_{n+1}(X))$$

for all n, and similarly with W_r. □

Corollary 5.5 *Let R be a word radical associated with a variety of groups $\mathcal{V}$, and let X be a connected space.*

(i) *If $\mathcal{V}$ has exponent zero, then $X_R^+ \simeq X$ if and only if $R^\infty \pi_1(X)$ is trivial.*

(ii) *If $\mathcal{V}$ has a prime exponent p, then $X_R^+ \simeq X$ if and only if $R^\infty \pi_1(X)$ is trivial and, for each $i \geq 2$, the group $\pi_i(X)$ is* Ext*-p-complete.*

PROOF. Let Φ_R be a locally free group sweeping R^∞, constructed as in [11, Theorem 3.3]. We recall that Φ_R is a free product of countable locally free groups, which is perfect if $\mathcal{V}$ has exponent zero, and

$$H_1(\Phi_R) \cong \bigoplus_{i \in I} \mathbf{Z}[1/p] \tag{5.4}$$

for some set of indices I if $\mathcal{V}$ has exponent p. Since $X_R^+ \simeq X$ if and only if X is $K(\Phi_R, 1)$-null, both statements (i) and (ii) are consequences of Theorem 5.4. □

Corollary 5.6 *Let R be a word radical corresponding to a variety of prime exponent p. Let X be a connected space such that $\pi_1(X)$ is abelian and p-divisible. Then X_R^+ is homotopy equivalent to the p-completion of X.*

Proof. It follows from Theorem 5.4 and the isomorphism (5.4) that, if $\pi_1(X)$ is abelian, then X is $K(\Phi_R,1)$-null if and only if it is $K(\mathbf{Z}[1/p],1)$-null, since $\mathbf{Z}[1/p]$ is also locally free. Hence, X_R^+ is homotopy equivalent to the localization of X with respect to the map $K(\mathbf{Z}[1/p],1) \longrightarrow *$, where $*$ denotes a point. The assumption that $\pi_1(X)$ is p-divisible is equivalent to imposing that $\mathbf{Z}[1/p]$ sweeps $\pi_1(X)$, and this guarantees that X_R^+ is homotopy equivalent to the p-completion of X, by [10, Theorem 4.4]. $\square$

Using this result, we show that the assumption that $B_R^+ \simeq B$ cannot be replaced with the assumption that $R^\infty \pi_1(B) = 1$ in Theorem 5.3 if the variety associated with R has nonzero exponent. The homotopy fibre sequence

$$K(\mathbf{Z},1) \longrightarrow K(\mathbf{Z}/2,1) \longrightarrow K(\mathbf{Z},2)$$

corresponding to a nonzero element in $H^2(\mathbf{Z}/2;\mathbf{Z})$ is transformed by the plus-construction with respect to any word radical of exponent p, where p is any odd prime, into the sequence

$$K(\mathbf{Z},1) \longrightarrow * \longrightarrow K(\widehat{\mathbf{Z}}_p,2),$$

where $\widehat{\mathbf{Z}}_p$ denotes the p-adic integers. This is of course not a homotopy fibre sequence.

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