Localizations as idempotent approximations to completions

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Abstract

Given a monad (also called a triple) $\mathbf{T}$ on an arbitrary category, an idempotent approximation to $\mathbf{T}$ is defined as an idempotent monad $\hat{\mathbf{T}}$ rendering invertible precisely the same class of morphisms which are rendered invertible by $\mathbf{T}$. One basic example is homological localization with coefficients in a ring R, which is an idempotent approximation to R-completion in the homotopy category of CW-complexes. We give general properties of idempotent approximations to monads using the machinery of orthogonal pairs, aiming to a better understanding of the relationship between localizations and completions.

0 Introduction

A monad on a category $\mathcal{C}$ consists of a functor $T: \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\mu: T^2 \to T$ and $\eta: \mathrm{Id} \to T$ satisfying the conditions of a multiplication and a unit; see [18, Ch. VI]. A monad is called idempotent if μ is an isomorphism. Idempotent monads are also called localizations, although the latter term is sometimes used with a more restrictive meaning.

It has long been known that, under suitable assumptions on the category $\mathcal{C}$, it is possible to universally associate with any given monad $\mathbf{T}$ an idempotent monad $\hat{\mathbf{T}}$. We propose to call $\hat{\mathbf{T}}$ an idempotent approximation to $\mathbf{T}$; this terminology is similar to the one used by Lambek and Rattray in [17]. The first construction of idempotent approximations was described by Fakir in [13], assuming that $\mathcal{C}$ is complete and well-powered. A similar idea, with different hypotheses, was exploited by Dror and Dwyer in [12]. In [7], the

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authors used Fakir's construction of idempotent approximations in order to extend P-localization of nilpotent groups over all groups in a universal (terminal) way.

We have observed that most of the properties of idempotent approximations to monads do not depend on the particular construction carried out, but turn out to be consequences of one primary property: A monad $\mathbf{T}$ and its idempotent approximation $\hat{\mathbf{T}}$ render invertible the same class of arrows. This suggests that one could get rid of any technical assumptions on $\mathcal{C}$, and study idempotent approximations to monads in arbitrary categories, whenever such approximations exist.

In Section 1 we prove that, indeed, if one defines an idempotent approximation to a monad $\mathbf{T}$ as an idempotent monad $\hat{\mathbf{T}}$ inverting the same class of arrows as $\mathbf{T}$, then whenever $\hat{\mathbf{T}}$ exists there is a unique morphism of monads $\lambda\colon \hat{\mathbf{T}} \to \mathbf{T}$, which is terminal among all morphisms from idempotent monads into $\mathbf{T}$. However, λ need not be a monomorphism in general, although it is so in categories which are complete and well-powered. We exhibit counterexamples in Section 3.

Our counterexamples arose from one of the motivations of our approach. Bousfield and Kan constructed, in the pointed homotopy category of simplicial sets, for each commutative ring R with 1, a monad called R-completion, which fails to be idempotent [5]. The class of maps rendered invertible by R-completion is the class of ordinary homology equivalences with coefficients in R. Now, although the pointed homotopy category is not complete, it is well known that there is an idempotent monad which renders invertible precisely the ordinary homology equivalences with coefficients in R, namely R-homology localization [2]. Hence, R-homology localization should be viewed as an idempotent approximation to R-completion in the pointed homotopy category. Some consequences of this fact are discussed in Section 3.

In Section 2 we prove that, if an idempotent approximation to a monad exists, then it can be constructed as the codensity monad of a suitable embedding. The codensity monad of a full embedding $E\colon \mathcal{A} \to \mathcal{C}$ is also called $\mathcal{A}$-completion. In fact we give a necessary and sufficient condition for a full subcategory $\mathcal{A}$ in order that $\mathcal{A}$-completion be idempotent, assuming its existence.

The basic tool that we use in discussing idempotent approximations in arbitrary categories is the concept of *orthogonality* between classes of arrows and classes of objects. This terminology is due to Freyd and Kelly [15]; see also [8], where the term *orthogonal pair* was introduced, inspired in earlier work by Adams [1]. The present paper also aims to illustrate further, along the lines marked in [7], the power and simplicity of the use of orthogonal pairs in the study of monads.

1 Idempotent approximation and its properties

A *monad* or *triple* on a category $\mathcal{C}$ consists of a functor $T:\mathcal{C} \to \mathcal{C}$ together with natural transformations $\eta: \mathrm{Id} \to T$ and $\mu: T^2 \to T$ such that $\mu\cdot T\mu = \mu\cdot\mu T$ and $\mu\cdot\eta T = \mu\cdot T\eta = \mathrm{Id}$. Morphisms of monads are defined in the obvious way (but see Lemma 1.4 below). A monad (T, η, μ) is called *idempotent* if μ is an isomorphism or, equivalently, if $T\eta = \eta T$; see [9].

We recall that a category $\mathcal{C}$ is called *complete* if limits of diagrams over small categories exist in $\mathcal{C}$, and it is called *well-powered* if for every object X in $\mathcal{C}$ the isomorphism classes of monic arrows $Y \to X$ form a set. The following result was obtained by Fakir in [13].

Theorem 1.1 *Assume that the category $\mathcal{C}$ is complete and well-powered. Then for every monad $\mathbf{T} = (T, \eta, \mu)$ on $\mathcal{C}$ there exists an idempotent monad $\hat{\mathbf{T}} = (\hat{T}, \hat{\eta}, \hat{\mu})$ with the following properties.*

(1) *There is a unique morphism of monads $\lambda: \hat{\mathbf{T}} \to \mathbf{T}$, and this morphism is terminal among all morphisms from idempotent monads into $\mathbf{T}$.*

(2) *Both $T\hat{\eta}: T \to T\hat{T}$ and $\hat{\eta}T: T \to \hat{T}T$ are isomorphisms.*

(3) *For morphisms f in $\mathcal{C}$, $\hat{T}f$ is an isomorphism if and only if Tf is an isomorphism.*

(4) *λ is a monomorphism.* $\sharp$

The idempotent monad $\hat{\mathbf{T}}$ was constructed pointwise in [13] by means of an inverse limit procedure. However, there are plenty of monads $\mathbf{T}$ on categories which are not complete or well-powered, still associated with an idempotent monad $\hat{\mathbf{T}}$ satisfying properties (1), (2), and (3) above. In fact, as we shall prove, properties (1) and (2) are consequences of (3). This motivates the following definition.

Definition 1.2 Given a monad $\mathbf{T} = (T, \eta, \mu)$ on a category $\mathcal{C}$, an *idempotent approximation* to $\mathbf{T}$ is an idempotent monad $\hat{\mathbf{T}} = (\hat{T}, \hat{\eta}, \hat{\mu})$ on $\mathcal{C}$ such that $\hat{T}f$ is an isomorphism if and only if Tf is an isomorphism, for morphisms f in $\mathcal{C}$.

Before proceeding to show that our definition entails that $\hat{\mathbf{T}}$ satisfies properties (1) and (2) above, we recall some terminology from [7] and [8]. Let $\mathcal{C}$, $\mathcal{C}'$ be any two categories. For a functor $T:\mathcal{C} \to \mathcal{C}'$, we denote by $\mathcal{S}(T)$ the class of morphisms f in $\mathcal{C}$ such that Tf is an isomorphism, and call such morphisms *T-equivalences*. The class of objects in $\mathcal{C}'$ which are isomorphic to TX for some X will be denoted by $\mathcal{D}(T)$. By a standard abuse of terminology, we often denote by the same letter $\mathcal{D}$ a class of objects and the full subcategory with these objects.

As in [15], we say that a morphism $f: A \to B$ and an object X in $\mathcal{C}$ are *orthogonal*, denoted $f \perp X$, if the function

$$\mathcal{C}(f, X): \mathcal{C}(B, X) \to \mathcal{C}(A, X)$$

is bijective. For a class of morphisms $\mathcal{S}$ (resp. a class of objects $\mathcal{D}$), we denote by $\mathcal{S}^\perp$ the class of objects X such that $f \perp X$ for all f in $\mathcal{S}$ (resp. by $\mathcal{D}^\perp$ the class of morphisms f such that $f \perp X$ for all X in $\mathcal{D}$). We call a class of objects $\mathcal{D}$ *saturated* if $\mathcal{D}^{\perp\perp} = \mathcal{D}$. Similarly, a class of morphisms $\mathcal{S}$ will be called *saturated* if $\mathcal{S}^{\perp\perp} = \mathcal{S}$. We warn the reader that this concept of saturation is not the same as the one used in earlier papers on Adams completion [10]. The extent to which the two notions are distinct is discussed in [6].

The proof of the following proposition is omitted; the first statement can be found in [15, 1.2.1].

Proposition 1.3 *Every saturated class of objects $\mathcal{D}$ in a category $\mathcal{C}$ is closed under all limits which exist in $\mathcal{C}$. Every saturated class of morphisms $\mathcal{S}$ is closed under colimits, in the following sense: For any natural transformation of functors $\alpha: F_1 \to F_2$ from a category $\mathcal{A}$ to $\mathcal{C}$ where αA is in $\mathcal{S}$ for every object A in $\mathcal{A}$, the induced morphism* $\operatorname{colim} F_1 \to \operatorname{colim} F_2$ *is in $\mathcal{S}$, provided that these colimits exist.* $\sharp$

We say that two classes $(\mathcal{S}, \mathcal{D})$ form an *orthogonal pair* if $\mathcal{S}^\perp = \mathcal{D}$ and $\mathcal{D}^\perp = \mathcal{S}$. Then both $\mathcal{S}$ and $\mathcal{D}$ are saturated.

If $\mathbf{T} = (T, \eta, \mu)$ is any monad on $\mathcal{C}$, then, by [7, Theorem 1.3],

$$\mathcal{D}(T)^\perp = \mathcal{S}(T).$$

This implies that the class $\mathcal{S}(T)$ is saturated. On the other hand, in general, we have only an inclusion $\mathcal{D}(T) \subseteq \mathcal{S}(T)^\perp$. If the monad $\mathbf{T}$ is idempotent, then $\mathcal{D}(T) = \mathcal{S}(T)^\perp$, so that $(\mathcal{S}(T), \mathcal{D}(T))$ is in fact an orthogonal pair; cf. [1]. However, as pointed out in [7], the equality $\mathcal{D}(T) = \mathcal{S}(T)^\perp$ does not imply the idempotence of $\mathbf{T}$.

The following technical observation was made by the authors in [7, (1.6)], in a slightly more restrictive form. Since it turns out to be quite useful in practice, we have adapted the proof to our current situation.

Lemma 1.4 *Let $\mathbf{R} = (R, \nu, \zeta)$ and $\mathbf{T} = (T, \eta, \mu)$ be monads on $\mathcal{C}$, with $\mathbf{R}$ idempotent. Suppose given a natural transformation of functors $\theta: R \to T$ such that $\theta \cdot \nu = \eta$. Then θ defines a morphism of monads, i.e., the relation $\theta \cdot \zeta = \mu \cdot T\theta \cdot \theta R$ also holds.*

PROOF. Since $\mathbf{T}$ is a monad, we have $\mu \cdot T\eta = \mathrm{Id}$. This gives

$$\theta \cdot \zeta = \mu \cdot T\eta \cdot \theta \cdot \zeta = \mu \cdot T(\theta \cdot \nu) \cdot \theta \cdot \zeta = \mu \cdot T\theta \cdot T\nu \cdot \theta \cdot \zeta.$$

Now the fact that θ is a natural transformation tells us that $T\nu \cdot \theta = \theta R \cdot R\nu$. But $R\nu \cdot \zeta = \mathrm{Id}$, since $\mathbf{R}$ is assumed to be idempotent. Therefore,

$$\mu \cdot T\theta \cdot T\nu \cdot \theta \cdot \zeta = \mu \cdot T\theta \cdot \theta R \cdot R\nu \cdot \zeta = \mu \cdot T\theta \cdot \theta R,$$

which yields the equation stated. $\sharp$

Theorem 1.5 *Let $\mathbf{R} = (R, \nu, \zeta)$ be an idempotent monad and $\mathbf{T} = (T, \eta, \mu)$ any monad. Suppose that a morphism of monads $\theta\colon \mathbf{R} \to \mathbf{T}$ exists. Then:*

(a) *$\nu T\colon T \to RT$ is an isomorphism.*

(b) *$T\nu\colon T \to TR$ is an isomorphism.*

(c) *$\mathcal{D}(T) \subseteq \mathcal{D}(R)$ and $\mathcal{S}(R) \subseteq \mathcal{S}(T)$.*

PROOF. To prove (a), observe that $\mu \cdot \theta T \cdot \nu T = \mu \cdot \eta T = \mathrm{Id}$, and hence νT is split monic. Since $\mathbf{R}$ is idempotent, this implies that νT is an isomorphism; cf. [11, Lemma 2.8]. What we have just proved tells us that $\mathcal{D}(T) \subseteq \mathcal{D}(R)$, and it follows that $\mathcal{S}(R) = \mathcal{D}(R)^{\perp} \subseteq \mathcal{D}(T)^{\perp} = \mathcal{S}(T)$, as claimed in (c). Finally, in order to prove (b), observe that, for every object X, the arrow ν_X is in $\mathcal{S}(R)$, since $\mathbf{R}$ is idempotent. By (c), ν_X is in $\mathcal{S}(T)$. But this means that $T\nu_X$ is invertible for all X, so that $T\nu$ is also an isomorphism. $\sharp$

A trivial instance of Theorem 1.5 is, of course, the case where $\mathbf{R}$ is the identity monad.

Theorem 1.6 *Any two morphisms from an idempotent monad $\mathbf{R}$ to any monad $\mathbf{T}$ necessarily coincide.*

PROOF. Let θ_1, θ_2 be two morphisms from $\mathbf{R} = (R, \nu, \zeta)$ to $\mathbf{T} = (T, \eta, \mu)$, where $\mathbf{R}$ is assumed to be idempotent. Then $\theta_1 \cdot \nu = \eta = \theta_2 \cdot \nu$. Since $\mathbf{R}$ is idempotent, ν_X is in $\mathcal{S}(R)$ for every object X. By part (c) of Theorem 1.5, $\mathcal{S}(R)$ is contained in $\mathcal{S}(T)$. Therefore, for all objects X, we have $\nu_X \perp TX$, and this forces $(\theta_1)_X = (\theta_2)_X$, as claimed. $\sharp$

Now we can prove our claim that, in any category, if an idempotent approximation exists (in the sense of Definition 1.2), then it has the properties (1) and (2) stated in Theorem 1.1.

Theorem 1.7 *Let $\mathbf{T} = (T, \eta, \mu)$ be any monad for which an idempotent approximation $\hat{\mathbf{T}} = (\hat{T}, \hat{\eta}, \hat{\mu})$ exists. Then:*

(1) *There is a unique morphism of monads $\lambda\colon \hat{\mathbf{T}} \to \mathbf{T}$, and this morphism is terminal among all morphisms from idempotent monads into $\mathbf{T}$.*

(2) *Both $T\hat{\eta}\colon T \to T\hat{T}$ and $\hat{\eta}T\colon T \to \hat{T}T$ are isomorphisms.*

Proof. If X is any object, then $\hat{\eta}_X$ is in $\mathcal{S}(\hat{T})$, and hence in $\mathcal{S}(T)$. Since TX is in $\mathcal{D}(T)$, there is a unique arrow $\lambda_X\colon \hat{T}X \to TX$ such that $\lambda_X \circ \hat{\eta}_X = \eta_X$. Moreover, by the same argument, λ is a natural transformation of functors. By Lemma 1.4, λ is in fact a morphism of monads. Now let $\theta\colon \mathbf{R} \to \mathbf{T}$ be any morphism of monads with $\mathbf{R}$ idempotent. Then, using Theorem 1.5 we have $\mathcal{S}(R) \subseteq \mathcal{S}(T) = \mathcal{S}(\hat{T})$, and this yields a unique morphism of monads $\phi\colon \mathbf{R} \to \hat{\mathbf{T}}$; cf. [7, Proposition 1.6]. The fact that $\lambda \cdot \phi = \theta$ is a consequence of Theorem 1.6. This proves Part (1). Then Part (2) follows as a special case of (a) and (b) in Theorem 1.5. $\sharp$

Theorem 1.7 implies that if $\hat{\mathbf{T}}_1$ and $\hat{\mathbf{T}}_2$ are two idempotent approximations to a monad $\mathbf{T}$, then there is a unique isomorphism of monads $\hat{\mathbf{T}}_1 \cong \hat{\mathbf{T}}_2$. Hence, we may speak of "the" idempotent approximation to $\mathbf{T}$, provided that it exists.

Our last result in this section aims to enlighten further the applications to homotopy theory discussed in Section 3.

Theorem 1.8 *Suppose that the monad $\mathbf{T} = (T, \eta, \mu)$ has an idempotent approximation $\hat{\mathbf{T}} = (\hat{T}, \hat{\eta}, \hat{\mu})$, and let $\lambda\colon \hat{\mathbf{T}} \to \mathbf{T}$ be the unique morphism. Then, for a given object X, the following statements are equivalent:*

(a) *$\eta_X\colon X \to TX$ is a $\hat{T}$-equivalence.*

(b) *$\eta_{TX}\colon TX \to T^2X$ is an isomorphism.*

(c) *$\lambda_X\colon \hat{T}X \to TX$ is an isomorphism.*

Proof. Under the assumption made in (b), we have $\eta_{TX} = T\eta_X = (\mu_X)^{-1}$. Hence, the assertion in (b) is equivalent to the assertion that η_X is a T-equivalence, which is in turn equivalent to (a). Next, since $\hat{\eta}$ is a natural transformation, we have

$$\hat{\eta}T \cdot \lambda \cdot \hat{\eta} = \hat{\eta}T \cdot \eta = \hat{T}\eta \cdot \hat{\eta}.$$

As $\hat{\mathbf{T}}$ is idempotent, it follows that $\hat{\eta}T \cdot \lambda = \hat{T}\eta$. By Theorem 1.5, $\hat{\eta}T$ is an isomorphism. Hence, for an object X, the arrow λ_X is invertible if and only if $\hat{T}\eta_X$ is invertible. This shows that (a) and (c) are equivalent. $\sharp$

2 Idempotent approximations as codensity monads

We recall that, for a full embedding $E\colon \mathcal{A} \to \mathcal{C}$, if the right Kan extension $\mathrm{Ran}_E E$ of E along itself exists, then it is part of a monad, called the *codensity monad* of E. The subcategory $\mathcal{A}$ (or the embedding E) is called *codense* if $\mathrm{Ran}_E E$ is the identity functor. The codensity monad of E *exists pointwise* if the limit of the functor EQ_X exists for all objects X in $\mathcal{C}$, where $Q_X\colon (X \downarrow E) \to \mathcal{A}$ is the projection sending $X \to EA$ to A.

The codensity monad of an embedding $E\colon \mathcal{A} \to \mathcal{C}$ will also be called $\mathcal{A}$-*completion*. For example, if E is the embedding of the full subcategory of finite p-groups into the category of groups, where p is a prime, then the codensity monad of E is the usual p-profinite completion.

The next theorem may be viewed both as an existence criterion for idempotent approximations and a general abstract method to construct them when they exist.

Theorem 2.1 *For a monad $\mathbf{T} = (T, \eta, \mu)$ on $\mathcal{C}$, the following statements are equivalent:*

(a) $\mathbf{T}$ *admits an idempotent approximation* $\hat{\mathbf{T}}$.

(b) *The full subcategory $\mathcal{S}(T)^{\perp}$ is reflective in $\mathcal{C}$, that is, the embedding $E\colon \mathcal{S}(T)^{\perp} \to \mathcal{C}$ has a left adjoint.*

(c) *The codensity monad of the embedding $E\colon \mathcal{S}(T)^{\perp} \to \mathcal{C}$ exists pointwise.*

Moreover, if these equivalent conditions hold, then $\hat{\mathbf{T}}$ is the codensity monad of E.

Proof. Statements (a) and (b) are equivalent as they both state that there is an idempotent monad $\hat{\mathbf{T}} = (\hat{T}, \hat{\eta}, \hat{\mu})$ on $\mathcal{C}$ with $\mathcal{D}(\hat{T}) = \mathcal{S}(T)^{\perp}$. The equivalence of (b) and (c) is shown in Theorem 1.10 of [7]; see also [18, X.7.2]. $\sharp$

Necessary and sufficient conditions for a codensity monad to be idempotent have been discussed in the literature; see [9], [16]. We next give a new criterion, which implies, as a special case, that if $\mathcal{A}$ is full and saturated then the associated codensity monad is idempotent.

Theorem 2.2 *Let $E\colon \mathcal{A} \to \mathcal{C}$ be a full embedding for which $R = \mathrm{Ran}_E E$ exists pointwise. Then R is part of an idempotent monad if and only if $\mathcal{A}$ is codense in $\mathcal{A}^{\perp\perp}$.*

Proof. Denote by $\mathbf{R} = (R, \nu, \zeta)$ the codensity monad of E. Since $\mathcal{A}$ is full, we may assume that $RE = E$. Hence, $\mathcal{A}$ embeds in $\mathcal{D}(R)$. By Lemma 1.9 in [7],

$$\mathcal{A}^{\perp} = \mathcal{S}(R) = \mathcal{D}(R)^{\perp}.$$

Let us label all the embeddings as follows:

$$\mathcal{A} \xrightarrow{E_1} \mathcal{D}(R) \xrightarrow{E_2} \mathcal{C}, \qquad\qquad \mathcal{A} \xrightarrow{E_3} \mathcal{A}^{\perp\perp} \xrightarrow{E_4} \mathcal{C}.$$

Assume first that E_3 is codense. Let X be any object in $\mathcal{A}^{\perp\perp}$, and denote by $(Q_3)_X\colon (X \downarrow E_3) \to \mathcal{A}$ the projection. By assumption, $\lim E_3(Q_3)_X = X$. Hence, using the fact that $\mathcal{A}^{\perp\perp}$ is closed under limits (Proposition 1.3), we obtain

$$\lim E(Q_3)_X = \lim E_4E_3(Q_3)_X = E_4(\lim E_3(Q_3)_X) = E_4X,$$

that is, $\mathrm{Ran}_{E_3}E = E_4$. Then, by [14, Lemma 1.2],

$$\mathrm{Ran}_E E = \mathrm{Ran}_{E_4}(\mathrm{Ran}_{E_3}E) = \mathrm{Ran}_{E_4}E_4.$$

Now, since $\mathcal{A}^{\perp\perp} = \mathcal{S}(R)^{\perp}$, it follows from Theorem 2.1 that $\mathbf{R}$ is its own idempotent approximation.

Conversely, if $\mathbf{R}$ is idempotent, then $\mathcal{D}(R)$ is saturated, and hence $\mathcal{A}^{\perp\perp} = \mathcal{D}(R)^{\perp\perp} = \mathcal{D}(R)$. Therefore, we are led to showing that $\mathcal{A}$ is codense in $\mathcal{D}(R)$. Observe that, for an arbitrary object X in $\mathcal{D}(R)$, we have a natural isomorphism $(X \downarrow E_1) \cong (E_2X \downarrow E)$, since E_2 is full. Hence, if we denote the corresponding projection by $(Q_1)_X\colon (X \downarrow E_1) \to \mathcal{A}$, we have

$$\begin{aligned} E_2X = RE_2X = (\mathrm{Ran}_E E)E_2X = \lim E(Q_1)_X = \\ \lim E_2E_1(Q_1)_X = E_2(\lim E_1(Q_1)_X) = E_2(\mathrm{Ran}_{E_1}E_1)X. \end{aligned}$$

This implies that $\mathrm{Ran}_{E_1}E_1 = \mathrm{Id}$, as desired. $\sharp$

3 Homological localization and completion

Given any category $\mathcal{C}$ and a full embedding $E\colon \mathcal{A} \to \mathcal{C}$ for which $\mathcal{A}$-completion exists, the $\mathcal{A}$-completion monad $\mathbf{T}$ need not be idempotent. If it admits an idempotent approximation $\hat{\mathbf{T}}$, then $\hat{\mathbf{T}}$ is precisely a localization onto the class $\mathcal{A}^{\perp\perp}$; cf. [7, Lemma 1.9]. An illuminating example in the category of groups was discussed in [7].

In spite of the lack of limits in the pointed homotopy category $\mathcal{C}$ of CW-complexes (or simplicial sets), examples of idempotent approximations to monads are encountered in practice. We next discuss a basic example. Let R be a subring of the rationals or a finite cyclic ring $\mathbf{Z}/n$. Then the Bousfield–Kan R-completion functor R_∞ is part of a monad $\mathbf{T} = (R_\infty, \eta, \mu)$ on $\mathcal{C}$, which is described in [5, I.5.6]. This monad is not idempotent.

Indeed, $\mu_X\colon R_\infty R_\infty X \to R_\infty X$ is not a homotopy equivalence if X is a wedge of two circles $S^1 \vee S^1$ and $R = \mathbf{Z}/p$ where p is a prime; see [4, §11].

According to [5, I.5.5], the class $\mathcal{S}(R_\infty)$ of maps f such that $R_\infty f$ is a homotopy equivalence is the class of all R-homology equivalences. Thus, the R-homology localization functor E_R of [2] is part of an idempotent monad $\hat{\mathbf{T}} = (E_R, \hat\eta, \hat\mu)$ satisfying $\mathcal{S}(E_R) = \mathcal{S}(R_\infty)$; that is, R-homology localization is the idempotent approximation to R-completion.

Now the theorems of Section 1 apply to this particular situation. For example, the equality $\mathcal{D}(R_\infty)^\perp = \mathcal{S}(R_\infty)$ tells us that a map $f\colon A \to B$ induces bijections

$$[B, R_\infty X] \cong [A, R_\infty X]$$

for all spaces X if and only if f is an R-homology equivalence.

Theorem 1.5 generalizes the well-known fact that the natural maps

$$R_\infty X \to E_R R_\infty X \qquad \text{and} \qquad R_\infty X \to R_\infty E_R X$$

are homotopy equivalences for all spaces X. Theorem 1.8 generalizes the statement that the following conditions are equivalent for a space X.

- The natural map $X \to R_\infty X$ is an R-homology equivalence.
- The natural map $R_\infty X \to R_\infty R_\infty X$ is a homotopy equivalence.
- The natural map $E_R X \to R_\infty X$ is a homotopy equivalence.

Spaces X for which these equivalent conditions hold were called *R-good* in [5]. Thus, R-completion restricts to an idempotent monad on the class of R-good spaces, where it coincides in fact with R-homology localization. By [5, VII.1], all simply connected spaces are R-good for any R, and so are many other classes of spaces, including nilpotent spaces, spaces with finite homotopy groups, and spaces X such

hat $H_1(X; R) = 0$.

We next show that the morphism $\lambda\colon \hat{\mathbf{T}} \to \mathbf{T}$ from the idempotent approximation to a monad need not be a monomorphism in general, although we know from Theorem 1.1 that it is necessarily a monomorphism in categories which are complete and well-powered. Thus, part (4) in Theorem 1.1 need not hold for idempotent approximations in arbitrary categories.

In fact we prove that the natural map $\lambda_X\colon E_R X \to R_\infty X$ need not be a monomorphism in the pointed homotopy category. Suppose it were. Then, for any CW-complex A, the induced function $[A, \lambda_X]: [A, E_R X] \to [A, R_\infty X]$ of pointed homotopy classes of maps would be injective. In particular, the induced homomorphism of fundamental groups

$$[S^1, \lambda_X]: \pi_1(E_R X) \to \pi_1(R_\infty X)$$

would be injective for all spaces X. But this is not the case if X is, for example, a wedge of two circles and R is the ring of integers; see [3, Proposition 4.4] and [5, IV.5.3].

We conclude with one last example. Let Σ denote the reduced suspension functor and Ω the loop space functor. Then $\Omega\Sigma$ is part of a monad in the pointed homotopy category of connected CW-complexes, which is not idempotent. The class of maps rendered invertible by this monad is the same as the class of maps rendered invertible by Σ (this happens for all adjoint pairs; see [7, Theorem 1.3]). This class of maps is precisely the class of all integral homology equivalences. Hence, it admits an idempotent approximation, which is in fact **Z**-homology localization. Thus, the theorems of Section 1 also particularize to this example. Observe that the natural map $\lambda_X: E_{\mathbf{Z}} X \to \Omega\Sigma X$ is not a monomorphism in general; indeed, if we apply the fundamental group functor to λ_X we obtain precisely the abelianization homomorphism from $\pi_1(E_{\mathbf{Z}} X)$ onto $H_1(X)$, if X is connected.

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