Extending localization functors

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Abstract

We prove that, under suitable restrictions, an idempotent monad $\mathbf{t}$ defined on a full subcategory $\mathcal{A}$ of a category $\mathcal{C}$ can be extended to an idempotent monad $\mathbf{T}$ on $\mathcal{C}$ in a universal (terminal) way. Our result applies in particular to the case when $\mathbf{t}$ is P-localization of nilpotent groups (where P denotes a set of primes) and $\mathcal{C}$ is the category of all groups. The corresponding monad $\mathbf{T}$ on $\mathcal{C}$ is, in a certain precise sense, the best idempotent approximation to the usual $\mathbf{Z}_P$-completion of groups; it turns out to be a (strict) epimorphic image of Bousfield's $H\mathbf{Z}_P$-localization.

0 Introduction

Most constructions which are called "localizations" in any branch of algebra or geometry share a common feature, namely they are idempotent functors, or, more properly, part of an *idempotent monad*. Category theorists have studied this concept extensively (see [2, ch. 3], [23, ch. VI], or [27, §21]).

When localization techniques (at a set of primes P) first became available in algebraic topology in the early seventies, the homotopy category of nilpotent CW-complexes was found to be the most natural setting in which to develop the theory, and, indeed, the setting where the main applications were discovered (see e.g. [7], [21], [28]). During the past two decades, several functors have appeared in the literature extending the classical P-localization over all groups and all spaces. Some of these functors were idempotent (for example, the ones described by Bousfield in [8] or by Ribenboim in [25]), while others were not (see Bousfield–Kan [7]). In general, "completion" functors fail to be idempotent, yet restrict to idempotent functors in sufficiently nice subcategories.

In [11], an attempt was started to understand the interrelations between the various localization and completion functors which exist in several subcategories of groups and spaces. It was observed that Bousfield's homological localization with $\mathbf{Z}_P$ coefficients (where $\mathbf{Z}_P$ denotes the ring of integers localized at P) is *terminal* among all idempotent extensions of P-localization of nilpotent spaces over all spaces. It was also observed that Ribenboim's localization turns out to be *initial* among all idempotent extensions of P-localization of nilpotent groups over all groups. These observations prompt two natural questions:

(1) Is there an initial idempotent extension of P-localization of nilpotent spaces over all spaces?

(2) Is there a terminal idempotent extension of P-localization of nilpotent groups over all groups?

In Section 3 of the present paper we answer affirmatively the second question. The same result has simultaneously been obtained with different techniques by Berrick and Tan ([5]).

Our approach is based on a procedure allowing to associate with a given monad an idempotent monad in a universal way, which works in categories which are complete and well-powered. It was first described by Fakir in [15] (see also [27, 21.8.9]). In fact, the two first sections of our paper contain results which hold in a very broad setting. We start by recalling from [11] the notion of *orthogonal pair*, and analyze the interplay between monads, adjunctions and orthogonal pairs in general. Orthogonal pairs are a quite useful tool to study the relationship between localization functors and other standard concepts, such as Kan extensions. Certain arguments become simpler by using that terminology.

Among other things, we prove that if $\mathcal{C}$ is complete and well-powered, $\mathcal{D}$ is any full subcategory of $\mathcal{C}$, and $K\colon \mathcal{D} \to \mathcal{C}$ is the inclusion, then the existence of the right Kan extension of K along itself suffices to ensure that the orthogonal pair generated by $\mathcal{D}$ admits a localization functor (Corollary 2.3 below); in other words, the saturation of $\mathcal{D}$, in the appropriate sense, is reflective in $\mathcal{C}$. This was also shown by Pfenniger in [24] under the assumption that $\mathcal{D}$ be small.

When particularizing to the category of groups, by choosing $\mathcal{D}$ to be the full subcategory of P-local nilpotent groups, this result ensures the existence of a functor L_P which is terminal among all idempotent extensions of P-localization of nilpotent groups over all groups. One may think of L_P as obtained "by approximating $\mathbf{Z}_P$-completion by an idempotent monad" (for a group G, the $\mathbf{Z}_P$-completion $\widehat{G}_P$ is the inverse limit of the tower $\{(G/\Gamma^i G)_P\}$, where $\Gamma^i G$ denotes the ith term of the lower central series of G).

The class of homomorphisms rendered invertible by L_P coincides with the class of homomorphisms rendered invertible by the $\mathbf{Z}_P$-completion functor (Theorem 3.5 below). In fact, $L_P G$ is isomorphic to $\widehat{G}_P$ in many cases, namely for all groups G for which $\widehat{G}_P \cong (\widehat{G}_P)\widehat{\ }_P$. This includes all finitely generated groups and, more generally, all groups G for which $H_1(G;\mathbf{Z}_P)$ is finitely generated as a $\mathbf{Z}_P$-module.

The functor L_P is *not* equivalent to the $H\mathbf{Z}_P$-localization functor of Bousfield —for example, their effect on a free group on two generators is different. This fact may be surprising, since it breaks the analogy with homotopy theory. Indeed, if one considers the Bousfield–Kan $\mathbf{Z}_P$-completion of spaces (which is not idempotent) and looks for an idempotent monad having the same class of equivalences in the pointed homotopy category of spaces, then one finds precisely the $H_*(\ \ ;\mathbf{Z}_P)$-localization functor, contrary to what we have found in the category of groups. See [14] to get a closer picture of the homotopy-theoretical case.

The referee has kindly indicated that the problem of universally extending an idempotent monad can (and should) be formulated, more generally, in the setting of 2-categories (see [19] for a review of this concept). It is well-known

that monads and adjunctions can be defined in any 2-category, and, in fact, some of the results presented in Sections 1 and 2 below hold in suitably restricted 2-categories. We are also indebted to Max Kelly for bringing to our attention the papers [6], [18], which discuss the existence of suprema and infima in the partially ordered set of reflective subcategories of sufficiently good categories (here the word "set" does not have the same meaning as in the rest of this paper, which implicitly uses the classical Gödel–Bernays theory).

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1 Monads, orthogonal pairs and Kan extensions

This section contains a survey of the basic categorical ingredients needed in the paper. The main source is [23], but some terminology has been borrowed from [11], [13], [17]. Results stated without a reference are new, to our knowledge.

Monads and adjunctions

Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. To any functor $F\colon \mathcal{A} \to \mathcal{B}$ we may associate the class $\mathcal{S}(F)$ of morphisms of $\mathcal{A}$ rendered invertible by F, which will be called *F-equivalences*. Also, we denote by $\mathcal{D}(F)$ the class of objects in $\mathcal{B}$ which are isomorphic to FX for some X in $\mathcal{A}$.

We often use the same letter to denote a class of objects in a category and the full subcategory with those objects. Thus,

$$\mathcal{A} \xrightarrow{F} \mathcal{D}(F) \xrightarrow{K} \mathcal{B} \tag{1.1}$$

is the *enlarged canonical factorization* of F (cf. [13]), where K is the inclusion.

Recall that a *monad* or *triple* $\mathbf{T} = (T, \eta, \mu)$ on a category $\mathcal{C}$ consists of a functor $T\colon \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\eta\colon \mathrm{Id} \to T$, $\mu\colon T^2 \to T$, such that $\mu \cdot T\mu = \mu \cdot \mu T$ and $\mu \cdot \eta T = \mu \cdot T\eta = \mathrm{Id}$. Any pair of adjoint functors $F\colon \mathcal{C} \to \mathcal{C}'$, $G\colon \mathcal{C}' \to \mathcal{C}$, with unit η and counit ε, gives rise to a monad (T, η, μ) on $\mathcal{C}$ by setting $T = GF$ and $\mu = G\varepsilon F$. Conversely, every monad is induced by some pair of adjoint functors, which is not uniquely determined in general. Indeed, among all adjoint pairs inducing a given monad, there is an initial one (supplied by the Kleisli construction) and a terminal one (supplied by the Eilenberg–Moore construction); see [2, ch. 3] or [23, ch. VI].

For a monad (T, η, μ), the following facts are equivalent (see e.g. [12]).

(a) $\mu\colon T^2 \to T$ is a natural equivalence of functors.

(b) For every object X in $\mathcal{C}$, $\eta_X \in \mathcal{S}(T)$.

(c) $T\eta = \eta T$.

The monad is called *idempotent* if any of these equivalent conditions is satisfied. Other characterizations of idempotent monads were given in [13]. We recall that, if $\mathbf{T}$ is idempotent, then the factorization (1.1)

$$\mathcal{C} \xrightarrow{T} \mathcal{D}(T) \xrightarrow{K} \mathcal{C}$$

is in fact an adjoint pair inducing $\mathbf{T}$. Moreover, the full subcategory $\mathcal{D}(T)$ is isomorphic to the Eilenberg–Moore category of $\mathbf{T}$ (cf. [13, Theorem 2.7]), and, in addition, it is equivalent to the Kleisli category of $\mathbf{T}$, by [16, Corollary 2.3]. In fact, the latter is isomorphic to the category of fractions of $\mathcal{C}$ with respect to the class $\mathcal{S}(T)$ (see [13, Theorem 2.6]).

Orthogonal pairs

A morphism $f\colon A \to B$ and an object X in a category $\mathcal{C}$ are called *orthogonal* (cf. [17], [30]) if the map

$$\mathcal{C}(f, X)\colon \mathcal{C}(B, X) \to \mathcal{C}(A, X)$$

is a bijection. For a class of morphisms $\mathcal{S}$, the class of objects orthogonal to all f in $\mathcal{S}$ is denoted by $\mathcal{S}^{\perp}$ (objects in $\mathcal{S}^{\perp}$ were called *left closed* for $\mathcal{S}$ in [13]). Analogously, given a class of objects $\mathcal{D}$, we denote by $\mathcal{D}^{\perp}$ the class of morphisms orthogonal to all X in $\mathcal{D}$. As in [11], we say that a class of morphisms $\mathcal{S}$ and a class of objects $\mathcal{D}$ form an *orthogonal pair* if $\mathcal{S}^{\perp} = \mathcal{D}$ and $\mathcal{D}^{\perp} = \mathcal{S}$.

Proposition 1.1 ([1]) *If the monad (T, η, μ) is idempotent, then $(\mathcal{S}(T), \mathcal{D}(T))$ is an orthogonal pair.* $\square$

Furthermore, if (T, η, μ) is idempotent, then $\mathcal{D}(T)$ coincides with the class of objects X in $\mathcal{C}$ such that $\eta_X \colon X \cong TX$. Given an orthogonal pair $(\mathcal{S}, \mathcal{D})$ on $\mathcal{C}$, there is an idempotent monad (T, η, μ) such that $\mathcal{S}(T) = \mathcal{S}$, $\mathcal{D}(T) = \mathcal{D}$ if and only if, for every object X, there is a morphism $\varphi \colon X \to Y$ in $\mathcal{S}$ with Y in $\mathcal{D}$. If this is the case we say that T is a *localization functor* associated with $(\mathcal{S}, \mathcal{D})$, and also that objects in $\mathcal{D}$ are T-*local*; see [1], [11].

For a monad (T, η, μ) which is not idempotent, the interplay between the classes $\mathcal{S}(T)$ and $\mathcal{D}(T)$ is less satisfactory. However, the following holds.

Lemma 1.2 *Let $\mathcal{C}$, $\mathcal{C}'$ be two categories and $F : \mathcal{C} \to \mathcal{C}'$ be left adjoint to $G \colon \mathcal{C}' \to \mathcal{C}$. Then $\mathcal{S}(F) = \mathcal{D}(G)^{\perp}$.*

Proof. Given a morphism $f \colon A \to B$ in $\mathcal{C}$ and an object X in $\mathcal{C}'$, there is a commutative diagram

$$\begin{array}{ccc} \mathcal{C}(B, GX) & \xrightarrow{\mathcal{C}(f,GX)} & \mathcal{C}(A, GX) \\ \cong\downarrow & & \downarrow\cong \\ \mathcal{C}'(FB, X) & \xrightarrow{\mathcal{C}'(Ff,X)} & \mathcal{C}'(FA, X), \end{array} \tag{1.2}$$

from which the result follows immediately. $\square$

Theorem 1.3 *Let $\mathbf{T} = (T, \eta, \mu)$ be any monad on a category $\mathcal{C}$. Let $F \colon \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C}' \to \mathcal{C}$ be an adjoint pair of functors inducing $\mathbf{T}$. Then $\mathcal{S}(T) = \mathcal{S}(F)$, $\mathcal{D}(T)^{\perp\perp} = \mathcal{D}(G)^{\perp\perp}$, and*

$$\mathcal{S}(T) = \mathcal{D}(T)^{\perp}.$$

PROOF. Since $T = GF$, we have $\mathcal{D}(T) \subseteq \mathcal{D}(G)$ and $\mathcal{S}(F) \subseteq \mathcal{S}(T)$. Now assume that $f\colon A \to B$ is a T-equivalence. Then, if we set $\psi = (Tf)^{-1} \circ \eta_B$, we obtain a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \arrow[dl, "\psi"'] \\ TA \arrow[r, "Tf"] & TB \end{tikzcd} \tag{1.3}$$

which corresponds under the adjunction to a commutative diagram

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\mathrm{Id}"'] & FB \arrow[d, "\mathrm{Id}"] \arrow[dl, "\varphi"'] \\ FA \arrow[r, "Ff"] & FB \end{tikzcd} \tag{1.4}$$

showing that f is an F-equivalence. This gives the equality $\mathcal{S}(T) = \mathcal{S}(F)$; cf. [27, 21.8.8]. Similarly, if $f\colon A \to B$ is orthogonal to all objects in the image of T, then there is a (unique) morphism $\psi\colon B \to TA$ rendering the upper triangle in (1.3) commutative. Then $(Tf) \circ \psi \circ f = (Tf) \circ \eta_A = \eta_B \circ f$, and hence the whole diagram (1.3) commutes. Then (1.4) also commutes, and this gives

$$\mathcal{D}(T)^{\perp} \subseteq \mathcal{S}(F) = \mathcal{S}(T). \tag{1.5}$$

On the other hand, by Lemma 1.2, $\mathcal{D}(G)^{\perp\perp} = \mathcal{S}(F)^{\perp} = \mathcal{S}(T)^{\perp}$. Hence, $\mathcal{S}(T) \subseteq \mathcal{S}(T)^{\perp\perp} = \mathcal{D}(G)^{\perp\perp\perp} = \mathcal{D}(G)^{\perp} \subseteq \mathcal{D}(T)^{\perp}$. In view of (1.5), this shows that $\mathcal{S}(T) = \mathcal{D}(T)^{\perp}$. The equality $\mathcal{D}(T)^{\perp\perp} = \mathcal{D}(G)^{\perp\perp}$ follows as well. □

As a consequence of Theorem 1.3, we have

$$\mathcal{D}(T) \subseteq \mathcal{S}(T)^{\perp}.$$

That is, for any monad (T, η, μ), objects of the form TX (or isomorphic to these) are orthogonal to all T-equivalences, but $\mathcal{S}(T)^{\perp}$ might be strictly larger if the monad (T, η, μ) is not idempotent. It would be interesting to determine precisely the class $\mathcal{S}(T)^{\perp}$ in general. We point out the following fact, omitting the easy proof.

Proposition 1.4 *Let* $\mathbf{T} = (T, \eta, \mu)$ *be a monad. If, for a given object* X, $\eta_X \colon X \to TX$ *is split monic (i.e., there is a morphism* $\xi \colon TX \to X$ *such that* $\xi \circ \eta_X = \mathrm{Id}_X$*), then* $X \in \mathcal{S}(T)^{\perp}$*.* $\square$

It seems natural to ask if for every monad $\mathbf{T} = (T, \eta, \mu)$, the class $\mathcal{S}(T)^{\perp}$ coincides with the class of objects X such that $\eta_X : X \to TX$ is split monic. (This class is contained in $\mathcal{S}(T)^{\perp}$, by Proposition 1.4, and contains $\mathcal{D}(T)$, because $\mu \cdot \eta T = \mathrm{Id}$.) However, the following example shows that it is *not* the case.

Example Let $\mathcal{C}$ be the pointed homotopy category of path-connected topological spaces having the homotopy type of a CW-complex, and $\mathcal{C}'$ the full subcategory of simply-connected spaces. Let $F = \Sigma$ be the reduced suspension functor and $G = \Omega$ the loop space functor. Then $F \colon \mathcal{C} \to \mathcal{C}'$, $G \colon \mathcal{C}' \to \mathcal{C}$ form an adjoint pair. Let $\mathbf{T} = (T, \eta, \mu)$ be the associated monad. The class $\mathcal{S}(T) = \mathcal{S}(F)$ is precisely the class of integral homology equivalences. Therefore, $\mathcal{S}(T)^{\perp}$ is the class of path-connected Bousfield $H_*(\quad; \mathbf{Z})$-local spaces ([8]). Now let X be a $K(G, 1)$ where G is a noncommutative $H\mathbf{Z}$-local group. Then $X \in \mathcal{S}(T)^{\perp}$, yet the natural map $\eta_X : X \to \Omega\Sigma X$ is not split monic (suppose it were; then the identity of G would factor through $\pi_1(\Omega\Sigma X)$, which is commutative, hence contradicting our choice of G).

One could also be tempted to ask whether the equality $\mathcal{S}(T)^{\perp} = \mathcal{D}(T)$ implies that the monad $\mathbf{T}$ is idempotent. In other words, if $(\mathcal{S}(T), \mathcal{D}(T))$ is an orthogonal pair, does it follow that $\mathbf{T}$ is idempotent? Again, the answer is no, as the following counterexample shows.

Example Let $\mathcal{C}$ be the category of countably infinite sets and $\mathcal{C}'$ the category of countably infinite vector spaces over the field $\mathbf{F}_2$. Let $F \colon \mathcal{C} \to \mathcal{C}'$ be the functor which assigns to every set X the vector space spanned by X, and $G \colon \mathcal{C}' \to \mathcal{C}$ the forgetful functor. Then F is left adjoint to G. Let $\mathbf{T} = (T, \eta, \mu)$ be the induced monad. The class $\mathcal{S}(T) = \mathcal{S}(F)$ is the class of all bijections in $\mathcal{C}$. Moreover, any two objects in $\mathcal{C}$ are isomorphic, so that $\mathcal{D}(T) = \mathcal{C} = \mathcal{S}(T)^{\perp}$. On the other hand, for every set X, the map $\eta_X \colon X \to TX$ is strictly injective, and hence $\mathbf{T}$ is not idempotent.

Extensions of monads and orthogonal pairs

Let $\mathcal{C}$ be a category, $\mathcal{A}$ a full subcategory, and $K : \mathcal{A} \to \mathcal{C}$ the inclusion. If $\mathbf{t} = (t, \nu, \zeta)$ is a monad on $\mathcal{A}$ and $\mathbf{T} = (T, \eta, \mu)$ is a monad on $\mathcal{C}$, then we say that $\mathbf{T}$ *extends* $\mathbf{t}$ if there is a natural equivalence $\phi \colon Kt \to TK$ which is compatible with the monad structure; that is, ϕ satisfies $\eta K = \phi \cdot K\nu$ and $\phi \cdot K\zeta = \mu K \cdot T\phi \cdot \phi t$. If $\mathbf{t}$ is idempotent (so that $t\nu \cdot \zeta = \mathrm{Id}$), then the second equality is a consequence of the first, as the following computation shows:

$$\begin{aligned} \phi \cdot K\zeta = (\mu \cdot T\eta)K \cdot \phi \cdot K\zeta = \mu K \cdot T\eta K \cdot \phi \cdot K\zeta = \\ \mu K \cdot T(\phi \cdot K\nu) \cdot \phi \cdot K\zeta = \mu K \cdot T\phi \cdot TK\nu \cdot \phi \cdot K\zeta = \\ \mu K \cdot T\phi \cdot \phi t \cdot Kt\nu \cdot K\zeta = \mu K \cdot T\phi \cdot \phi t \cdot K(t\nu \cdot \zeta) = \mu K \cdot T\phi \cdot \phi t. \end{aligned} \tag{1.6}$$

In the sequel, we shall drop K from most expressions if there is no risk of confusion.

If (s, d) is an orthogonal pair on $\mathcal{A}$ and $(\mathcal{S}, \mathcal{D})$ is an orthogonal pair on $\mathcal{C}$, we say that $(\mathcal{S}, \mathcal{D})$ *extends* (s, d) if $s \subseteq \mathcal{S}$ and $d \subseteq \mathcal{D}$.

Proposition 1.5 *Let $\mathcal{A}$ be a full subcategory of $\mathcal{C}$. Assume given idempotent monads $\mathbf{t} = (t, \nu, \zeta)$ on $\mathcal{A}$ and $\mathbf{T} = (T, \eta, \mu)$ on $\mathcal{C}$. Then $\mathbf{T}$ extends $\mathbf{t}$ if and only if the orthogonal pair $(\mathcal{S}(T), \mathcal{D}(T))$ extends $(\mathcal{S}(t), \mathcal{D}(t))$.* $\square$

In any category, orthogonal pairs can be partially ordered by setting

$$(\mathcal{S}_1, \mathcal{D}_1) \leq (\mathcal{S}_2, \mathcal{D}_2) \quad \Leftrightarrow \quad \mathcal{D}_1 \subseteq \mathcal{D}_2. \tag{1.7}$$

Proposition 1.6 *Let $\mathbf{T}_1 = (T_1, \eta_1, \mu_1)$ and $\mathbf{T}_2 = (T_2, \eta_2, \mu_2)$ be idempotent monads on $\mathcal{C}$. Then $(\mathcal{S}(T_1), \mathcal{D}(T_1)) \leq (\mathcal{S}(T_2), \mathcal{D}(T_2))$ if and only if there is a morphism of monads $\mathbf{T}_2 \to \mathbf{T}_1$, which is then unique.*

Proof. Assume first that $(\mathcal{S}(T_1), \mathcal{D}(T_1)) \leq (\mathcal{S}(T_2), \mathcal{D}(T_2))$. Then for every object X in $\mathcal{C}$ there is a unique commutative diagram

$$\begin{tikzcd} X \arrow[r, "(\eta_1)_X"] \arrow[d, "(\eta_2)_X"'] & T_1X \\ T_2X \arrow[ur, "\lambda_X"'] & \end{tikzcd} \tag{1.8}$$

as $(\eta_2)_X \in \mathcal{S}(T_2)$ and $T_1X \in \mathcal{D}(T_2)$. The compatibility of λ with μ_1 and μ_2 (i.e., $\lambda \cdot \mu_2 = \mu_1 \cdot T_1\lambda \cdot \lambda T_2$) follows from the fact that $\mathbf{T}_2$ is idempotent, as in (1.6).

Conversely, assume given a natural transformation $\lambda\colon T_2 \to T_1$ such that $\lambda \cdot \eta_2 = \eta_1$. Let X be any object of $\mathcal{D}(T_1)$. Then $(\eta_1)_X : X \to T_1X$ is an isomorphism. Hence, (1.8) shows that λ_X is epic. As $\mathbf{T}_2$ is idempotent, $T_2(\eta_2)_X$ is an isomorphism, and hence so is $T_2\lambda_X$, by (1.8). Now we have $T_2\lambda \cdot \eta_2 T_2 = \eta_2 T_1 \cdot \lambda$ (by naturality of η_2) and, since both $T_2\lambda_X$ and $(\eta_2)_{T_2X}$ are isomorphisms, λ_X is split monic. Therefore, λ_X is an isomorphism and, by (1.8), $(\eta_2)_X$ is also an isomorphism, which implies that X is in $\mathcal{D}(T_2)$. This shows that $(\mathcal{S}(T_1), \mathcal{D}(T_1)) \leq (\mathcal{S}(T_2), \mathcal{D}(T_2))$. $\square$

As observed in [11, Proposition 2.2], every orthogonal pair (s, d) on a full subcategory $\mathcal{A}$ of $\mathcal{C}$ has a *minimal* and a *maximal* extension to $\mathcal{C}$ (with respect to the partial order defined in (1.7)); namely, for every extension $(\mathcal{S}, \mathcal{D})$ of (s, d), the following holds:

$$(d^{\perp}, d^{\perp\perp}) \leq (\mathcal{S}, \mathcal{D}) \leq (s^{\perp\perp}, s^{\perp}), \tag{1.9}$$

where, of course, orthogonality is meant in $\mathcal{C}$. Note that, even assuming that the pair (s, d) is associated with some idempotent monad on $\mathcal{A}$, the pairs $(d^{\perp}, d^{\perp\perp})$ and $(s^{\perp\perp}, s^{\perp})$ need not correspond to idempotent monads on $\mathcal{C}$. Hence, it does *not* follow from (1.9) that every idempotent monad on $\mathcal{A}$ has an initial extension and a terminal extension over $\mathcal{C}$. However, we can state a weaker fact. From (1.9) and Proposition 1.5 we obtain the following:

Theorem 1.7 *Let $\mathcal{A}$ be a subcategory of $\mathcal{C}$. Assume given an idempotent monad*

$\mathbf{t} = (t, \nu, \zeta)$ *on* $\mathcal{A}$ *and an idempotent monad* $\mathbf{T} = (T, \eta, \mu)$ *on* $\mathcal{C}$ *extending* $\mathbf{t}$.

(a) *If* $(\mathcal{S}(T), \mathcal{D}(T)) = (\mathcal{D}(t)^{\perp}, \mathcal{D}(t)^{\perp\perp})$, *then* $\mathbf{T}$ *is terminal among all idempotent extensions of* $\mathbf{t}$ *over* $\mathcal{C}$, *meaning that if* $\hat{\mathbf{T}}$ *is any idempotent monad on* $\mathcal{C}$ *extending* $\mathbf{t}$, *then there is a unique morphism of monads* $\hat{\mathbf{T}} \to \mathbf{T}$.

(b) *If* $(\mathcal{S}(T), \mathcal{D}(T)) = (\mathcal{S}(t)^{\perp\perp}, \mathcal{S}(t)^{\perp})$, *then* $\mathbf{T}$ *is initial among all idempotent extensions of* $\mathbf{t}$ *over* $\mathcal{C}$ (*in the same sense as above*). $\square$

Kan extensions

Let $K\colon \mathcal{A} \to \mathcal{C}$ and $F\colon \mathcal{A} \to \mathcal{M}$ be functors. If the category $\mathcal{M}$ is complete and for any object X in $\mathcal{C}$ the comma category $(X \downarrow K)$ has a small initial subcategory, then the *right Kan extension* $R = \mathrm{Ran}_K F$ of F along K can be computed as a pointwise limit (see [23, ch. X]):

$$RX = \varprojlim FQ_X, \tag{1.10}$$

where $Q_X\colon (X \downarrow K) \to \mathcal{A}$ is the projection sending $X \to KA$ to A. If K is a full embedding and R is constructed as in (1.10), then RK is naturally equivalent to F. Actually, R can be so chosen that $RK = F$ ([23, X.3.3 and X.3.4]).

If $K\colon \mathcal{A} \to \mathcal{C}$ has a left adjoint $L\colon \mathcal{C} \to \mathcal{A}$, then for every $F\colon \mathcal{A} \to \mathcal{M}$ and each $X \in \mathcal{C}$ one finds that $\varprojlim FQ_X = FLX$. Hence, the right Kan extension of F along K exists pointwise and is given by

$$\mathrm{Ran}_K F = FL. \tag{1.11}$$

This says, in particular, that if $\mathbf{T} = (T, \eta, \mu)$ is the monad induced by the adjoint pair L, K, then $T = \mathrm{Ran}_K K$. More generally, the following holds.

Proposition 1.8 *Let* $K : \mathcal{A} \to \mathcal{C}$ *be a functor and* $\mathbf{t} = (t, \nu, \zeta)$ *be a monad on* $\mathcal{A}$. *Let* $F\colon \mathcal{A} \to \mathcal{B}$, $G\colon \mathcal{B} \to \mathcal{A}$ *be any pair of adjoint functors inducing* $\mathbf{t}$. *Then* $\mathrm{Ran}_K Kt = \mathrm{Ran}_{KG} KG$.

Proof. Since G has a left adjoint, we may infer from (1.11) that

$$\mathrm{Ran}_G KG = KGF = Kt.$$

Hence, $\mathrm{Ran}_K Kt = \mathrm{Ran}_K(\mathrm{Ran}_G KG) = \mathrm{Ran}_{KG} KG.$ $\square$

The right Kan extension of a functor E along itself (if it exists) is always part of a monad. It is called the *codensity monad* of E (see [23, p. 246]); cf. also [12], [22].

Proposition 1.8 tells us that, in order to study extensions of idempotent monads over larger categories via Kan extensions, it suffices to consider codensity monads of certain embeddings. Indeed, if we assume that the monad $\mathbf{t}$ is idempotent, then it is induced by an adjoint pair

$$\mathcal{A} \xrightarrow{F} \mathcal{D}(t) \xrightarrow{G} \mathcal{A},$$

where $F = t$ and G is the inclusion. If in addition we suppose that $K\colon \mathcal{A} \to \mathcal{C}$ is a full embedding, then the right Kan extension R of Kt along K in the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{K} & \mathcal{C} \\ {\scriptstyle t}\downarrow & & \\ \mathcal{A} & \xrightarrow{K} & \mathcal{C} \end{array}$$

is part of the codensity monad $\mathbf{R} = (R, \eta, \mu)$ of the embedding of $\mathcal{D}(t)$ in $\mathcal{C}$, by Proposition 1.8 (provided that R exists). Moreover, $\mathbf{R}$ is an extension of $\mathbf{t}$ over $\mathcal{C}$.

The next observation will be useful in the sequel.

Lemma 1.9 *Let $K : \mathcal{D} \to \mathcal{C}$ be a full embedding. Assume that $R = \mathrm{Ran}_K K$ exists pointwise. Then $\mathcal{S}(R) = \mathcal{D}^{\perp}$.*

Proof. If a morphism $f : X \to Y$ is in $\mathcal{D}^{\perp}$, then f induces an isomorphism of categories $(Y \downarrow K) \cong (X \downarrow K)$ and hence it follows from (1.10) that f is an R-equivalence. To prove the converse, note that R is the identity on objects of

$\mathcal{D}$, so that $\mathcal{D} \subseteq \mathcal{D}(R)$. Thus it follows from Theorem 1.3 that $\mathcal{S}(R) = \mathcal{D}(R)^{\perp} \subseteq \mathcal{D}^{\perp}$. $\square$

As a consequence, we obtain an alternative proof of the following special case of [23, X.7.2].

Theorem 1.10 *Let $\mathcal{C}$ be any category, $(\mathcal{S}, \mathcal{D})$ an orthogonal pair on $\mathcal{C}$, and $K\colon \mathcal{D} \to \mathcal{C}$ the inclusion. Then $(\mathcal{S}, \mathcal{D})$ admits a localization functor if and only if $\mathrm{Ran}_K K$ exists pointwise. If this is the case, then $\mathrm{Ran}_K K$ is the localization functor associated with $(\mathcal{S}, \mathcal{D})$.*

PROOF. If T is a localization functor for $(\mathcal{S}, \mathcal{D})$, then $T = \mathrm{Ran}_K K$, by (1.11). Conversely, if $R = \mathrm{Ran}_K K$ exists pointwise, then Lemma 1.9 tells us that $\mathcal{S} = \mathcal{D}^{\perp} = \mathcal{S}(R)$. Furthermore, $\mathcal{D} \subseteq \mathcal{D}(R)$ and $\mathcal{D}(R) \subseteq \mathcal{D}(R)^{\perp\perp} = \mathcal{S}(R)^{\perp} = \mathcal{S}^{\perp} = \mathcal{D}$, so that $\mathcal{D} = \mathcal{D}(R)$ as well. Only the idempotence of R remains to be established. For an object X, the object RX is in $\mathcal{D}(R)$ and hence in $\mathcal{D}$. Therefore, $R^2X = RX$, as desired. $\square$

A class of objects $\mathcal{D}$ is called *saturated* if $\mathcal{D}^{\perp\perp} = \mathcal{D}$. Theorem 1.10 says precisely that, if $\mathcal{D}$ is saturated, then the existence of a localization functor for $(\mathcal{D}^{\perp}, \mathcal{D}^{\perp\perp})$ is equivalent to the existence of $\mathrm{Ran}_K K$. For an arbitrary class $\mathcal{D}$, not necessarily saturated, one implication is true, under suitable restrictions on the category $\mathcal{C}$ (cf. Corollary 2.3).

2 Extending idempotent monads

Suppose given a category $\mathcal{C}$ which is complete and well-powered, i.e., for every object X the isomorphism classes of monic arrows $Z \to X$ form a set. Then there is a general procedure to associate with any monad on $\mathcal{C}$ an idempotent monad in a universal way. Specifically,

Theorem 2.1 ([15]) *Let $\mathcal{C}$ be complete and well-powered. Let $\mathbf{R} = (R, \eta, \mu)$ be a monad on $\mathcal{C}$. Then there is an idempotent monad $\mathbf{R}_\infty = (R_\infty, \eta_\infty, \mu_\infty)$ on $\mathcal{C}$ and a monomorphism of monads $\lambda\colon \mathbf{R}_\infty \to \mathbf{R}$ with the following properties:*

(i) *λ is universal, in the sense that, given a morphism $\phi\colon \hat{\mathbf{R}} \to \mathbf{R}$ with $\hat{\mathbf{R}}$ idempotent, there is a unique morphism $\phi_\infty\colon \hat{\mathbf{R}} \to \mathbf{R}_\infty$ with $\lambda\phi_\infty = \phi$.*

(ii) *$\eta_\infty R : R \to R_\infty R$ is an isomorphism.*

(iii) *A morphism $f\colon X \to Y$ in $\mathcal{C}$ is an R_∞-equivalence if and only if it is an R-equivalence.* $\square$

We recall the construction of $\mathbf{R}_\infty$ for future use. Set $(R_0, \eta_0, \mu_0) = (R, \eta, \mu)$. For a successor ordinal $\alpha + 1$, define $R_{\alpha+1}$ to be the equalizer of $R_\alpha\eta_\alpha$ and $\eta_\alpha R_\alpha$, together with the unique natural transformations $\eta_{\alpha+1}$, $\mu_{\alpha+1}$ rendering commutative the diagrams

$$\begin{tikzcd} & \mathrm{Id} \arrow[dl, "\eta_{\alpha+1}"'] \arrow[dr, "\eta_\alpha"] & \\ R_{\alpha+1} \arrow[rr, "\kappa_{\alpha+1}"] & & R_\alpha \end{tikzcd} \qquad \begin{tikzcd} (R_{\alpha+1})^2 \arrow[r, "\kappa_{\alpha+1}R_{\alpha+1}"] \arrow[d, "\mu_{\alpha+1}"'] & R_\alpha R_{\alpha+1} \arrow[r, "R_\alpha\kappa_{\alpha+1}"] & (R_\alpha)^2 \arrow[d, "\mu_\alpha"] \\ R_{\alpha+1} \arrow[rr, "\kappa_{\alpha+1}"] & & R_\alpha \end{tikzcd}$$

If ω is a limit ordinal, set

$$R_\omega = \lim_{\longleftarrow} R_\alpha,$$

where the limit is taken over all ordinals $\alpha < \omega$. Thus we obtain, for each object X, an inverse system of monic arrows which must stabilize at some ordinal because $\mathcal{C}$ is well-powered. We denote by $R_\infty X$ the object obtained upon stabilization. The functor R_∞ is part of a monad $\mathbf{R}_\infty = (R_\infty, \eta_\infty, \mu_\infty)$. Furthermore, $R_\infty\eta_\infty = \eta_\infty R_\infty$, which means that the monad $\mathbf{R}_\infty$ is indeed idempotent.

We can now state the main result of this section.

Theorem 2.2 *Let $\mathcal{C}$ be a complete well-powered category, $\mathcal{A}$ a full subcategory, and $K : \mathcal{A} \to \mathcal{C}$ the inclusion. Let $\mathbf{t} = (t, \nu, \zeta)$ be an idempotent monad on the subcategory $\mathcal{A}$. Assume that $R = \mathrm{Ran}_K Kt$ exists pointwise. Then R is part of a monad $\mathbf{R} = (R, \eta, \mu)$ on $\mathcal{C}$, and the associated idempotent monad $\mathbf{R}_\infty = (R_\infty, \eta_\infty, \mu_\infty)$ satisfies:*

(i) $\mathbf{R}_\infty$ *extends* $\mathbf{t}$ *over* $\mathcal{C}$.

(ii) $(\mathcal{S}(R_\infty), \mathcal{D}(R_\infty)) = (\mathcal{D}(t)^\perp, \mathcal{D}(t)^{\perp\perp})$.

(iii) $\mathbf{R}_\infty$ *is terminal among all idempotent extensions of* $\mathbf{t}$ *over* $\mathcal{C}$.

Proof. By the remarks made after Proposition 1.8, the functor R is part of a monad $\mathbf{R} = (R, \eta, \mu)$ on $\mathcal{C}$ extending $\mathbf{t}$. Let $\mathbf{R}_\infty$ be the idempotent monad obtained by applying Theorem 2.1 to $\mathbf{R}$. We can choose R so that the natural equivalence $Kt \to RK$ is the identity. Thus, if A is any object of $\mathcal{A}$, then $R\eta_A = t\nu_A = \nu_{tA} = \eta_{RA}$, since $\mathbf{t}$ is idempotent. It follows from the construction of $\mathbf{R}_\infty$ that $R_1 A = RA$, and hence

$$R_\infty A = RA = tA, \tag{2.1}$$

so that $\mathbf{R}_\infty$ extends $\mathbf{t}$ as well. By Lemma 1.9 and part (iii) of Theorem 2.1, we have $\mathcal{D}(t)^\perp = \mathcal{S}(R) = \mathcal{S}(R_\infty)$, so that the two orthogonal pairs in (ii) are identical. Finally, Theorem 1.7 guarantees that $\mathbf{R}_\infty$ is the terminal idempotent extension of $\mathbf{t}$ over $\mathcal{C}$. $\square$

We record the following consequences of Theorem 2.2.

Corollary 2.3 *Assume that $\mathcal{C}$ is complete and well-powered. Let $\mathcal{D}$ be any full subcategory of $\mathcal{C}$, and denote by $K\colon \mathcal{D} \to \mathcal{C}$ the inclusion. If* $\mathrm{Ran}_K K$ *exists pointwise, then the orthogonal pair $(\mathcal{D}^\perp, \mathcal{D}^{\perp\perp})$ admits a localization functor.*

Proof. Take $\mathcal{A} = \mathcal{D}$ and $t = \mathrm{Id}$ in Theorem 2.2. Then $\mathcal{D}(t)$ is the closure of $\mathcal{D}$ under isomorphisms, and hence $(\mathcal{S}(R_\infty), \mathcal{D}(R_\infty)) = (\mathcal{D}^\perp, \mathcal{D}^{\perp\perp})$. This tells us that R_∞ is a localization functor associated with the pair $(\mathcal{D}^\perp, \mathcal{D}^{\perp\perp})$. $\square$

An important special case of this result is Theorem 3.2 below, where $\mathcal{C}$ is the category of groups, and $\mathcal{D}$ is the full subcategory of P-local nilpotent groups. In that situation, $\mathrm{Ran}_K K$ is $\mathbf{Z}_P$-completion and the localization functor associated with $(\mathcal{D}^\perp, \mathcal{D}^{\perp\perp})$, which we denote by L_P, will be the subject of our discussion in Section 3.

Recall from Theorem 1.10 that, if $\mathcal{D}$ is saturated, then the existence of a localization functor for $(\mathcal{D}^\perp, \mathcal{D})$ is in fact *equivalent* to the existence of $\mathrm{Ran}_K K$. Moreover, in that special case, $\mathrm{Ran}_K K$ is idempotent.

For an arbitrary class of objects $\mathcal{D}$, we say, as in [11], that the orthogonal pair $(\mathcal{D}^\perp, \mathcal{D}^{\perp\perp})$ is *generated* by $\mathcal{D}$. With this terminology, we have

Corollary 2.4 *If $\mathcal{C}$ is complete and well-powered, and $(\mathcal{S}, \mathcal{D})$ is an orthogonal pair on $\mathcal{C}$ generated by a* **set** *of objects, then $(\mathcal{S}, \mathcal{D})$ admits a localization functor.*

PROOF. Let $\mathcal{D}_0$ be a full small subcategory of $\mathcal{C}$ whose objects generate $(\mathcal{S}, \mathcal{D})$. Let $K\colon \mathcal{D}_0 \to \mathcal{C}$ be the inclusion. Then, for each object X in $\mathcal{C}$, the comma category $(X \downarrow K)$ is small and therefore $\mathrm{Ran}_K K$ can be computed on X by (1.10). Now the result follows from Corollary 2.3. $\square$

This result was first obtained by Pfenniger ([24]). It should be compared with [10, Theorem 3.4], where a similar conclusion was derived in certain cocomplete categories.

We end this section with a remark which is immediate in view of Theorem 2.1, and gives some additional insight on the problem of characterizing the class $\mathcal{S}(T)^\perp$ for a nonidempotent monad $\mathbf{T} = (T, \eta, \mu)$ (see the discussion before and after Proposition 1.4).

Proposition 2.5 *Let $\mathcal{C}$ be complete and well-powered. Let $\mathbf{T} = (T, \eta, \mu)$ be any monad on $\mathcal{C}$. Then $\mathcal{S}(T)^\perp = \mathcal{D}(T_\infty)$.* $\square$

3 Applications to localization of groups

Let $\mathcal{G}$ denote the category of groups, $\mathcal{N}$ the full subcategory of nilpotent groups, and $\mathcal{N}_c$ the full subcategory of nilpotent groups of class $\leq c$; that is, objects in $\mathcal{N}_c$ are groups G such that $\Gamma^{c+1}G$ is trivial (we denote by $\Gamma^2 G$ the commutator subgroup $[G, G]$ and $\Gamma^i G = [G, \Gamma^{i-1}G]$ for each $i > 2$).

Throughout this section, $\mathbf{t}_P = (t_P, \nu, \zeta)$ stands for the idempotent monad on $\mathcal{N}$ corresponding to P-localization, where P is a fixed set of primes (see [21]). This monad restricts to an idempotent monad on $\mathcal{N}_c$ for each c, and we shall denote the restrictions by the same letter $\mathbf{t}_P$. The usual notation for $\nu_N \colon N \to t_P N$ is in fact $l \colon N \to N_P$.

Our aim is to discuss the initial and terminal idempotent extensions of $\mathbf{t}_P$ over the category of all groups. The answer depends on whether we view $\mathbf{t}_P$ as a monad on $\mathcal{N}$ or on $\mathcal{N}_c$ for some c. Let us consider first the latter case, which is much easier.

Theorem 3.1 *The monad $\mathbf{t}_P$ on $\mathcal{N}_c$ has a terminal idempotent extension $\mathbf{T}_P$ over $\mathcal{G}$, which is given by $T_P G = (G/\Gamma^{c+1}G)_P$.*

Proof. The inclusion $K \colon \mathcal{N}_c \to \mathcal{G}$ has a left adjoint $F \colon \mathcal{G} \to \mathcal{N}_c$; namely,

$$FG = G/\Gamma^{c+1}G.$$

Hence, by (1.11),

$$\mathrm{Ran}_K K t_P = K t_P F;$$

if we call it T_P, then $T_P G = (G/\Gamma^{c+1}G)_P$. This functor T_P is part of a monad $\mathbf{T}_P$ on $\mathcal{G}$, which is idempotent. By Proposition 1.8 and Lemma 1.9, $\mathcal{S}(T_P) = \mathcal{D}(t_P)^{\perp}$. Hence, by Theorem 1.7, the monad $\mathbf{T}_P$ is terminal among all idempotent extensions of $\mathbf{t}_P$ over $\mathcal{G}$. $\square$

In other words, among all idempotent monads extending $\mathbf{t}_P$ over $\mathcal{G}$, the one given by Theorem 3.1 has as few local objects as possible, and turns as many arrows as possible into isomorphisms.

On the other hand, by [11, Example 3.3], Ribenboim's localization, which will be denoted by $l \colon G \to G_P$, is initial among all idempotent extensions of $\mathbf{t}_P$ over $\mathcal{G}$. Recall that a group G is *P-local* in Ribenboim's sense if the qth power map $x \mapsto x^q$ is bijective in G for all primes $q \notin P$. Thus, among all idempotent extensions of $\mathbf{t}_P$ over $\mathcal{G}$, the latter one has as many local objects as possible and renders invertible as few arrows as possible.

We now consider the same situation when $\mathcal{N}_c$ is replaced by the whole category $\mathcal{N}$ of nilpotent groups. The initial idempotent extension of $\mathbf{t}_P$ over $\mathcal{G}$ is again Ribenboim's localization, by the same argument given in [11]. However, in this case the inclusion $K\colon \mathcal{N} \to \mathcal{G}$ fails to have a left adjoint, and hence it is not obvious, in principle, that a terminal idempotent extension should exist. We next exploit our results in the previous sections to show that such a terminal extension indeed exists.

Let $\mathcal{D}$ be a full subcategory of $\mathcal{G}$ such that, for every group G, the comma category $(G \downarrow K)$ has a small initial subcategory, where $K\colon \mathcal{D} \to \mathcal{C}$ denotes the inclusion. Then, by (1.10), $\mathrm{Ran}_K K$ exists and is called the $\mathcal{D}$-*completion* functor. We specialize to the class $\mathcal{D}(t_P)$ of P-local nilpotent groups. In this case, for a group G, the category $(G \downarrow K)$ has a small initial subcategory consisting of the compositions

$$G \twoheadrightarrow G/\Gamma^i G \xrightarrow{l} \left(G/\Gamma^i G\right)_P \qquad\qquad 1 \le i < \infty.$$

Thus $R = \mathrm{Ran}_K K$ exists and is given by

$$RG = \lim_{\longleftarrow} \left(G/\Gamma^i G\right)_P$$

which is usually denoted by $\widehat{G}_P$, and called the P-*local nilpotent completion* or the $\mathbf{Z}_P$-*completion* of the group G. The corresponding monad $\mathbf{R} = (R, \eta, \mu)$ is *not* idempotent on the category of groups. This statement requires some comment: in [7, IV.5.4] a proof of the nonidempotence of $\mathbf{R}$ is given for P containing all primes (i.e., $\mathbf{Z}_P = \mathbf{Z}$), and mention is made that it should be possible to prove it for any set P. In [29, III.1.4], a detailed proof for arbitrary P is given.

To the nonidempotent monad $\mathbf{R}$ above we associate an idempotent monad $\mathbf{R}_\infty$ as described in Section 2, which is terminal among all idempotent extensions of $\mathbf{t}_P$ over $\mathcal{G}$, by Theorem 2.2. We shall use the notation L_P instead of R_∞. Thus we have proved

Theorem 3.2 *The monad $\mathbf{t}_P$ on $\mathcal{N}$ has a terminal idempotent extension $\mathbf{L}_P = (L_P, \eta, \mu)$ over the category $\mathcal{G}$.* $\square$

For a group G, the localization $L_P G$ is a subgroup of $\widehat{G}_P$ which is constructed as follows (cf. the procedure explained after Theorem 2.1). Let $\eta_G \colon G \to \widehat{G}_P$ be the $\mathbf{Z}_P$-completion homomorphism. Let R_1 denote the equalizer of $R\eta$ and ηR, i.e.,

$$R_1 G \longrightarrow \widehat{G}_P \underset{\eta_{RG}}{\overset{R\eta_G}{\rightrightarrows}} (\widehat{G}_P)\widehat{\ }_P. \tag{3.1}$$

Then η_G factors through $R_1 G$, and we can use the same letter to denote the homomorphism $\eta_G : G \to R_1 G$. This process can be iterated by transfinite induction. The tower $\{R_\alpha G\}$ must stabilize at some ordinal and in this way we obtain a subgroup $L_P G = \cap_\alpha R_\alpha G$ of $\widehat{G}_P$ still containing the image of $\eta_G \colon G \to \widehat{G}_P$.

Theorem 3.3 *The natural transformation η induces isomorphisms*

$$\widehat{G}_P \cong L_P(\widehat{G}_P) \qquad \text{and} \qquad \widehat{G}_P \cong (L_P G)\widehat{\ }_P.$$

Proof. The first isomorphism is given by part (ii) of Theorem 2.1. To prove the second isomorphism, observe that $\eta_G \colon G \to L_P G$ is an L_P-equivalence, and hence also an R-equivalence, by part (iii) of Theorem 2.1. $\square$

Proposition 3.4 *Let $\mathcal{C}$ be any category and $\mathbf{R} = (R, \eta, \mu)$ a monad on $\mathcal{C}$. Let X be an object of $\mathcal{C}$ for which $\eta_X \colon X \to RX$ is split monic. Then for every G the map $\mathcal{C}(\eta_G, X) \colon \mathcal{C}(RG, X) \to \mathcal{C}(G, X)$ is onto. Moreover, if η_G factorizes as $G \xrightarrow{\pi} Q \to RG$ with π epic, then $\mathcal{C}(\pi, X) \colon \mathcal{C}(Q, X) \cong \mathcal{C}(G, X)$.* $\square$

Of course, if $X \in \mathcal{D}(R)$, then η_X is split monic. Hence, Proposition 3.4 tells us that every morphism of the form $f \colon G \to RK$ factors through η_G, possibly not in a unique way. In our context, if we let $\mathbf{R}$ be the $\mathbf{Z}_P$-completion monad in the category of groups, then every homomorphism $f \colon G \to \widehat{K}_P$ factors through $\eta_G \colon G \to \widehat{G}_P$, possibly not in a unique way; cf. [5, Lemma 2.4].

Moreover, the second part of Proposition 3.4 tells us that the surjection $\pi \colon G \twoheadrightarrow \operatorname{Im} \eta_G$ is orthogonal to all groups in $\mathcal{D}(R)$ and hence it is an R-equivalence, by Theorem 1.3. That is, $\widehat{G}_P \cong (\operatorname{Im} \eta_G)\widehat{\ }_P$.

Theorem 3.5 *For a group homomorphism $\varphi\colon G \to H$, the following assertions are equivalent.*

(a) $\varphi_*\colon L_P G \cong L_P H$.

(b) $\varphi_*\colon \widehat{G}_P \cong \widehat{H}_P$.

(c) *φ is orthogonal to $\widehat{K}_P$ for every group K.*

(d) $\varphi_*\colon (G/\Gamma^i G)_P \cong (H/\Gamma^i H)_P$ *for* $1 \le i < \infty$.

Proof. Part (iii) of Theorem 2.1 tells us that (a) and (b) are equivalent. The equivalence between (b) and (c) is precisely the equality $\mathcal{S}(R) = \mathcal{D}(R)^\perp$ obtained in Theorem 1.3. Now, by choosing $K = G/\Gamma^i G$ in (c) for a fixed i, we obtain a homomorphism $\psi\colon H \to (G/\Gamma^i G)_P$ such that $\psi \circ \varphi$ is the natural map $G \to (G/\Gamma^i G)_P$. Now ψ factors to a map $\bar{\psi}\colon (H/\Gamma^i H)_P \to (G/\Gamma^i G)_P$ which is right and left inverse to $\varphi_*\colon (G/\Gamma^i G)_P \to (H/\Gamma^i H)_P$. Hence, (c) implies (d). Finally, (d) implies (b) by passing to the inverse limit. $\square$

The equivalence between (b) and (d) in Theorem 3.5 is remarkable, for it tells us that if a group homomorphism $\varphi\colon G \to H$ induces an isomorphism of the inverse limits of the towers $\{(G/\Gamma^i G)_P\}$ and $\{(H/\Gamma^i H)_P\}$, then it induces in fact a stepwise isomorphism of the whole towers. As a special case, $\eta_G\colon G \to L_P G$ induces

$$(\eta_G)_*\colon (G/\Gamma^i G)_P \cong (L_P G/\Gamma^i L_P G)_P \qquad \text{for } 1 \le i < \infty.$$

In particular, for $i = 2$, this gives an isomorphism

$$(\eta_G)_*\colon H_1(G; \mathbf{Z}_P) \cong H_1(L_P G; \mathbf{Z}_P). \tag{3.2}$$

We next analyze the relationship of L_P with the $H\mathbf{Z}_P$-localization functor $E^{\mathbf{Z}_P}$ of Bousfield ([8], [9]). If $(\mathcal{S}, \mathcal{D})$ is an orthogonal pair on any category, then the class $\mathcal{D}$ is closed under inverse limits. Therefore, since $\widehat{G}_P$ is an inverse limit of P-local nilpotent groups, it is L_P-local, and hence also T-local for any idempotent monad $\mathbf{T} = (T, \eta, \mu)$ extending $\mathbf{t}_P$ over $\mathcal{G}$. The $H\mathbf{Z}_P$-localization functor $E^{\mathbf{Z}_P}$ is part of such an idempotent monad. Hence, there is a natural homomorphism

$$\rho\colon E^{\mathbf{Z}_P} G \to \widehat{G}_P, \tag{3.3}$$

which factors through $L_P G$ and fits into a commutative diagram

$$\begin{tikzcd} & G \arrow[dl] \arrow[d] \arrow[dr] & \\ E^{\mathbf{Z}_P} G \arrow[r] & L_P G \arrow[r] & \widehat{G}_P, \end{tikzcd} \tag{3.4}$$

where all groups in the bottom row are $H\mathbf{Z}_P$-local.

Now recall from [9, Corollary 2.13] that a homomorphism $f : X \to Y$ between $H\mathbf{Z}_P$-local groups is onto if and only if the induced homomorphism $f_*\colon H_1(X; \mathbf{Z}_P) \to H_1(Y; \mathbf{Z}_P)$ is onto. Since the arrows in the left-hand triangle in (3.4) become isomorphisms after applying $H_1(\quad; \mathbf{Z}_P)$, it follows that the map $E^{\mathbf{Z}_P} G \to L_P G$ in (3.4) is onto for all groups G. In other words, $L_P G$ is precisely the image of ρ. Also, the following assertions are equivalent:

(i) $L_P G = \widehat{G}_P$.

(ii) The maps $R\eta_G$ and η_{RG} in (3.1) are isomorphisms and hence coincide (in particular, $(\widehat{G}_P)\widehat{\ }_P$ is isomorphic to $\widehat{G}_P$).

(iii) $\rho\colon E^{\mathbf{Z}_P} G \to \widehat{G}_P$ is onto.

(iv) $\rho_*\colon H_1(E^{\mathbf{Z}_P} G; \mathbf{Z}_P) \to H_1(\widehat{G}_P; \mathbf{Z}_P)$ is onto.

(v) $(\eta_G)_*\colon H_1(G; \mathbf{Z}_P) \to H_1(\widehat{G}_P; \mathbf{Z}_P)$ is onto.

(vi) $(\eta_G)_*\colon H_1(G; \mathbf{Z}_P) \cong H_1(\widehat{G}_P; \mathbf{Z}_P)$.

(A circle of ideas which is easy to follow is (ii) $\Leftrightarrow$ (i) $\Rightarrow$ (vi) $\Rightarrow$ (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i).)

By the comment preceding Theorem 3.2, condition (ii) —and hence all the equivalent conditions (i) to (vi)— fail to hold in general; they do fail e.g. if G is a free group on a countably infinite number of generators.

On the other hand, the above equivalent conditions are satisfied whenever $H_1(G; \mathbf{Z}_P)$ is finitely generated as a $\mathbf{Z}_P$-module, by [9, Theorem 13.3]. In particular, if G is finitely generated, then $L_P G = \widehat{G}_P$. However, for a free group on two generators, $E^{\mathbf{Z}_P} G$ is not isomorphic to $\widehat{G}_P$, at least if $2 \in P$ ([9, Proposition 4.4]). This shows that the functors $E^{\mathbf{Z}_P}$ and L_P are distinct. In fact, they coincide on very restricted classes of groups. For example, on those groups for which the tower $\{(G/\Gamma^i G)_P\}$ stabilizes; indeed, in that case, if we denote by $(G/\Gamma^s G)_P$ the first stable term, then $E^{\mathbf{Z}_P} G = (G/\Gamma^s G)_P$ by [8, Lemma 7.5], and $\widehat{G}_P = (G/\Gamma^s G)_P$ is nilpotent, so that $L_P G = (G/\Gamma^s G)_P$ as well. This happens for all perfect groups (for which, of course, $L_P G = \{1\}$) and all finite groups, among others (e.g. all polycyclic-by-finite groups when $P = \emptyset$; see [9, Theorem 4.9]).

The following fact is a direct consequence of the universality of L_P.

Theorem 3.6 *Let (T, λ, ξ) be any idempotent monad on $\mathcal{G}$ extending P-localization of nilpotent groups. If $\eta_G \colon G \to L_P G$ is injective, then $\lambda_G \colon G \to TG$ is also injective. If $\eta_G \colon G \to L_P G$ is surjective, then the natural homomorphism $TG \to L_P G$ is also surjective.* □

Theorem 3.7 *The map $\eta_G \colon G \to L_P G$ is injective if and only if G is residually P'-torsion-free nilpotent (where P' denotes the complement of P).*

PROOF. Both assertions are equivalent to $\eta_G \colon G \to \widehat{G}_P$ being injective. □

In particular, Theorems 3.6 and 3.7 give an elementary proof of the fact that the P-localization homomorphism $l \colon F \to F_P$ in the sense of Ribenboim is injective when F is a free group. This is an old result of Baumslag [3]; see also [20, Corolario 2.1.7] and [26, Proposition 9.4].

We do not know if the natural homomorphism $F_P \to L_P F$ is injective when F is free. This question is related to an open problem proposed by Baumslag in [4]; namely, it is not known if free P-local groups are residually P'-torsion-free nilpotent.

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