ARE ALL LOCALIZING SUBCATEGORIES OF STABLE HOMOTOPY CATEGORIES COREFLECTIVE?

CARLES CASACUBERTA, JAVIER J. GUTIÉRREZ AND JIŘÍ ROSICKÝ*

ABSTRACT. We prove that, in a triangulated category with combinatorial models, every localizing subcategory is coreflective and every colocalizing subcategory is reflective if a certain large-cardinal axiom (Vopěnka's principle) is assumed true. It follows that, under the same assumptions, orthogonality sets up a bijective correspondence between localizing subcategories and colocalizing subcategories. The existence of such a bijection was left as an open problem by Hovey, Palmieri and Strickland in their axiomatic study of stable homotopy categories and also by Neeman in the context of well-generated triangulated categories.

INTRODUCTION

The main purpose of this article is to address a question asked in [39, p. 35] of whether every localizing subcategory (i.e., a full triangulated subcategory closed under coproducts) of a stable homotopy category $\mathcal{T}$ is the kernel of a localization on $\mathcal{T}$ (or, equivalently, the image of a colocalization). We prove that the answer is affirmative if $\mathcal{T}$ arises from a combinatorial model category, assuming the truth of a large-cardinal axiom from set theory called Vopěnka's principle [2], [40]. A model category (in the sense of Quillen) is called *combinatorial* if it is cofibrantly generated [35], [38] and its underlying category is locally presentable [2], [28]. Many triangulated categories of interest admit combinatorial models, including derived categories of rings, the homotopy category of spectra or the stable motivic homotopy category.

More precisely, we show that, if $\mathcal{K}$ is a stable combinatorial model category, then every semilocalizing subcategory $\mathcal{C}$ of the homotopy category $\text{Ho}(\mathcal{K})$ is coreflective under Vopěnka's principle, and the coreflection is exact if $\mathcal{C}$ is localizing. We call $\mathcal{C}$ *semilocalizing* if it is closed under coproducts, cofibres and extensions, but not necessarily under fibres. Examples include kernels of nullifications in the sense of [12] or [20].

We also prove that, under the same hypotheses, every semilocalizing subcategory $\mathcal{C}$ is *singly generated*; that is, there is an object A such that $\mathcal{C}$ is the smallest semilocalizing subcategory containing A. The same result is true for localizing subcategories. The

question of whether every localizing subcategory is singly generated in a well-generated triangulated category was asked in [50, Problem 7.2]. We note that, as shown in [52, Proposition 6.10], triangulated categories with combinatorial models are well generated.

In an arbitrary triangulated category $\mathcal{T}$, localizing subcategories need neither be singly generated nor coreflective. Indeed, the existence of a coreflection onto a localizing subcategory $\mathcal{C}$ is equivalent to the existence of a right adjoint for the Verdier functor $\mathcal{T} \to \mathcal{T}/\mathcal{C}$; see [48, Proposition 9.1.18]. Hence, if a coreflection onto $\mathcal{C}$ exists, then $\mathcal{T}/\mathcal{C}$ has small hom-sets. This need not happen if no restriction is imposed on $\mathcal{T}$; a counterexample was given in [15].

Dually, we prove that every full subcategory $\mathcal{L}$ closed under products and fibres in a triangulated category with locally presentable models is reflective under Vopěnka's principle. The reflection is semiexact if $\mathcal{L}$ is closed under extensions, and it is exact if $\mathcal{L}$ is colocalizing, as in the dual case. However, we have not been able to prove that colocalizing (or semicolocalizing) subcategories are necessarily singly generated, not even under large-cardinal assumptions.

This apparent lack of symmetry is not entirely surprising, in view of some well-known facts involving torsion theories. In abelian categories, a full subcategory closed under co-products, quotients and extensions is called a *torsion class*, and one closed under products, subobjects and extensions is called a *torsion-free class*. These are analogues of semilocalizing and semicolocalizing subcategories of triangulated categories. Torsion theories have also been considered in triangulated categories by Beligiannis and Reiten in [6], in connection with t-structures. In well-powered abelian categories, torsion classes are necessarily coreflective and torsion-free classes are reflective [19]. As shown in [22] and [30], Vopěnka's principle implies that every torsion class of abelian groups is singly generated. However, there exist torsion-free classes that are not singly generated in ZFC; for example, the class of abelian groups whose countable subgroups are free [21, Theorem 5.4]. (In this article, we do not make a distinction between the terms "singly generated" and "singly cogenerated".)

Our results imply that, if $\mathcal{T}$ is the homotopy category of a stable combinatorial model category and Vopěnka's principle holds, then there is a bijection between localizing and colocalizing subcategories of $\mathcal{T}$, given by orthogonality. This was asked in [57, § 6] and in [50, Problem 7.3]. In fact, we prove that there is a bijection between semilocalizing and semicolocalizing subcategories as well, and each of those determines a t-structure in $\mathcal{T}$.

The lack of symmetry between reflections and coreflections also shows up in the fact that singly generated semilocalizing subcategories are coreflective (in ZFC) in triangulated categories with combinatorial models. A detailed proof of this claim is given in Theorem 3.7 below; the argument goes back to Bousfield [10], [11] in the case of spectra, and has subsequently been adapted to other special cases in [4], [6], [39], [41], [45] —our version generalizes some of these. However, we do not know if singly generated semicolocalizing subcategories can be shown to be reflective in ZFC. A positive answer would imply the existence of cohomological localizations of spectra, which is so far unsettled in ZFC.

It remains of course to decide if Vopěnka's principle (or any other large-cardinal principle) is really needed in order to answer all these questions. Although we cannot ascertain this, we prove that there is a full subcategory of the homotopy category of spectra closed under retracts and products which fails to be weakly reflective, assuming that there are no measurable cardinals. This follows from the existence of a full subcategory of abelian groups with the same property, and hence solves an open problem proposed in [2, p. 296]. In connection with this problem, see also [51].

1. Reflections and coreflections in triangulated categories

In this first section we recall basic concepts and fix our terminology, which is mostly standard, except for small discrepancies in the notation for orthogonality and localization in a number of recent articles and monographs about triangulated categories, such as [6], [7], [39], [43], [48] or [50]. The essentials of triangulated categories can be found in [48].

For a category $\mathcal{T}$, we denote by $\mathcal{T}(X, Y)$ the set of morphisms from X to Y. We tacitly assume that subcategories are isomorphism-closed, and denote with the same letter a full subcategory and the class of its objects.

1.1. **Reflections and coreflections.** A full subcategory $\mathcal{L}$ of a category $\mathcal{T}$ is *reflective* if the inclusion $\mathcal{L} \hookrightarrow \mathcal{T}$ has a left adjoint $\mathcal{T} \to \mathcal{L}$. Then the composite $L \colon \mathcal{T} \to \mathcal{T}$ is called a *reflection* onto $\mathcal{L}$. Such a functor L will be called a *localization* and objects in $\mathcal{L}$ will be called *L-local*. There is a natural transformation $l \colon \mathrm{Id} \to L$ (namely, the unit of the adjunction) such that $Ll \colon L \to LL$ is an isomorphism, lL is equal to Ll, and, for each X, the morphism $l_X \colon X \to LX$ is initial in $\mathcal{T}$ among morphisms from X to objects in $\mathcal{L}$.

Similarly, a full subcategory $\mathcal{C}$ of $\mathcal{T}$ is *coreflective* if the inclusion $\mathcal{C} \hookrightarrow \mathcal{T}$ has a right adjoint. The composite $C \colon \mathcal{T} \to \mathcal{T}$ is called a *coreflection* or a *colocalization* onto $\mathcal{C}$, and it is equipped with a natural transformation $c \colon C \to \mathrm{Id}$ (the counit of the adjunction) such that $Cc \colon CC \to C$ is an isomorphism, cC is equal to Cc, and $c_X \colon CX \to X$ is terminal in $\mathcal{T}$, for each X, among morphisms from objects in $\mathcal{C}$ (which are called *C-colocal*) into X.

A full subcategory $\mathcal{L}$ of a category $\mathcal{T}$ is called *weakly reflective* if for every object X of $\mathcal{T}$ there is a morphism $l_X \colon X \to X^*$ with X^* in $\mathcal{L}$ and such that the function

$$\mathcal{T}(l_X, Y) \colon \mathcal{T}(X^*, Y) \longrightarrow \mathcal{T}(X, Y)$$

is surjective for all objects Y of $\mathcal{L}$. Thus, every morphism from X to an object of $\mathcal{L}$ factors through l_X, not necessarily in a unique way. If such a factorization is unique for all objects X, then the morphisms $l_X \colon X \to X^*$ for all X define together a reflection, so $\mathcal{L}$ is then reflective. One defines *weakly coreflective* subcategories dually.

If a weakly reflective subcategory is closed under retracts, then it is closed under all products that exist in $\mathcal{T}$; see [2, Remark 4.5(3)]. Dually, weakly coreflective subcategories closed under retracts are closed under coproducts. Reflective subcategories are closed under limits, while coreflective subcategories are closed under colimits.

If L is a reflection on an additive category $\mathcal{T}$, then the objects X such that $LX = 0$ are called *L-acyclic*. The full subcategory of L-acyclic objects is closed under colimits. For a

coreflection C, the class of objects X such that $CX = 0$ is closed under limits, and such objects are called C-*acyclic*.

1.2. Closure properties in triangulated categories.

From now on, we assume that $\mathcal{T}$ is a triangulated category with products and coproducts. Motivated by topology, we denote by Σ the shift operator and call it *suspension*. Distinguished triangles in $\mathcal{T}$ will simply be called *triangles* and will be denoted by

$$(1.1) \qquad X \xrightarrow{\ u\ } Y \xrightarrow{\ v\ } Z \xrightarrow{\ w\ } \Sigma X,$$

or shortly by (u, v, w).

A full subcategory $\mathcal{S}$ of $\mathcal{T}$ will be called

(i) *closed under fibres* if X is in $\mathcal{S}$ for every triangle (1.1) where Y and Z are in $\mathcal{S}$;
(ii) *closed under cofibres* if Z is in $\mathcal{S}$ for every triangle (1.1) where X and Y are in $\mathcal{S}$;
(iii) *closed under extensions* if Y is in $\mathcal{S}$ for every triangle (1.1) where X and Z are in $\mathcal{S}$;
(iv) *triangulated* if it is closed under fibres, cofibres and extensions.

A full subcategory of $\mathcal{T}$ is called *localizing* if it is triangulated and closed under coproducts, and *colocalizing* if it is triangulated and closed under products. If a triangulated subcategory $\mathcal{S}$ is closed under countable coproducts or under countable products, then $\mathcal{S}$ is automatically closed under retracts; see [39, Lemma 1.4.9] or [48, Proposition 1.6.8].

More generally, a full subcategory of $\mathcal{T}$ will be called *semilocalizing* if it is closed under coproducts, cofibres, and extensions (hence under retracts and suspension), but not necessarily under fibres. And a full subcategory will be called *semicolocalizing* if it is closed under products, fibres, and extensions (therefore under retracts and desuspension as well). Semilocalizing subcategories are also called *cocomplete pre-aisles* elsewhere [4], [56], and semicolocalizing subcategories are called *complete pre-coaisles* —the terms "aisle" and "coaisle" originated in [42]. See also [6] for a related discussion of torsion pairs and t-structures in triangulated categories.

A reflection L on $\mathcal{T}$ will be called *semiexact* if the subcategory of L-local objects is semicolocalizing, and *exact* if it is colocalizing. Dually, a coreflection C will be called *semiexact* if the subcategory of C-colocal objects is semilocalizing and *exact* if it is localizing.

If L is a semiexact reflection with unit l, then, since the class of L-local objects is closed under desuspension, there is a natural morphism $\nu_X \colon LX \to \Sigma^{-1}L\Sigma X$ such that $\nu_X \circ l_X = \Sigma^{-1}l_{\Sigma X}$ for all X, and hence a natural morphism

$$(1.2) \qquad \Sigma\nu_X \colon \Sigma LX \longrightarrow L\Sigma X$$

such that $\Sigma\nu_X \circ \Sigma l_X = l_{\Sigma X}$. As we next show, if $\Sigma LX \cong L\Sigma X$ for a given object X, then $\Sigma\nu_X$ is automatically an isomorphism.

Lemma 1.1. *Suppose that L is a semiexact reflection. If ΣLX is L-local for a given object X, then $\Sigma\nu_X$ is an isomorphism.*

Proof. If ΣLX is L-local, then there is a (unique) morphism $h\colon L\Sigma X \to \Sigma LX$ such that $h \circ l_{\Sigma X} = \Sigma l_X$. Thus $\Sigma \nu_X \circ h \circ l_{\Sigma X} = l_{\Sigma X}$, which implies that $\Sigma \nu_X \circ h = \mathrm{id}$, by the universal property of L. Similarly, $h \circ \Sigma \nu_X \circ \Sigma l_X = \Sigma l_X$, and hence $\Sigma^{-1} h \circ \nu_X \circ l_X = l_X$, from which it follows that $\Sigma^{-1} h \circ \nu_X = \mathrm{id}$, or $h \circ \Sigma \nu_X = \mathrm{id}$. This proves that $\Sigma \nu_X$ has indeed an inverse. $\square$

Theorem 1.2. *Let $\mathcal{T}$ be a triangulated category. For a semiexact reflection L on $\mathcal{T}$, the following assertions are equivalent:*

   (i) *L is exact.*
   (ii) *The class of L-local objects is closed under Σ.*
   (iii) *$\Sigma LX \cong L\Sigma X$ for all X.*
   (iv) *$\Sigma \nu_X \colon \Sigma LX \to L\Sigma X$ is an isomorphism for all X.*
   (v) *L preserves all triangles.*

Proof. The equivalence between (i) and (ii) follows from the definitions. The fact that (ii) $\Rightarrow$ (iv) is given by Lemma 1.1, and obviously (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii).

   In order to prove that (ii) $\Rightarrow$ (v), let (u, v, w) be a triangle, and let C be a cofibre of Lu. Thus we can choose a morphism φ yielding a commutative diagram of triangles

$$
\begin{array}{ccccccccc}
X & \xrightarrow{\ u\ } & Y & \xrightarrow{\ v\ } & Z & \xrightarrow{\ w\ } & \Sigma X & \xrightarrow{\ -\Sigma u\ } & \Sigma Y \\
\downarrow{\scriptstyle l_X} & & \downarrow{\scriptstyle l_Y} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \Sigma l_X} & & \downarrow{\scriptstyle \Sigma l_Y} \\
LX & \xrightarrow{\ Lu\ } & LY & \longrightarrow & C & \longrightarrow & \Sigma LX & \xrightarrow{\ -\Sigma Lu\ } & \Sigma LY.
\end{array}
$$

Since C is a fibre of a morphism between L-local objects, it is itself L-local, since L is semiexact. From the five-lemma it follows that the morphism $\mathcal{T}(C, W) \to \mathcal{T}(Z, W)$ induced by φ is an isomorphism for every L-local object W, and therefore φ is an L-localization, so $C \cong LZ$. Then the induced morphisms $LY \to LZ$ and $LZ \to L\Sigma X$ (using $\Sigma \nu_X$) are equal to Lv and Lw respectively, by the universal property of L. This proves that (Lu, Lv, Lw) is a triangle.

   Finally, (v) $\Rightarrow$ (iii) by considering triangles of the form $X \to X \to 0 \to \Sigma X$, so the argument is complete. $\square$

   There is of course a dual result for semiexact coreflections, with a similar proof. We omit the details.

Theorem 1.3. *Let $\mathcal{T}$ be a triangulated category. A reflection L on $\mathcal{T}$ is semiexact if and only if, for every triangle*

$$
\tag{1.3} X \xrightarrow{\ u\ } Y \xrightarrow{\ v\ } Z \xrightarrow{\ w\ } \Sigma X
$$

in which Z is L-local, the sequence

$$
\tag{1.4} LX \xrightarrow{\ Lu\ } LY \xrightarrow{\ (l_Z)^{-1} \circ Lv\ } Z \xrightarrow{\ \Sigma l_X \circ w\ } \Sigma LX
$$

is a triangle.

Proof. Assume that L is a semiexact reflection and let C be a cofibre of $l_X \circ (-\Sigma^{-1}w)$ for a given triangle (1.3). Then there is a commutative diagram of triangles

$$\begin{array}{ccccccc}
\Sigma^{-1}Z & \xrightarrow{-\Sigma^{-1}w} & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\| & & \downarrow{l_X} & & \downarrow{\varphi} & & \| & & \downarrow{\Sigma l_X} \\
\Sigma^{-1}Z & \longrightarrow & LX & \longrightarrow & C & \longrightarrow & Z & \longrightarrow & \Sigma LX
\end{array}$$

where C is L-local since L is semiexact. Again by the five-lemma, φ induces an isomorphism $\mathcal{T}(C,W) \cong \mathcal{T}(Y,W)$ for every L-local object W. Hence $C \cong LY$, and the universal property of L implies then that the resulting arrows $LX \to LY$ and $LY \to Z$ are Lu and $(l_Z)^{-1} \circ Lv$ respectively, as claimed.

Conversely, suppose given any triangle (1.3) where X and Z are L-local. Then, by assumption, we have a commutative diagram of triangles

$$\begin{array}{ccccccc}
\Sigma^{-1}Z & \xrightarrow{-\Sigma^{-1}w} & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\downarrow{\Sigma^{-1}l_Z} & & \downarrow{l_X} & & \downarrow{l_Y} & & \downarrow{l_Z} & & \downarrow{\Sigma l_X} \\
\Sigma^{-1}LZ & \longrightarrow & LX & \xrightarrow{Lu} & LY & \xrightarrow{Lv} & LZ & \longrightarrow & \Sigma LX
\end{array}$$

where l_X and l_Z are isomorphisms. It then follows that l_Y is also an isomorphism and hence Y is L-local. Similarly, if Y and Z are L-local, then X is L-local. Therefore, the subcategory of L-local objects is closed under fibres and extensions, as claimed. $\square$

There is again a dual result for coreflections.

1.3. Orthogonality and semiorthogonality.

Several kinds of orthogonality can be considered in a triangulated category. In this article it will be convenient to use the same notation as in [7]. Thus, for a class of objects $\mathcal{D}$ in a triangulated category $\mathcal{T}$ with products and coproducts, we write

$$^{\perp}\mathcal{D} = \{X \mid \mathcal{T}(X, \Sigma^k D) = 0 \text{ for all } D \in \mathcal{D} \text{ and } k \in \mathbb{Z}\},$$

$$\mathcal{D}^{\perp} = \{Y \mid \mathcal{T}(\Sigma^k D, Y) = 0 \text{ for all } D \in \mathcal{D} \text{ and } k \in \mathbb{Z}\}.$$

For every class of objects $\mathcal{D}$, the class $^{\perp}\mathcal{D}$ is localizing and $\mathcal{D}^{\perp}$ is colocalizing. A localizing subcategory $\mathcal{C}$ is called *closed* if $\mathcal{C} = {}^{\perp}\mathcal{D}$ for some $\mathcal{D}$, or equivalently if $\mathcal{C} = {}^{\perp}(\mathcal{C}^{\perp})$, and a colocalizing subcategory $\mathcal{L}$ is called *closed* if $\mathcal{L} = \mathcal{D}^{\perp}$ for some $\mathcal{D}$, or equivalently if $\mathcal{L} = ({}^{\perp}\mathcal{L})^{\perp}$.

For example, if we work in the homotopy category of spectra and E is a spectrum, then the statement $X \in {}^{\perp}E$ holds if and only if $E^*(X) = 0$, where E^* is the reduced cohomology theory represented by E. Thus, $^{\perp}E$ is the class of E^*-acyclic spectra. (Here and later, we write $^{\perp}E$ instead of $^{\perp}\{E\}$ for simplicity.)

Let us introduce the following variant, which we call *semiorthogonality*:

$$^{\lrcorner}\mathcal{D} = \{X \mid \mathcal{T}(X, \Sigma^k D) = 0 \text{ for all } D \in \mathcal{D} \text{ and } k \leq 0\},$$

$$\mathcal{D}^{\llcorner} = \{Y \mid \mathcal{T}(\Sigma^k D, Y) = 0 \text{ for all } D \in \mathcal{D} \text{ and } k \geq 0\}.$$

Similarly as above, for every class of objects $\mathcal{D}$ the class ${}^{\lrcorner}\mathcal{D}$ is semilocalizing, while $\mathcal{D}^{\llcorner}$ is semicolocalizing. A semilocalizing subcategory $\mathcal{C}$ will be called *closed* if $\mathcal{C} = {}^{\lrcorner}\mathcal{D}$ for some class of objects $\mathcal{D}$, or equivalently if $\mathcal{C} = {}^{\lrcorner}(\mathcal{C}^{\llcorner})$. A semicolocalizing subcategory $\mathcal{L}$ will be called *closed* if $\mathcal{L} = \mathcal{D}^{\llcorner}$ for some $\mathcal{D}$, or equivalently if $\mathcal{L} = ({}^{\lrcorner}\mathcal{L})^{\llcorner}$.

Note that, if a class $\mathcal{D}$ is preserved by Σ and Σ^{-1}, then ${}^{\lrcorner}\mathcal{D} = {}^{\perp}\mathcal{D}$ and $\mathcal{D}^{\llcorner} = \mathcal{D}^{\perp}$. Therefore, if a localizing subcategory $\mathcal{C}$ is closed, then it is also closed as a semilocalizing subcategory, since $\mathcal{C} = {}^{\perp}(\mathcal{C}^{\perp}) = {}^{\lrcorner}(\mathcal{C}^{\llcorner})$. The dual assertion is of course also true.

Semiexact reflections and semiexact coreflections are linked through the following basic result, which generalizes Lemma 3.1.6 in [39]; compare also with [6, Proposition 2.3], [7, Lemma 1.2], and [43, Proposition 4.12.1].

Theorem 1.4. *In every triangulated category $\mathcal{T}$ there is a bijective correspondence between semiexact reflections and semiexact coreflections such that, if a reflection L is paired with a coreflection C under this bijection, then the following hold:*

(i) *For every X, the morphisms $l_X \colon X \to LX$ and $c_X \colon CX \to X$ fit into a triangle*

$$CX \longrightarrow X \longrightarrow LX \longrightarrow \Sigma CX.$$

(ii) *The class $\mathcal{L}$ of L-local objects coincides with the class of C-acyclics, and the class $\mathcal{C}$ of C-colocal objects coincides with the class of L-acyclics.*

(iii) *The class $\mathcal{C}$ is equal to ${}^{\lrcorner}\mathcal{L}$, and $\mathcal{L}$ is equal to $\mathcal{C}^{\llcorner}$.*

(iv) *L is exact if and only if C is exact. In this case, $\mathcal{C} = {}^{\perp}\mathcal{L}$ and $\mathcal{L} = \mathcal{C}^{\perp}$.*

Proof. Let L be a semiexact reflection. For every X in $\mathcal{T}$, choose a fibre CX of the unit morphism $l_X \colon X \to LX$. Thus, for every X in $\mathcal{T}$ we have a triangle

$$(1.5) \qquad\qquad CX \xrightarrow{c_X} X \xrightarrow{l_X} LX \longrightarrow \Sigma CX.$$

If we apply L to (1.5), then, since LX is L-local, Theorem 1.3 implies that

$$LCX \xrightarrow{Lc_X} LX \xrightarrow{\text{id}} LX \longrightarrow \Sigma LCX$$

is a triangle, and hence $LCX = 0$. For each morphism $f \colon X \to Y$, choose a morphism $Cf \colon CX \to CY$ such that the following diagram commutes:

$$
\begin{array}{ccccccc}
\Sigma^{-1}LX & \longrightarrow & CX & \xrightarrow{c_X} & X & \xrightarrow{l_X} & LX \\
{\scriptstyle \Sigma^{-1}Lf}\downarrow & & {\scriptstyle Cf}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Lf} \\
\Sigma^{-1}LY & \longrightarrow & CY & \xrightarrow[c_Y]{} & Y & \xrightarrow[l_Y]{} & LY.
\end{array}
$$

Then Cf is unique, since $LCX = 0$ implies that $\mathcal{T}(CX, \Sigma^k LY) = 0$ for $k \leq 0$, and therefore $c_Y \colon CY \to Y$ induces a bijection $\mathcal{T}(CX, CY) \cong \mathcal{T}(CX, Y)$. This yields functoriality

of C and naturality of c. Moreover, for each X the following diagram commutes:

$$
\begin{array}{ccc}
CCX & \xrightarrow{c_{CX}} CX \xrightarrow{l_{CX}} & LCX \\
{\scriptstyle Cc_X}\downarrow & {\scriptstyle c_X}\downarrow & \downarrow{\scriptstyle Lc_X} \\
CX & \xrightarrow{\;\;c_X\;\;} X \xrightarrow{\;\;l_X\;\;} & LX.
\end{array}
$$

Here the fact that $LCX = 0$ implies that c_{CX} is an isomorphism. And, since c_X induces a bijection $\mathcal{T}(CCX, CX) \cong \mathcal{T}(CCX, X)$, we infer that $c_{CX} = Cc_X$ and therefore C is a coreflection.

In order to prove that C is semiexact, we need to show that the class of C-colocal objects is closed under cofibres and extensions. For this, let $X \to Y \to Z$ be a triangle and assume first that X and Z are C-colocal. Consider the following diagram, where all the columns and the central row are triangles:

$$
\begin{array}{ccc}
CX & \longrightarrow CY \longrightarrow & CZ \\
\downarrow & \downarrow & \downarrow \\
X & \longrightarrow Y \longrightarrow & Z \\
\downarrow & \downarrow & \downarrow \\
LX & \longrightarrow LY \longrightarrow & LZ.
\end{array}
$$

Since $CX \cong X$ and $CZ \cong Z$, we have that $LX = 0$ and $LZ = 0$. Since L is semiexact, if W is any L-local object then $\mathcal{T}(\Sigma X, W) \cong \mathcal{T}(X, \Sigma^{-1}W) \cong \mathcal{T}(LX, \Sigma^{-1}W) = 0$, which tells us that $L\Sigma X = 0$ as well. This implies that $\mathcal{T}(Y, W) \cong \mathcal{T}(Z, W)$ for every L-local object W, and hence the morphism $LY \to LZ$ is an isomorphism. Therefore $LY = 0$ and $CY \cong Y$, as needed. Second, assume that X and Y are C-colocal. Then $LX = 0$ and $LY = 0$ and the same argument tells us that $LZ = 0$, so $CZ \cong Z$.

Part (ii) follows directly from (i). The first claim of (iii) is proved as follows. Let $X \in \mathcal{C}$. Since $LX = 0$ and the class of L-local objects is closed under desuspension, we have that

$$
\mathcal{T}(X, \Sigma^k D) \cong \mathcal{T}(LX, \Sigma^k D) = 0 \text{ for } k \leq 0 \text{ and all } D \in \mathcal{L}.
$$

This tells us that $X \in {}^{\dashv}\mathcal{L}$. Conversely, if $X \in {}^{\dashv}\mathcal{L}$, then $\mathcal{T}(LX, LX) \cong \mathcal{T}(X, LX) = 0$. Therefore $LX = 0$, so $X \in \mathcal{C}$. The second part is proved dually.

The first claim of (iv) follows by considering the commutative diagram

$$
\begin{array}{ccc}
\Sigma CX & \xrightarrow{\Sigma c_X} \Sigma X \xrightarrow{\Sigma l_X} \Sigma LX \longrightarrow & \Sigma\Sigma CX \\
\downarrow & \| \quad\quad \downarrow & \downarrow \\
C\Sigma X & \xrightarrow{\;\;c_{\Sigma X}\;\;} \Sigma X \xrightarrow{\;\;l_{\Sigma X}\;\;} L\Sigma X \longrightarrow & \Sigma C\Sigma X,
\end{array}
$$

and using Theorem 1.2 (and its dual). The rest is proved with the same arguments as in part (iii). $\qquad\square$

In the homotopy category of spectra, every f-localization functor L_f in the sense of [12] or [20] is a reflection, and cellularizations Cell_A are coreflections. Classes of f-local spectra are closed under fibres, but not under cofibres nor extensions, in general. Dually, A-cellular classes are closed under cofibres. Nullification functors P_A (i.e., f-localizations where $f \colon A \to 0$, such as Postnikov sections) are semiexact reflections. Homological localizations of spectra (or, more generally, nullifications P_A where $\Sigma A \simeq A$) are exact reflections. One proves as in [17] that the kernel of a nullification P_A is precisely the closure under extensions of the image of Cell_A; see also [33].

1.4. **Torsion pairs and t-structures.** For a class of objects $\mathcal{D}$ in a triangulated category $\mathcal{T}$ with products and coproducts, we denote by $\mathrm{loc}(\mathcal{D})$ the smallest localizing subcategory of $\mathcal{T}$ that contains $\mathcal{D}$; that is, the intersection of all the localizing subcategories of $\mathcal{T}$ that contain $\mathcal{D}$. We use the terms $\mathrm{coloc}(\mathcal{D})$, $\mathrm{sloc}(\mathcal{D})$, and $\mathrm{scoloc}(\mathcal{D})$ analogously, and we say that each of these is *generated* by $\mathcal{D}$. If $\mathcal{D}$ consists of only one object, then we say that the respective classes are *singly generated*.

Note that if $\mathcal{D} = \{D_i\}_{i \in I}$ is a set (not a proper class), then

$$\mathrm{loc}(\mathcal{D}) = \mathrm{loc}\left(\coprod_{i \in I} D_i\right) \ \text{and} \ \mathrm{coloc}(\mathcal{D}) = \mathrm{coloc}\left(\prod_{i \in I} D_i\right),$$

and similarly with $\mathrm{sloc}(\mathcal{D})$ and $\mathrm{scoloc}(\mathcal{D})$. Thus, in the presence of products and coproducts, "generated by a set" and "singly generated" mean the same thing.

It is important to relate classes generated by $\mathcal{D}$ in this sense with the corresponding closures of $\mathcal{D}$ under orthogonality or semiorthogonality. Although this seems to be difficult in general, it follows from Theorem 1.4 that reflective colocalizing or semicolocalizing subcategories are closed, and coreflective localizing or semilocalizing subcategories are also closed. This has the following consequence.

Proposition 1.5. *Let $\mathcal{D}$ be any class of objects in a triangulated category with products and coproducts.*

   (i) *If $\mathrm{scoloc}(\mathcal{D})$ is reflective, then $\mathrm{scoloc}(\mathcal{D}) = ({}^{\lrcorner}\mathcal{D})^{\llcorner}$, and if $\mathrm{sloc}(\mathcal{D})$ is coreflective, then $\mathrm{sloc}(\mathcal{D}) = {}^{\lrcorner}(\mathcal{D}^{\llcorner})$.*

   (ii) *If $\mathrm{coloc}(\mathcal{D})$ is reflective, then $\mathrm{coloc}(\mathcal{D}) = ({}^{\perp}\mathcal{D})^{\perp}$, and if $\mathrm{loc}(\mathcal{D})$ is coreflective, then $\mathrm{loc}(\mathcal{D}) = {}^{\perp}(\mathcal{D}^{\perp})$.*

Proof. We only prove the first claim, as the others follow similarly. Since $\mathcal{D} \subseteq \mathrm{scoloc}(\mathcal{D})$, we have $({}^{\lrcorner}\mathcal{D})^{\llcorner} \subseteq ({}^{\lrcorner}\mathrm{scoloc}(\mathcal{D}))^{\llcorner}$. If $\mathrm{scoloc}(\mathcal{D})$ is reflective, then it follows from part (iii) of Theorem 1.4 that $\mathrm{scoloc}(\mathcal{D})$ is closed. Hence, $({}^{\lrcorner}\mathcal{D})^{\llcorner} \subseteq \mathrm{scoloc}(\mathcal{D})$. The reverse inclusion follows from the fact that $({}^{\lrcorner}\mathcal{D})^{\llcorner}$ is a semicolocalizing subcategory containing $\mathcal{D}$, and $\mathrm{scoloc}(\mathcal{D})$ is minimal with this property. $\qquad\square$

We remark that, for every class $\mathcal{D}$, we have $\mathcal{D}^{\llcorner} = \mathrm{sloc}(\mathcal{D})^{\llcorner}$, and similarly with left semiorthogonality or orthogonality in either side. To prove this assertion, only the inclusion $\mathcal{D}^{\llcorner} \subseteq \mathrm{sloc}(\mathcal{D})^{\llcorner}$ needs to be checked, and this is done as follows. Since ${}^{\lrcorner}(\mathcal{D}^{\llcorner})$ is semilocalizing and contains $\mathcal{D}$, it also contains $\mathrm{sloc}(\mathcal{D})$. Hence,

$$(1.6) \qquad\qquad \mathcal{D}^{\llcorner} = ({}^{\lrcorner}(\mathcal{D}^{\llcorner}))^{\llcorner} \subseteq \mathrm{sloc}(\mathcal{D})^{\llcorner},$$

as claimed.

Proposition 1.6. *Let $\mathcal{D}$ be any class of objects in a triangulated category $\mathcal{T}$. Suppose that for each $X \in \mathcal{T}$ there is a triangle $CX \to X \to LX \to \Sigma CX$ where $CX \in \mathrm{sloc}(\mathcal{D})$ and $LX \in \mathrm{sloc}(\mathcal{D})^{\mathsf{L}}$. Then C defines a semiexact coreflection onto $\mathrm{sloc}(\mathcal{D})$.*

Proof. If Y is any object in $\mathrm{sloc}(\mathcal{D})$, then $\mathcal{T}(Y, LX) = 0$ and $\mathcal{T}(Y, \Sigma^{-1}LX) = 0$. Hence, the morphism $CX \to X$ induces a bijection $\mathcal{T}(Y, CX) \cong \mathcal{T}(Y, X)$, so C is a coreflection onto $\mathrm{sloc}(\mathcal{D})$, hence semiexact. $\square$

This fact will be used in Section 3. There is of course a dual result, and there are corresponding facts for exact coreflections and exact reflections; cf. [48, Theorem 9.1.13].

Under the hypotheses of Proposition 1.6, the classes $\mathrm{sloc}(\mathcal{D})$ and $\mathrm{sloc}(\mathcal{D})^{\mathsf{L}}$ form a *torsion pair* as defined in [6, I.2.1]. This yields the following fact, which also relates these notions with t-structures; see [4, § 1] as well.

Theorem 1.7. *In every triangulated category $\mathcal{T}$ there is a bijective correspondence between the following classes:*

(i) *Reflective semicolocalizing subcategories.*
(ii) *Coreflective semilocalizing subcategories.*
(iii) *Torsion pairs.*
(iv) *t-structures.*

Proof. The bijective correspondence between (i) and (ii) has been established in Theorem 1.4. The bijective correspondence between torsion pairs and t-structures is proved in [6, I.2.13]. If $\mathcal{C}$ is a coreflective semilocalizing subcategory, then, by Proposition 1.6, $(\mathcal{C}, \mathcal{C}^{\mathsf{L}})$ is a torsion pair. Conversely, if $(\mathcal{X}, \mathcal{Y})$ is a torsion pair, then [6, I.2.3] tells us that $\mathcal{X}$ is coreflective and semilocalizing, while $\mathcal{Y}$ is reflective and semicolocalizing. $\square$

1.5. **Tensor triangulated categories.** To conclude this introductory section, let $\mathcal{T}$ be a *tensor triangulated category*, in the sense of [5], [39], [57]. More precisely, we assume that $\mathcal{T}$ has a closed symmetric monoidal structure with a unit object S, tensor product denoted by $\wedge$ and internal hom bifunctor $F(-, -)$, compatible with the triangulated structure and such that $\mathcal{T}(X, F(Y, Z)) \cong \mathcal{T}(X \wedge Y, Z)$ naturally in all variables; cf. [39, A.2.1].

Then a full subcategory $\mathcal{C}$ of $\mathcal{T}$ is called an *ideal* if $E \wedge X$ is in $\mathcal{C}$ for every X in $\mathcal{C}$ and all E in $\mathcal{T}$, and a full subcategory $\mathcal{L}$ is called a *coideal* if $F(E, X)$ is in $\mathcal{L}$ for every X in $\mathcal{L}$ and all E in $\mathcal{T}$. A *localizing ideal* is a localizing subcategory that is also an ideal, and similarly in the dual case.

In the homotopy category of spectra, all localizing subcategories are ideals and all colocalizing subcategories are coideals. As shown in [39, Lemma 1.4.6], the same happens in any *monogenic* stable homotopy category (i.e., such that the unit of the monoidal structure is a small generator). In [39], for stable homotopy categories, the terms *localization* and *colocalization* were used in a more restrictive sense than in the present article. Thus an exact reflection L was called a localization in [39, Definition 3.1.1] if $LX = 0$ for an object X implies that $L(E \wedge X) = 0$ for every E; in other words, if the class of L-acyclic objects is a localizing ideal. Dually, an exact coreflection C was called a colocalization if

$CX = 0$ implies that $C(F(E, X)) = 0$ for every E, i.e., if the class of C-acyclic objects is a colocalizing coideal.

For each class of objects $\mathcal{D}$, the class of those X such that $F(X, D) = 0$ for all $D \in \mathcal{D}$ is a localizing ideal, and the class of those Y such that $F(D, Y) = 0$ for all $D \in \mathcal{D}$ is a colocalizing coideal. If $\mathcal{C}$ is a localizing ideal, then $\mathcal{C}^{\perp}$ is a colocalizing coideal, and, if $\mathcal{L}$ is a colocalizing coideal, then $^{\perp}\mathcal{L}$ is a localizing ideal. Thus, under the bijective correspondence given by Theorem 1.4, localizations in the sense of [39] are also paired with colocalizations.

2. Reflective colocalizing subcategories

Background on locally presentable and accessible categories can be found in [2] and [47]. The basic definitions are as follows. Let λ be a regular cardinal. A nonempty small category is λ-*filtered* if, given any set of objects $\{A_i \mid i \in I\}$ where $|I| < \lambda$, there is an object A and a morphism $A_i \to A$ for each $i \in I$, and, moreover, given any set of parallel arrows between any two objects $\{\varphi_j \colon B \to C \mid j \in J\}$ where $|J| < \lambda$, there is a morphism $\psi \colon C \to D$ such that $\psi \circ \varphi_j$ is the same morphism for all $j \in J$.

Let $\mathcal{K}$ be any category. A diagram $D \colon I \to \mathcal{K}$ where I is a λ-filtered small category is called a λ-*filtered diagram*, and, if D has a colimit, then $\operatorname{colim}_I D$ is called a λ-*filtered colimit*. An object X of $\mathcal{K}$ is λ-*presentable* if the functor $\mathcal{K}(X, -)$ preserves λ-filtered colimits. The category $\mathcal{K}$ is λ-*accessible* if all λ-filtered colimits exist in $\mathcal{K}$ and there is a set $\mathcal{S}$ of λ-presentable objects such that every object of $\mathcal{K}$ is a λ-filtered colimit of objects from $\mathcal{S}$. It is called *accessible* if it is λ-accessible for some λ. A cocomplete accessible category is called *locally presentable*.

Thus, the passage from locally presentable to accessible categories amounts to weakening the assumption of cocompleteness by imposing only that enough colimits exist. As explained in [2, § 2.1], using directed colimits instead of filtered colimits in the definitions leads to the same concepts of accessibility and local presentability.

As explained in [2, 6.3], *Vopěnka's principle* is equivalent to the statement that, *given any family of objects X_s of an accessible category indexed by the class of all ordinals, there is a morphism $X_s \to X_t$ for some ordinals $s < t$.*

A functor $\gamma \colon \mathcal{K} \to \mathcal{T}$ between two categories will be called *essentially surjective on sources* if, for every object X and every collection of morphisms $\{f_i \colon X \to X_i \mid i \in I\}$ in $\mathcal{T}$ (where I is any discrete category, possibly a proper class), there is an object K and a collection of morphisms $\{g_i \colon K \to K_i \mid i \in I\}$ in $\mathcal{K}$ together with isomorphisms $h \colon \gamma K \to X$ and $h_i \colon \gamma K_i \to X_i$ for all i rendering the following diagram commutative:

$$(2.1) \qquad \begin{array}{ccc} \gamma K & \xrightarrow{\gamma g_i} & \gamma K_i \\ {\scriptstyle h} \downarrow {\scriptstyle \cong} & & {\scriptstyle \cong} \downarrow {\scriptstyle h_i} \\ X & \xrightarrow{f_i} & X_i. \end{array}$$

For example, if $\mathcal{K}$ is a model category [38] and $\gamma \colon \mathcal{K} \to \operatorname{Ho}(\mathcal{K})$ is the canonical functor onto the corresponding homotopy category (which we assume to be the identity on objects),

then γ is essentially surjective on sources —and also on sinks; here "sources" and "sinks" are meant as in [1]. Indeed, given a collection $\{f_i\colon X \to X_i \mid i \in I\}$ in $\mathrm{Ho}(\mathcal{K})$, we can choose a cofibrant replacement $q\colon K \to X$ and a fibrant replacement $r_i\colon X_i \to K_i$ for each i. Then we can pick a morphism $g_i\colon K \to K_i$ for each i such that the zig-zag

$$X \xleftarrow{\;q\;} K \xrightarrow{\;g_i\;} K_i \xleftarrow{\;r_i\;} X_i$$

represents f_i. Then the choices $h = \gamma q$ and $h_i = (\gamma r_i)^{-1}$ render (2.1) commutative.

Theorem 2.1. *Let $\mathcal{T}$ be a category with products and suppose given a functor $\gamma\colon \mathcal{K} \to \mathcal{T}$ where $\mathcal{K}$ is accessible and γ is essentially surjective on sources. If Vopěnka's principle holds, then every full subcategory $\mathcal{L} \subseteq \mathcal{T}$ closed under products is weakly reflective.*

Proof. Write $\mathcal{L}$ as the union of an ascending chain of full subcategories indexed by the ordinals,

$$\mathcal{L} = \bigcup_{i \in \mathrm{Ord}} \mathcal{L}_i,$$

where each $\mathcal{L}_i$ is the closure under products of a small subcategory $\mathcal{A}_i$. For each object X of $\mathcal{T}$, let X_i be the product of the codomains of all morphisms from X to objects of $\mathcal{A}_i$, and let $f_i\colon X \to X_i$ be the induced morphism. Then every morphism from X to some object of $\mathcal{L}_i$ factors through f_i and hence f_i is a weak reflection of X onto $\mathcal{L}_i$.

Now, as in [2, 6.26], in order to prove that $\mathcal{L}$ is weakly reflective, it suffices to find an ordinal i such that, for all $j \geq i$, the morphism f_j can be factorized as $f_j = \varphi_{ij} \circ f_i$ for some $\varphi_{ij}\colon X_i \to X_j$. In other words,

$$(X \downarrow \mathcal{L})(f_i, f_j) \neq \emptyset$$

for all $j \geq i$, where $(X \downarrow \mathcal{L})$ denotes the comma category of $\mathcal{L}$ under X. Suppose the contrary. Then there are ordinals $i_0 < i_1 < \cdots < i_s < \cdots$, where s ranges over all the ordinals, such that

(2.2) $$(X \downarrow \mathcal{L})(f_{i_s}, f_{i_t}) = \emptyset$$

if $s < t$. Since γ is essentially surjective on sources, there is a morphism $g_i\colon K \to K_i$ in $\mathcal{K}$ for each ordinal i, and there are isomorphisms $h\colon \gamma K \to X$ and $h_i\colon \gamma K_i \to X_i$ such that $f_i \circ h = h_i \circ \gamma g_i$ for all i. Then $(K \downarrow \mathcal{K})(g_{i_s}, g_{i_t}) = \emptyset$ if $s < t$, since, if there is a morphism $G\colon K_{i_s} \to K_{i_t}$ with $G \circ g_{i_s} = g_{i_t}$, then $F = h_{i_t} \circ \gamma G \circ (h_{i_s})^{-1}$ satisfies $F \circ f_{i_s} = f_{i_t}$, contradicting (2.2). Since the category $(K \downarrow \mathcal{K})$ is accessible by [2, 2.44], this is incompatible with Vopěnka's principle, according to [2, 6.3]. $\qquad\square$

We next show that the existence of a weak reflection implies the existence of a reflection under assumptions that do not require any further input from large-cardinal theory. We say that *idempotents split* in a category $\mathcal{T}$ if for every morphism $e\colon A \to A$ such that $e \circ e = e$ there are morphisms $f\colon A \to B$ and $g\colon B \to A$ such that $e = g \circ f$ and $f \circ g = \mathrm{id}$. This is automatic in a category with coequalizers, since f can be chosen to be a coequalizer of e and the identity, and g is determined by the universal property of the coequalizer. It also holds in other important cases; for instance, by [48, Proposition 1.6.8], idempotents split in any triangulated category with countable coproducts or countable products.

We will need the following result, which, as pointed out to us by B. Chorny, can be derived from [34, VII.28H]. A self-contained proof is given here for the sake of completeness.

Recall that a *weak limit* of a diagram $D\colon I \to \mathcal{T}$, where $\mathcal{T}$ is any category and I is a small category, is an object X of $\mathcal{T}$ together with a natural transformation $\nu\colon X \to D$ (where X is seen as a constant functor, so ν is a cone to D) such that any other natural transformation $Y \to D$ with Y in $\mathcal{T}$ factorizes through ν, not necessarily in a unique way. Weak colimits are defined dually.

Theorem 2.2. *Let $\mathcal{T}$ be a category with products where idempotents split. Let $\mathcal{L}$ be a weakly reflective subcategory of $\mathcal{T}$ closed under retracts, and assume that every pair of parallel arrows in $\mathcal{L}$ has a weak equalizer that lies in $\mathcal{L}$. Then $\mathcal{L}$ is reflective.*

Proof. Note first that, by [2, Remark 4.5(3)], since $\mathcal{L}$ is weakly reflective and closed under retracts, it is also closed under products. Recall also that reflective or weakly reflective subcategories are tacitly assumed to be full.

Let A be any object of $\mathcal{T}$ and let $r_0\colon A \to A_0$ be a weak reflection of A onto $\mathcal{L}$. Let I denote the set of all pairs of morphisms $(f, g)\colon A_0 \rightrightarrows A_0$ such that $f \circ r_0 = g \circ r_0$, and let $u_1\colon A_1 \to A_0$ be a weak equalizer of the pair $(\prod_{i\in I} f_i, \prod_{i\in I} g_i)\colon A_0 \rightrightarrows \prod_{i\in I} A_0$. By hypothesis, we may choose u_1 in $\mathcal{L}$.

Since u_1 is a weak equalizer, there is a morphism $r_1\colon A \to A_1$ such that $u_1 \circ r_1 = r_0$. Moreover, since r_0 is a weak reflection and A_1 is in $\mathcal{L}$, there is a morphism $t_1\colon A_0 \to A_1$ such that $t_1 \circ r_0 = r_1$. Then $(u_1 \circ t_1, \mathrm{id}) \in I$ and hence $u_1 \circ t_1 \circ u_1 = u_1$. It follows that $t_1 \circ u_1$ is idempotent and hence it splits. That is, there are morphisms $u_2\colon A_2 \to A_1$ and $t_2\colon A_1 \to A_2$ such that $u_2 \circ t_2 = t_1 \circ u_1$ and $t_2 \circ u_2 = \mathrm{id}$.

We next prove that, if we pick $r_2 = t_2 \circ r_1$, then r_2 is a reflection of A onto $\mathcal{L}$. First of all, A_2 is a retract of A_1 and hence A_2 is in $\mathcal{L}$. Second, from the equality $r_0 = u_1 \circ u_2 \circ r_2$ it follows that r_2 is a weak reflection of A onto $\mathcal{L}$. Now, given a morphism $f\colon A \to X$ with X in $\mathcal{L}$, since r_2 is a weak reflection, there is a morphism $g\colon A_2 \to X$ such that $g \circ r_2 = f$. Suppose that there is another $h\colon A_2 \to X$ with $h \circ r_2 = g \circ r_2$. Let $w\colon B \to A_2$ be a weak equalizer of g and h with B in $\mathcal{L}$. Then, as we next show, w has a right inverse, so the equality $g \circ w = h \circ w$ implies that $g = h$, as needed.

In order to prove that w has a right inverse, note that, since $h \circ r_2 = g \circ r_2$, there is a morphism $t\colon A \to B$ with $w \circ t = r_2$. Since r_0 is a weak reflection of A onto $\mathcal{L}$, there is a morphism $s\colon A_0 \to B$ such that $s \circ r_0 = t$. Now $u_1 \circ u_2 \circ w \circ s \circ r_0 = r_0$; hence $(u_1 \circ u_2 \circ w \circ s, \mathrm{id}) \in I$, from which it follows that $u_1 \circ u_2 \circ w \circ s \circ u_1 = u_1$. Finally, note that $t_2 \circ t_1 \circ u_1 \circ u_2 = t_2 \circ u_2 \circ t_2 \circ u_2 = \mathrm{id}$ and therefore $w \circ s \circ u_1 = t_2 \circ t_1 \circ u_1 = t_2$, so $w \circ s \circ u_1 \circ u_2 = \mathrm{id}$, as claimed. $\qquad\square$

Corollary 2.3. *Every weakly reflective subcategory closed under retracts and fibres in a triangulated category with products is reflective.*

Proof. This is implied by Theorem 2.2, since a fibre of $f - g$ is a weak equalizer of two given parallel arrows f and g, and idempotents split in a triangulated category if countable products exist, according to [48, Remark 1.6.9]. $\qquad\square$

Dually, every weakly coreflective subcategory closed under retracts and cofibres in a triangulated category $\mathcal{T}$ with coproducts is coreflective. Neeman proved this fact in [49, Proposition 1.4] for thick subcategories, without assuming the existence of coproducts in $\mathcal{T}$, but imposing that idempotents split in $\mathcal{T}$.

Putting together Theorem 2.1 and Theorem 2.2, we state the main result of this section. A model category $\mathcal{K}$ is called *stable* [54, 2.1.1] if it is pointed (i.e., the unique map from the initial object to the terminal object is an isomorphism) and the suspension and loop operators are inverse equivalences on the homotopy category $\mathrm{Ho}(\mathcal{K})$. It then follows that $\mathrm{Ho}(\mathcal{K})$ is triangulated, where the triangles come from fibre or cofibre sequences in $\mathcal{K}$ (see [38, 6.2.6]), and has products and coproducts over arbitrary index sets, coming from those of $\mathcal{K}$. In fact, $\mathrm{Ho}(\mathcal{K})$ has weak limits and colimits, but it is neither complete nor cocomplete in general; see [39, 2.2]. Quillen equivalences of stable model categories preserve fibre and cofibre sequences and hence the triangulated structure of their homotopy categories.

Theorem 2.4. *Let $\mathcal{K}$ be a locally presentable category with a stable model category structure. If Vopěnka's principle holds, then every full subcategory $\mathcal{L}$ of $\mathrm{Ho}(\mathcal{K})$ closed under products and fibres is reflective. If $\mathcal{L}$ is semicolocalizing, then the reflection is semiexact. If $\mathcal{L}$ is colocalizing, then the reflection is exact.*

Proof. The canonical functor $\gamma \colon \mathcal{K} \to \mathrm{Ho}(\mathcal{K})$ is essentially surjective on sources and sinks. Hence γ satisfies the assumptions of Theorem 2.1, from which it follows that $\mathcal{L}$ is weakly reflective. Closure of $\mathcal{L}$ under retracts follows from the Eilenberg swindle, as in [39, Lemma 1.4.9]. Then Corollary 2.3 implies that $\mathcal{L}$ is in fact reflective. The other statements hold by the definitions of the terms involved. □

We do not know if the assumption that $\mathcal{L}$ be closed under fibres is necessary for the validity of Theorem 2.4. We note however that an important kind of reflective subcategories, namely classes of f-local objects in the sense of [20] or [35] in homotopy categories of suitable model categories, are closed under fibres.

Under the assumptions of Theorem 2.4, if $\mathrm{Ho}(\mathcal{K})$ is tensor triangulated and a given colocalizing subcategory $\mathcal{L} \subseteq \mathrm{Ho}(\mathcal{K})$ is a coideal, then the exact reflection L given by Theorem 2.4 is automatically a localization in the sense of [39], that is, the class of L-acyclic objects is then a localizing ideal.

Corollary 2.5. *Let $\mathcal{K}$ be a locally presentable stable model category. If Vopěnka's principle holds, then every closed semilocalizing subcategory of $\mathrm{Ho}(\mathcal{K})$ is coreflective.*

Proof. Let $\mathcal{C}$ be a closed semilocalizing subcategory of $\mathrm{Ho}(\mathcal{K})$. Then $\mathcal{C}^{\mathsf{L}}$ is a semicolocalizing subcategory, which is reflective by Theorem 2.4. Hence $^{\mathsf{J}}(\mathcal{C}^{\mathsf{L}})$ is coreflective by Theorem 1.4, and it is equal to $\mathcal{C}$ since $\mathcal{C}$ is closed by assumption. □

As observed in Subsection 1.3, if a localizing subcategory $\mathcal{C}$ is closed, then it is also closed if viewed as a semilocalizing subcategory. Hence, the statement of Corollary 2.5 is also true for closed localizing subcategories.

It would be very interesting to have a counterexample (if there is one) to the statement of Theorem 2.4 under some set-theoretical assumption incompatible with Vopěnka's principle. We next give a partial result in this direction, based on [16] and [21], which shows that Theorem 2.1 cannot be proved in ZFC. This result answers the second part of Open Problem 5 from [2, p. 296].

Proposition 2.6. *Assuming the nonexistence of measurable cardinals, there is a full subcategory of the category of abelian groups which is closed under products and retracts but not weakly reflective.*

Proof. Let $\mathcal{C}$ be the closure of the class of groups $\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}$ under products and retracts, where κ runs over all cardinals, and $\mathbb{Z}^\kappa$ denotes a product of copies of the integers indexed by κ while $\mathbb{Z}^{<\kappa}$ denotes the subgroup of sequences whose support (i.e., the set of nonzero entries) has cardinality smaller than κ. Assume that $w\colon \mathbb{Z} \to A$ is a weak reflection of $\mathbb{Z}$ onto $\mathcal{C}$. Then there is a retraction

$$\prod_{i \in I} \mathbb{Z}^{\kappa_i}/\mathbb{Z}^{<\kappa_i} \xrightarrow{\ r\ } A$$

for some set of cardinals $\{\kappa_i\}_{i \in I}$. Choose a regular cardinal λ bigger than the sum $\Sigma_{i \in I}\,\kappa_i$. Let $d\colon \mathbb{Z} \to \mathbb{Z}^\lambda$ be the diagonal and $p\colon \mathbb{Z}^\lambda \to \mathbb{Z}^\lambda/\mathbb{Z}^{<\lambda}$ the projection. Since w is a weak reflection, there is a homomorphism $f\colon A \to \mathbb{Z}^\lambda/\mathbb{Z}^{<\lambda}$ with $f \circ w = p \circ d$. Following [16, Lemma 6.1], there is a homomorphism $g\colon A \to \mathbb{Z}^\lambda$ such that $f = p \circ g$. Since the image of d is not contained in $\mathbb{Z}^{<\lambda}$, we have $f \neq 0$ and thus $g \neq 0$. Since r is an epimorphism, $g \circ r \neq 0$. Now, since $\mathbb{Z}^\lambda$ maps onto $\prod_{i \in I} \mathbb{Z}^{\kappa_i}$, there is a nonzero homomorphism $h\colon \mathbb{Z}^\lambda \to \mathbb{Z}^\lambda$ which vanishes on the direct sum $\oplus_{i<\lambda} \mathbb{Z}$, since it factors through $\prod_{i \in I} \mathbb{Z}^{\kappa_i}/\mathbb{Z}^{<\kappa_i}$. Hence, by composing h with a suitable projection, we obtain a nonzero homomorphism $\mathbb{Z}^\lambda \to \mathbb{Z}$ which vanishes on the direct sum $\oplus_{i<\lambda} \mathbb{Z}$. According to [27, 94.4], this fact implies the existence of measurable cardinals. This contradiction proves the statement. $\square$

Corollary 2.7. *Assuming the nonexistence of measurable cardinals, there is a full subcategory of the homotopy category of spectra which is closed under products and retracts but not weakly reflective.*

Proof. Consider the full embedding H of the category of abelian groups into the homotopy category of spectra given by assigning to each abelian group A an Eilenberg–Mac Lane spectrum HA representing ordinary cohomology with coefficients in A. Since H preserves products and its image is closed under retracts, it sends the class $\mathcal{C}$ considered in the proof of Proposition 2.6 to a class $H\mathcal{C}$ of spectra closed under products and retracts. This class $H\mathcal{C}$ is not weakly reflective, since the above argument shows that $H\mathbb{Z}$ does not admit a weak reflection onto $H\mathcal{C}$ if there are no measurable cardinals. $\square$

3. Coreflective localizing subcategories

In Section 2 we proved that, if $\mathcal{K}$ is a locally presentable category with a stable model category structure, then Vopěnka's principle implies that all colocalizing subcategories

(in fact, all full subcategories closed under products and fibres) of $\text{Ho}(\mathcal{K})$ are reflective. Our purpose in this section is to study if Vopěnka's principle also implies that all localizing or semilocalizing subcategories of $\text{Ho}(\mathcal{K})$ are coreflective. By Corollary 2.5, this is equivalent to asking if they are closed.

We provide an affirmative answer by assuming that $\mathcal{K}$ be cofibrantly generated [38, Definition 2.1.17], in addition to being locally presentable. Recall that a cofibrantly generated model category whose underlying category is locally presentable is called *combinatorial*. It was shown in [24] and [25] that a model category is combinatorial if and only if it is Quillen equivalent to a localization of some category of diagrams of simplicial sets with respect to a set of morphisms. Hence, examples abound. When $\mathcal{K}$ is combinatorial, we not only prove that semilocalizing subcategories of $\text{Ho}(\mathcal{K})$ are coreflective, but we moreover show that they are singly generated.

Although, for the validity of our arguments, we need that $\mathcal{K}$ be *simplicial* [31, II.3], it is not necessary to impose this as a restriction, due to the following fact.

Proposition 3.1. *Every (stable) combinatorial model category is Quillen equivalent to a (stable) simplicial combinatorial model category.*

Proof. For a small category $\mathcal{C}$, denote by $U_+\mathcal{C}$, as in [26], the category of functors from $\mathcal{C}^{\text{op}}$ to the category $\mathbf{sSets}_*$ of pointed simplicial sets. According to [24, Corollary 6.4] and [26, Proposition 5.2], for every combinatorial model category $\mathcal{K}$ there is a small category $\mathcal{C}$ such that $\mathcal{K}$ is Quillen equivalent to the left Bousfield localization of $U_+\mathcal{C}$ with respect to a certain set of morphisms. The category $U_+\mathcal{C}$ is combinatorial, simplicial and pointed, and so is any of its localizations. Since Quillen equivalences preserve the suspension and loop functors, every pointed model category which is Quillen equivalent to a stable one is itself stable. $\square$

One crucial property of combinatorial model categories that we will use in this section is the following. For every combinatorial model category $\mathcal{K}$ there is a regular cardinal λ such that, if $X \colon I \to \mathcal{K}$ and $Y \colon I \to \mathcal{K}$ are diagrams where I is a small λ-filtered category, and a morphism of diagrams $f \colon X \to Y$ is given such that $f_i \colon X_i \to Y_i$ is a weak equivalence for each $i \in I$, then the induced map $\text{colim}_I X \to \text{colim}_I Y$ is also a weak equivalence. This fact is stated in [25, Proposition 2.3], although the proof relies on unpublished work of J. H. Smith. Details can be found in [44, Corollary A.2.6.6] or [52, Proposition 3.1].

Another feature of combinatorial model categories is that, if $\mathcal{K}$ is combinatorial and I is any small category, then the *projective model structure* (in which weak equivalences and fibrations are objectwise) and the *injective model structure* (in which weak equivalences and cofibrations are objectwise) exist on the diagram category $\mathcal{K}^I$; see [44, Proposition A.2.8.2]. In fact, as shown in [35, Theorem 11.6.1], for the existence of the projective model structure it is enough that $\mathcal{K}$ be cofibrantly generated.

If $\mathcal{K}^I$ is equipped with the projective model structure, then the constant functor $\mathcal{K} \to \mathcal{K}^I$ is right Quillen and therefore its left adjoint $\text{colim}_I \colon \mathcal{K}^I \to \mathcal{K}$ is left Quillen, so it preserves cofibrations, trivial cofibrations, and weak equivalences between cofibrant diagrams [31, II.8.9]. Hence, its total left derived functor hocolim_I exists.

Since we will need to use explicit formulas to compute homotopy colimits, we recall, before going further, a number of basic facts about homotopy colimits in model categories. Our main sources are [13], [29], [31], [35], [36], [38], [55]. For simplicity, we restrict our discussion to pointed simplicial model categories, which is sufficient for our purposes. The unpointed case would be treated analogously.

3.1. **A review of homotopy colimits.** Let $\mathcal{K}$ be a pointed simplicial model category. Let $*$ be the initial and terminal object, and let $\otimes$ denote the tensoring $\mathcal{K} \times \mathbf{sSets}_* \to \mathcal{K}$ over pointed simplicial sets given as part of the simplicial model category structure. For each simplicial set W, we denote by W_+ its union with a disjoint basepoint.

Let $\mathbf{\Delta}$ denote the category whose objects are finite ordered sets $[n] = (0, 1, \ldots, n)$ for $n \geq 0$, and whose morphisms are nondecreasing functions. Let $\Delta[n]$ be the simplicial set whose set of k-simplices is the set of morphisms $[k] \to [n]$ in $\mathbf{\Delta}$, and denote by $\Delta_+ \colon \mathbf{\Delta} \to \mathbf{sSets}_*$ the functor that sends $[n]$ to $\Delta[n]_+$. If X is a cofibrant object in $\mathcal{K}$, then $X \otimes \partial\Delta[1]_+ \to X \otimes \Delta[1]_+$ is a cofibration yielding a cylinder for X, and hence $\Sigma X \simeq X \otimes S^1$; cf. [38, 6.1.1].

The *realization* $|B|$ of a simplicial object $B \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{K}$ is the coequalizer of the two maps

$$(3.1) \qquad \coprod_{[m] \to [n]} B_n \otimes \Delta[m]_+ \;\Longrightarrow\; \coprod_{[n]} B_n \otimes \Delta[n]_+$$

induced by $B_n \to B_m$ and $\Delta[m]_+ \to \Delta[n]_+$, respectively, for each morphism $[m] \to [n]$ in $\mathbf{\Delta}$; see [31, VII.3.1]. Using coend notation [46, IX.6], this can be written as

$$|B| = \int^n B_n \otimes \Delta[n]_+ = B \otimes_{\mathbf{\Delta}^{\mathrm{op}}} \Delta_+.$$

Suppose given functors $X \colon I \to \mathcal{K}$ and $W \colon I^{\mathrm{op}} \to \mathbf{sSets}_*$, where W will be called a *weight*. The (two-sided) *bar construction* $B(W, I, X) \in \mathcal{K}^{\mathbf{\Delta}^{\mathrm{op}}}$ is the simplicial object with

$$(3.2) \qquad B(W, I, X)_n = \coprod_{i_n \to \cdots \to i_0} X_{i_n} \otimes W_{i_0},$$

whose kth face map omits i_k using the identity on $X_{i_n} \otimes W_{i_0}$ if $0 < k < n$, and using $W_{i_0} \to W_{i_1}$ if $k = 0$ and $X_{i_n} \to X_{i_{n-1}}$ if $k = n$. Degeneracies are given by insertions of the identity. If we choose as weight the constant diagram S at the 0th sphere S^0, then we denote $B_I X = B(S, I, X)$ and call it a *simplicial replacement* of X.

The (pointed) *homotopy colimit* of a functor $X \colon I \to \mathcal{K}$ is defined as

$$(3.3) \qquad \operatorname{hocolim}_I X = |B_I X|.$$

It follows that homotopy colimits commute; that is, given $X \colon I \times J \to \mathcal{K}$,

$$\operatorname{hocolim}_I \operatorname{hocolim}_J X \cong \operatorname{hocolim}_{I \times J} X \cong \operatorname{hocolim}_J \operatorname{hocolim}_I X.$$

From (3.3) and (3.1) one obtains the *Bousfield–Kan formula* [13, XII.2.1], [35, 18.1.2], as follows. Let $N(i \downarrow I)^{\mathrm{op}}$ be the nerve of the category $(i \downarrow I)^{\mathrm{op}}$ for each $i \in I$. Thus, $N(i \downarrow I)^{\mathrm{op}}_+$ is the realization of the simplicial space $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{sSets}_*$ that consists in degree n

of a coproduct of copies of S^0 indexed by the set of sequences $i \to i_n \to \cdots \to i_0$ of morphisms in I. Since coequalizers commute, we have

$$\operatorname{hocolim}_I X \cong \operatorname{coeq}\left[\coprod_{i \to j} X_i \otimes N(j \downarrow I)^{\mathrm{op}}_+ \rightrightarrows \coprod_i X_i \otimes N(i \downarrow I)^{\mathrm{op}}_+\right] = X \otimes_I N(- \downarrow I)^{\mathrm{op}}_+.$$

(We note that $(I \downarrow i)$ was used in [13] instead of $(i \downarrow I)^{\mathrm{op}}$.) In other words, $\operatorname{hocolim}_I X$ is a weighted colimit of X with weight $N(- \downarrow I)^{\mathrm{op}}_+ \colon I^{\mathrm{op}} \to \mathbf{sSets}_*$. This point of view goes back to [9] and is discussed further in [29, §3] and [53, §2].

For simplicial objects $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{K}$, the *nth skeleton* sk_n is the composite of the truncation functor $\mathcal{K}^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{K}^{\mathbf{\Delta}_n^{\mathrm{op}}}$ with its left adjoint, where $\mathbf{\Delta}_n$ is the full subcategory of $\mathbf{\Delta}$ with objects $\{[0], \ldots, [n]\}$; see [31, VII.1.3]. Thus $(\operatorname{sk}_n B)_m \cong B_m$ if $m \leq n$, and hence

$$B \cong \operatorname{colim}_n \operatorname{sk}_n B$$

for every simplicial object B.

The *nth latching object* of $B \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{K}$ is defined as $L_n B = (\operatorname{sk}_{n-1} B)_n$ for each n. As explained in [31, VII.2], the category $\mathcal{K}^{\mathbf{\Delta}^{\mathrm{op}}}$ of simplicial objects in $\mathcal{K}$ admits a model structure (called *Reedy model structure*) where weak equivalences are objectwise and cofibrations are maps $f \colon X \to Y$ such that $L_n Y \coprod_{L_n X} X_n \to Y_n$ is a cofibration for all n. Thus an object B is Reedy cofibrant if and only if the natural maps $L_n B \to B_n$ are cofibrations in $\mathcal{K}$ for all n.

If B is Reedy cofibrant, then each skeleton $\operatorname{sk}_n B$ is also Reedy cofibrant, and the inclusions $\operatorname{sk}_{n-1} B \hookrightarrow \operatorname{sk}_n B$ are Reedy cofibrations; see e.g. [8, Proposition 6.5].

As shown in [31, VII.3.6], the realization functor is left Quillen if $\mathcal{K}^{\mathbf{\Delta}^{\mathrm{op}}}$ is equipped with the Reedy model structure. The following consequence is crucial.

Lemma 3.2. *Let $\mathcal{K}$ be a pointed simplicial model category and let I be small.*

    (a) *If a diagram $X \colon I \to \mathcal{K}$ is objectwise cofibrant, then $B_I X$ is Reedy cofibrant.*

    (b) *If $f \colon X \to Y$ is an objectwise weak equivalence in $\mathcal{K}^I$ and the diagrams X and Y are objectwise cofibrant, then the induced map $\operatorname{hocolim}_I X \to \operatorname{hocolim}_I Y$ is a weak equivalence of cofibrant objects.*

Proof. Let $B = B_I X$. Thus $B_n = \coprod_{i_n \to \cdots \to i_0} X_{i_n}$ for all $n \geq 0$, and we may write

$$(3.4) \qquad\qquad\qquad\qquad B_n = L_n B \coprod Z_n B,$$

where $L_n B$ includes the "degenerate" summands of B_n, i.e., those labelled by sequences $i_n \to \cdots \to i_0$ where some arrow is an identity, and $Z_n B$ collects the rest. Then the inclusion $L_n B \to B_n$ is a coproduct of the identity $L_n B \to L_n B$ and $* \to Z_n B$, which is a cofibration since X takes cofibrant values. This proves part (a). Then part (b) follows from the fact that realization is left Quillen, since $B_I f \colon B_I X \to B_I Y$ is a weak equivalence between Reedy cofibrant objects. $\qquad\square$

Thus, if defined as in (3.3), the homotopy colimit is only homotopy invariant on *objectwise cofibrant* diagrams. For this reason, it is often convenient to "correct" $\operatorname{hocolim}_I$ by composing it with a cofibrant replacement functor in $\mathcal{K}$, as in [55, Definition 8.2].

The fundamental fact that, if made homotopy invariant, $\operatorname{hocolim}_I$ yields a total left derived functor of colim_I is explained as follows. For each diagram $X \colon I \to \mathcal{K}$ there is a natural map

$$(3.5) \qquad \operatorname{hocolim}_I X \longrightarrow \operatorname{colim}_I X,$$

since $\operatorname{colim}_I X$ is the coequalizer of the two face maps $\coprod_{i \to j} X_i \rightrightarrows \coprod_i X_i$; that is, (3.5) takes the form

$$(3.6) \qquad X \otimes_I N(- \downarrow I)_+^{\mathrm{op}} \longrightarrow X \otimes_I S.$$

This map is a weak equivalence only in some cases; cf. [13, XII.2]. For instance, it is so if I has a terminal object. More importantly, (3.6) is a weak equivalence if X is *cofibrant* in the projective model structure of $\mathcal{K}^I$. To show this, use the fact, proved in [29, Theorem 3.2], that $(-) \otimes_I (-)$ is a left Quillen functor in two variables if the projective model structure exists and is chosen on $\mathcal{K}^I$ and the injective model structure is considered in $\mathbf{sSets}_*^{I^{\mathrm{op}}}$. Accordingly, if X is a projectively cofibrant diagram, then $X \otimes_I (-)$ preserves weak equivalences between (objectwise) cofibrant objects, so (3.6) is indeed a weak equivalence.

It is also true, as shown in [29, Theorem 3.3], that $(-) \otimes_I (-)$ is left Quillen in two variables if the projective model structure is considered in $\mathbf{sSets}_*^{I^{\mathrm{op}}}$ and the injective model structure exists and is chosen on $\mathcal{K}^I$. Thus, since $N(- \downarrow I)_+^{\mathrm{op}} \to S$ is a projectively cofibrant approximation in $\mathbf{sSets}_*^{I^{\mathrm{op}}}$, the Bousfield–Kan formula displays in fact $\operatorname{hocolim}_I$ as a left derived functor of colim_I, provided that we restrict it to objectwise cofibrant diagrams (i.e., cofibrant in the injective model structure).

For some purposes it is useful to consider the following functorial projectively cofibrant replacement of a given diagram $X \colon I \to \mathcal{K}$. Assume that X takes cofibrant values (or compose it with a cofibrant replacement functor in $\mathcal{K}$ otherwise). Consider the functor $B_{(I \downarrow -)} X \colon I \times \boldsymbol{\Delta}^{\mathrm{op}} \to \mathcal{K}$ given by

$$(B_{(I \downarrow -)} X)(j, [n]) = (B_{(I \downarrow j)}(X \circ U_j))_n = \coprod_{i_n \to \cdots \to i_0 \to j} X_{i_n},$$

where $U_j \colon (I \downarrow j) \to I$ sends each arrow $i \to j$ to i, and let $\widetilde{X} = |B_{(I \downarrow -)} X|$. Thus,

$$(3.7) \qquad \widetilde{X}_j = |B_{(I \downarrow j)}(X \circ U_j)| = \operatorname{hocolim}_{(I \downarrow j)}(X \circ U_j)$$

for all $j \in I$. Since $(I \downarrow j)$ has a terminal object for each j, the natural morphism $\widetilde{X} \to X$ is an objectwise weak equivalence. Using the fact that realization is a left adjoint and hence commutes with colimits, one obtains a canonical isomorphism

$$(3.8) \quad \operatorname{colim}_I \widetilde{X} = \operatorname{colim}_j \widetilde{X}_j = \operatorname{colim}_j \operatorname{hocolim}_{(I \downarrow j)}(X \circ U_j)$$
$$= \operatorname{colim}_j |B_{(I \downarrow j)}(X \circ U_j)| \cong |\operatorname{colim}_j B_{(I \downarrow j)}(X \circ U_j)| \cong |B_I X| = \operatorname{hocolim}_I X.$$

In order to prove that the diagram $\widetilde{X}$ is indeed projectively cofibrant, view $B_{(I \downarrow -)} X$ as an object in $(\mathcal{K}^I)^{\boldsymbol{\Delta}^{\mathrm{op}}}$ and check that it is Reedy cofibrant if the projective model structure is chosen in $\mathcal{K}^I$, similarly as in part (a) of Lemma 3.2.

Although projectively cofibrant diagrams are not easy to characterize in general, we note the following well-known special case for subsequent reference.

Lemma 3.3. *Let λ be an infinite ordinal and let $\mathcal{K}$ be a model category. Suppose that, for an objectwise cofibrant diagram $X\colon \lambda \to \mathcal{K}$, each map $X_i \to X_{i+1}$ with $i < \lambda$ is a cofibration and the induced map $\mathrm{colim}_{i<\alpha} X_i \to X_\alpha$ is also a cofibration for every limit ordinal $\alpha < \lambda$. Then the diagram X is projectively cofibrant in $\mathcal{K}^\lambda$.*

Proof. Suppose given an objectwise trivial fibration $f\colon A \to B$ in $\mathcal{K}^\lambda$ and a morphism $v\colon X \to B$. Then a lifting $u\colon X \to A$ is constructed by transfinite induction, as follows. Since $f_0\colon A_0 \to B_0$ is a trivial fibration and X_0 is cofibrant, there is a map $u_0\colon X_0 \to A_0$ such that $f_0 \circ u_0 = v_0$. For the inductive step, let $u_i\colon X_i \to A_i$ be a map with $f_i \circ u_i = v_i$ for some ordinal $i < \lambda$. Then, since $X_i \to X_{i+1}$ is a cofibration and $f_{i+1}\colon A_{i+1} \to B_{i+1}$ is a trivial fibration, there is a map $u_{i+1}\colon X_{i+1} \to A_{i+1}$ compatible with u_i and such that $f_{i+1} \circ u_{i+1} = v_{i+1}$. For a limit ordinal $\alpha < \lambda$, a suitable map $u_\alpha\colon X_\alpha \to A_\alpha$ is obtained similarly, using the assumption that $\mathrm{colim}_{i<\alpha} X_i \to X_\alpha$ is a cofibration. Then the collection $\{u_i \mid i < \lambda\}$ defines a morphism $u\colon X \to A$ with $f \circ u = v$, as needed. $\square$

For an objectwise cofibrant diagram $X\colon I \to \mathcal{K}$, the homotopy colimit $\mathrm{hocolim}_I X$ can be filtered as follows. Let $B = B_I X$, and denote $F_n = |\,\mathrm{sk}_n B\,|$. Since B is Reedy cofibrant, the *Bousfield–Kan map*

$$\mathrm{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} B = B \otimes_{\mathbf{\Delta}^{\mathrm{op}}} N(- \downarrow \mathbf{\Delta}^{\mathrm{op}})_+^{\mathrm{op}} \longrightarrow B \otimes_{\mathbf{\Delta}^{\mathrm{op}}} \Delta_+ = |B|$$

is a weak equivalence; cf. [13, XII.3.4], [35, 18.7.1]. Since homotopy colimits commute,

$$(3.9) \quad \mathrm{hocolim}_I X = |B| \simeq \mathrm{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} B \simeq \mathrm{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{hocolim}_n \mathrm{sk}_n B$$
$$\cong \mathrm{hocolim}_n \mathrm{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{sk}_n B \simeq \mathrm{hocolim}_n F_n.$$

This equivalence, which was our main goal in this subsection, is relevant in the context of triangulated categories, since it allows us to replace a homotopy colimit indexed by an arbitrary small category by another one indexed by a countably infinite ordinal, which fits into a well-known triangle involving countable coproducts and the shift map, as in [48, Definition 1.6.4].

3.2. **Singly generated semilocalizing subcategories are coreflective.** The filtration displayed in (3.9) of a homotopy colimit was used in [10], [11] to show that the class of acyclics of any homology theory on spectra is closed under homotopy colimits, as a key ingredient of the proof of the existence of homological localizations. The validity of the same argument for localizing subcategories of stable homotopy categories was suggested in [39, Remark 2.2.5]. A similar argument in derived categories of Grothendieck categories was used for filtered homotopy colimits in [3, Theorem 3.1]. We generalize it as follows.

Proposition 3.4. *Let $\mathcal{K}$ be a stable simplicial model category and let $\gamma\colon \mathcal{K} \to \mathrm{Ho}(\mathcal{K})$ denote the canonical functor. Let $\mathcal{C}$ be a semilocalizing subcategory of $\mathrm{Ho}(\mathcal{K})$. If a diagram $X\colon I \to \mathcal{K}$ is objectwise cofibrant and $\gamma X_i \in \mathcal{C}$ for all $i \in I$, then $\gamma\,\mathrm{hocolim}_I X \in \mathcal{C}$.*

Proof. Let $B = B_I X$ be the simplicial replacement of X, as in (3.3), and let $F_n = |\operatorname{sk}_n B|$. As explained in [31, VII.3.8] or [32, 5.2], since realization commutes with colimits, there is a natural pushout diagram

(3.10)
$$(B_n \otimes \partial\Delta[n]_+) \coprod_{(L_n B \otimes \partial\Delta[n]_+)} (L_n B \otimes \Delta[n]_+) \longrightarrow B_n \otimes \Delta[n]_+$$
$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$
$$F_{n-1} \longrightarrow F_n.$$

According to part (a) of Lemma 3.2, since the diagram X is objectwise cofibrant, B is Reedy cofibrant. Hence, by Quillen's SM7 axiom for a simplicial model category [31, II.3.12], the upper arrow in (3.10) is a cofibration. Therefore, the cofibre F_n/F_{n-1} is isomorphic to the cofibre of the upper arrow in (3.10), which is isomorphic to $Z_n B \otimes S^n$ if we write, as in (3.4), $B_n = L_n B \coprod Z_n B$, where $Z_n B$ contains the nondegenerate summands of B_n. Hence, the sequence

$$\gamma F_{n-1} \longrightarrow \gamma F_n \longrightarrow \Sigma^n \gamma Z_n B$$

yields a triangle in $\operatorname{Ho}(\mathcal{K})$. Since $Z_n B$ is a coproduct of objects X_i with $i \in I$, it follows inductively that $\gamma F_n \in \mathcal{C}$ for all n.

Since $F_{n-1} \to F_n$ is a cofibration between cofibrant objects for every n, Lemma 3.3 implies that

$$\gamma \operatorname{hocolim}_n F_n \cong \gamma \operatorname{colim}_n F_n,$$

and $\gamma \operatorname{colim}_n F_n$ is a cofibre of a morphism $\coprod_n \gamma F_n \to \coprod_n \gamma F_n$ in $\operatorname{Ho}(\mathcal{K})$, namely the difference between the identity and the shift map, from which it follows that $\gamma \operatorname{hocolim}_n F_n$ is in $\mathcal{C}$, because $\mathcal{C}$ is semilocalizing. Since, as observed in (3.9), $\operatorname{hocolim}_n F_n \simeq \operatorname{hocolim}_I X$, the claim is proved. $\qquad\square$

Corollary 3.5. *If $\mathcal{K}$ is a pointed simplicial combinatorial model category and we denote by $\gamma \colon \mathcal{K} \to \operatorname{Ho}(\mathcal{K})$ the canonical functor, then there is a regular cardinal λ such that:*
   (a) *For every λ-filtered objectwise cofibrant diagram $X \colon I \to \mathcal{K}$, the natural map $\operatorname{hocolim}_I X \to \operatorname{colim}_I X$ is a weak equivalence.*
   (b) *If $\mathcal{K}$ is stable and $\mathcal{C}$ is a semilocalizing subcategory of $\operatorname{Ho}(\mathcal{K})$, then, for every λ-filtered diagram $X \colon I \to \mathcal{K}$ with $\gamma X_i \in \mathcal{C}$ for all i, we have $\gamma \operatorname{colim}_I X \in \mathcal{C}$.*

Proof. By [25, Proposition 7.3], for a combinatorial category $\mathcal{K}$ there is a regular cardinal λ such that λ-filtered colimits of weak equivalences are weak equivalences. Let $X \colon I \to \mathcal{K}$ be an objectwise cofibrant diagram where I is λ-filtered. Let $\widetilde{X} \to X$ be the objectwise weak equivalence defined in (3.7). Then, by our choice of λ, the induced morphism

$$\operatorname{colim}_I \widetilde{X} \longrightarrow \operatorname{colim}_I X$$

is a weak equivalence. Since $\operatorname{colim}_I \widetilde{X} \cong \operatorname{hocolim}_I X$ by (3.8), part (a) is proved.

Now let $X \colon I \to \mathcal{K}$ be any diagram where I is λ-filtered, and let Q be a cofibrant replacement functor in $\mathcal{K}$. From our choice of λ we infer that $\operatorname{colim}_I X$ is weakly equivalent to $\operatorname{colim}_I QX$ and hence to $\operatorname{hocolim}_I QX$, by part (a). Therefore, if $\mathcal{K}$ is stable, then

for every semilocalizing subcategory $\mathcal{C}$ of $\mathrm{Ho}(\mathcal{K})$ it follows from Proposition 3.4 that $\gamma \operatorname{colim}_I X \in \mathcal{C}$ if $\gamma X_i \in \mathcal{C}$ for all $i \in I$. □

We emphasize that the cardinal λ in the statement of Corollary 3.5 depends only on $\mathcal{K}$, not on the subcategory $\mathcal{C}$.

The following is another useful property of triangulated categories with models. A special case is discussed in [39, Remark 2.2.8]. (The assumption that $\mathcal{K}$ be simplicial is not really necessary here nor in Corollary 3.5, since homotopy colimits can be used, with the same basic properties, in all model categories; see [35, Chapter 19].)

Lemma 3.6. *Let $\mathcal{K}$ be a stable simplicial model category and denote by $\gamma\colon \mathcal{K} \to \mathrm{Ho}(\mathcal{K})$ the canonical functor. Let I be any small category such that the projective model structure exists on $\mathcal{K}^I$. Suppose given morphisms of objectwise cofibrant diagrams $X \to Y \to Z$ in $\mathcal{K}^I$ such that $\gamma X \to \gamma Y \to \gamma Z$ yields a triangle in $\mathrm{Ho}(\mathcal{K}^I)$. Then*

$$\gamma \operatorname{hocolim}_I X \longrightarrow \gamma \operatorname{hocolim}_I Y \longrightarrow \gamma \operatorname{hocolim}_I Z$$

yields a triangle in $\mathrm{Ho}(\mathcal{K})$.

Proof. Since colim_I is left Quillen if the projective model structure is considered on $\mathcal{K}^I$, its total left derived functor preserves triangles, as shown in [38, Proposition 6.4.1].

Alternatively, this result follows from the fact that homotopy colimits commute, since, by assumption, Z is weakly equivalent to the homotopy cofibre in $\mathcal{K}^I$ of the given morphism $X \to Y$, i.e., the homotopy pushout of $* \leftarrow X \to Y$, and $\operatorname{hocolim}_I$ is homotopy invariant on objectwise cofibrant diagrams. □

Special cases or variants of the next result have been described in [4, Theorem 3.4] for derived categories of Grothendieck categories; in [6, Proposition III.2.6] for compactly generated torsion pairs; in [39, Proposition 2.3.1] for algebraic stable homotopy categories; in [41, Theorem 3.1] for derivators; and in [45, Proposition 16.1] for stable ∞-categories. The core of the argument was first used by Bousfield in [10].

We note that, if the dual statement could be proved without large-cardinal assumptions, namely that singly generated colocalizing subcategories are reflective in ZFC, this would imply the existence of cohomological localizations of spectra in ZFC, a long-standing unsolved problem.

Theorem 3.7. *If $\mathcal{K}$ is a stable combinatorial model category, then every singly generated semilocalizing subcategory of $\mathrm{Ho}(\mathcal{K})$ is coreflective.*

Proof. By Proposition 3.1, we may assume that $\mathcal{K}$ is simplicial. Let $\gamma\colon \mathcal{K} \to \mathrm{Ho}(\mathcal{K})$ denote the canonical functor. Let $\mathcal{C}$ be a semilocalizing subcategory of $\mathrm{Ho}(\mathcal{K})$ and suppose that $\mathcal{C} = \mathrm{sloc}(A)$ for some object A. Pick, for each $n \geq 0$, a cofibrant object B_n in $\mathcal{K}$ such that $\gamma B_n \cong \Sigma^n A$, and choose a regular cardinal λ and a fibrant replacement functor R in $\mathcal{K}$ such that:

    (i) B_n is λ-presentable for every $n \geq 0$;
    (ii) all λ-filtered colimits of weak equivalences are weak equivalences;
    (iii) the functor R preserves λ-filtered colimits.

This is possible according to [25, Proposition 2.3] and [25, Proposition 7.3], due to the assumption that $\mathcal{K}$ is combinatorial.

In order to construct a coreflection onto $\mathcal{C}$, we proceed similarly as in [10, Proposition 1.5] or as in the proof of [39, Proposition 2.3.17]. For any object X of $\mathcal{K}$ —which we assume fibrant and cofibrant—, take $Y_0 = X$ and let W_0 be a coproduct of copies of B_n for $n \geq 0$ indexed by all morphisms in $\mathcal{K}(B_n, Y_0)$. Let $u_0 \colon W_0 \to Y_0$ be given by $f \colon B_n \to Y_0$ on the summand corresponding to f.

Next, let Y_1 be the homotopy cofibre of u_0. More precisely, factor u_0 into a cofibration $\widetilde{u}_0 \colon W_0 \to \widetilde{Y}_0$ followed by a trivial fibration $\phi_0 \colon \widetilde{Y}_0 \to Y_0$; let Y_1' be the pushout of $\widetilde{u}_0$ and $W_0 \to *$, and let $Y_1' \to Y_1''$ be a trivial cofibration with Y_1'' fibrant. Since Y_0 is cofibrant, there is a left inverse $Y_0 \to \widetilde{Y}_0$ to ϕ_0 and hence a morphism $Y_0 \to Y_1''$, which we factor again into a cofibration $v_0 \colon Y_0 \to Y_1$ followed by a trivial fibration $Y_1 \to Y_1''$. Thus it follows from our choices that Y_1 is both fibrant and cofibrant, and

$$W_0 \xrightarrow{u_0} Y_0 \xrightarrow{v_0} Y_1$$

yields a triangle in $\mathrm{Ho}(\mathcal{K})$, since

$$\gamma W_0 \xrightarrow{\gamma u_0} \gamma Y_0 \xrightarrow{\gamma v_0} \gamma Y_1 \quad \text{is isomorphic to} \quad \gamma W_0 \xrightarrow{\gamma \widetilde{u}_0} \gamma \widetilde{Y}_0 \longrightarrow \gamma Y_1',$$

which is a canonical triangle.

Now repeat the process with Y_1 in the place of Y_0. In this way we construct inductively, for every ordinal i, a sequence

$$(3.11) \qquad\qquad W_i \xrightarrow{u_i} Y_i \xrightarrow{v_i} Y_{i+1}$$

yielding a triangle in $\mathrm{Ho}(\mathcal{K})$, where Y_{i+1} is fibrant and cofibrant, v_i is a cofibration, and W_i is a coproduct of copies of B_n for $n \geq 0$, together with a morphism $w_{i+1} \colon X \to Y_{i+1}$ such that $w_{i+1} = v_i \circ w_i$ (with $w_1 = v_0$). If α is a limit ordinal, take $Z_\alpha = \mathrm{colim}_{i<\alpha} Y_i$, and let $Z_\alpha \to Y_\alpha$ be a trivial cofibration with Y_α fibrant. Since every (possibly transfinite) composition of cofibrations is a cofibration, the morphism $X \to Z_\alpha$ given by w_i for $i < \alpha$ is a cofibration, and hence the composite $w_\alpha \colon X \to Y_\alpha$ is also a cofibration.

Let $Y \colon \lambda \to \mathcal{K}$ be the diagram given by the objects Y_i and the maps v_i for $i < \lambda$. Then Y is cofibrant in $\mathcal{K}^\lambda$, by Lemma 3.3, and the constant diagram $X \colon \lambda \to \mathcal{K}$ at the object X is also cofibrant in $\mathcal{K}^\lambda$. Let F be the homotopy pullback of the map $X \to Y$ given by the morphisms w_i and the trivial map $* \to Y$ in $\mathcal{K}^\lambda$. Thus,

$$\gamma F \longrightarrow \gamma X \longrightarrow \gamma Y$$

yields a triangle in $\mathrm{Ho}(\mathcal{K}^\lambda)$, since $\mathcal{K}^\lambda$ is stable. Let Q be a cofibrant replacement functor in $\mathcal{K}$, and let QF be the composite of Q and F. Thus QF is objectwise cofibrant and, by Lemma 3.6,

$$(3.12) \qquad\qquad \gamma \operatorname{hocolim}_{i<\alpha} QF_i \longrightarrow \gamma X \longrightarrow \gamma \operatorname{hocolim}_{i<\alpha} Y_i$$

yields a triangle for each limit ordinal $\alpha \leq \lambda$.

By the octahedral axiom in $\mathrm{Ho}(\mathcal{K})$ and (3.11), there is a triangle

$$\gamma QF_i \longrightarrow \gamma QF_{i+1} \longrightarrow \gamma W_i$$

for each ordinal i. Therefore, it follows from transfinite induction that $\gamma QF_i \in \mathrm{sloc}(A)$ for all ordinals i, since each γW_i is constructed from A by means of suspensions and coproducts. If α is a limit ordinal, then $\gamma Y_\alpha \cong \gamma\,\mathrm{colim}_{i<\alpha} Y_i \cong \gamma\,\mathrm{hocolim}_{i<\alpha} Y_i$ because Y is cofibrant, and it then follows from (3.12) that $\gamma QF_\alpha \cong \gamma\,\mathrm{hocolim}_{i<\alpha} QF_i$, which is in $\mathrm{sloc}(A)$ by Proposition 3.4.

Let $CX = \mathrm{colim}_{i<\lambda} QF_i$ and let $LX = \mathrm{colim}_{i<\lambda} Y_i$, and note that the natural morphisms $\mathrm{hocolim}_{i<\lambda} QF_i \to CX$ and $\mathrm{hocolim}_{i<\lambda} Y_i \to LX$ are weak equivalences, by part (a) of Corollary 3.5 and by our choice of λ. By Lemma 3.6, the sequence

$$\gamma CX \longrightarrow \gamma X \longrightarrow \gamma LX$$

yields a triangle in $\mathrm{Ho}(\mathcal{K})$. By Proposition 3.4, $\gamma CX \in \mathrm{sloc}(A)$.

Again by our choice of λ, we have $RLX \cong \mathrm{colim}_{i<\lambda} RY_i$. Now every $f \colon \Sigma^n A \to \gamma LX$ in $\mathrm{Ho}(\mathcal{K})$ can be lifted to a morphism $\widetilde{f} \colon B_n \to RLX$ in $\mathcal{K}$, as RLX is fibrant. Once more by our choice of λ, this morphism $\widetilde{f}$ factors through RY_k for some $k < \lambda$, since B_n is λ-presentable. Since (3.11) yields a triangle for all i, the composite

$$\Sigma^n A \longrightarrow \gamma RY_k \longrightarrow \gamma RY_{k+1}$$

is zero. This implies that $f \colon \Sigma^n A \to \gamma LX$ is zero. Therefore, $\gamma LX \in A^\mathsf{L}$, and, by (1.6), $A^\mathsf{L} = \mathrm{sloc}(A)^\mathsf{L}$. This proves that C is a coreflection onto $\mathcal{C}$, using Proposition 1.6. $\square$

We note that the reflection L obtained in the previous proof is a nullification P_A in the sense of [12] and [20], and the subcategory $\mathcal{C}$ is thus the closure under extensions of the class of A-cellular objects.

We have given the argument in full detail to stress the fact that it works for *semilocalizing* subcategories. It then also works for localizing subcategories, since, if $\mathcal{C}$ is generated by an object A as a localizing subcategory, then it is generated by $\coprod_{n\leq 0} \Sigma^n A$ as a semilocalizing subcategory. However, in the case of a localizing subcategory, there is an alternative, much shorter proof of Theorem 3.7 which does not require the existence of models. Instead, it is based on Brown representability. A similar argument can be found in [43, Theorem 7.2.1].

Proposition 3.8. *Let $\mathcal{T}$ be a well-generated triangulated category with coproducts. Then every singly generated localizing subcategory of $\mathcal{T}$ is coreflective.*

Proof. By [48, Proposition 8.4.2], the category $\mathcal{T}$ satisfies Brown representability, and $\mathcal{T} = \cup_\alpha \mathcal{T}^\alpha$, i.e., every object of $\mathcal{T}$ is α-compact for some infinite cardinal α.

Let $\mathcal{C}$ be a localizing subcategory of $\mathcal{T}$ generated by some object A. Then $A \in \mathcal{T}^\alpha$ for some infinite cardinal α. Hence, it follows from [48, Corollary 4.4.3] that the Verdier quotient category $\mathcal{T}/\mathcal{C}$ has small hom-sets.

The existence of a coreflection onto $\mathcal{C}$ amounts to the existence of a right adjoint to the inclusion $\mathcal{C} \hookrightarrow \mathcal{T}$, and this is equivalent to the existence of a right adjoint to the functor

$F\colon \mathcal{T} \to \mathcal{T}/\mathcal{C}$ (see [48, Proposition 9.1.18]). Since $\mathcal{T}/\mathcal{C}$ has small hom-sets, a right adjoint $G\colon \mathcal{T}/\mathcal{C} \to \mathcal{T}$ can be defined as follows. If X is any object of $\mathcal{T}/\mathcal{C}$, then GX is obtained by Brown representability, namely $(\mathcal{T}/\mathcal{C})(F(-), X) \cong \mathcal{T}(-, GX)$. $\qquad\square$

Recall from [52, Proposition 6.10] that, if $\mathcal{K}$ is a stable combinatorial model category, then $\mathrm{Ho}(\mathcal{K})$ is indeed well generated.

3.3. Semilocalizing subcategories are singly generated.

We remark that the way in which Vopěnka's principle is used in Theorem 3.9 below is different from the way in which it was used in Section 2. What we need here is the fact that, by [2, Theorem 6.6 and Corollary 6.18], *if Vopěnka's principle holds, then every full subcategory of a locally presentable category closed under λ-filtered colimits for some regular cardinal λ is accessible.* The following argument was used similarly in [14, Lemma 1.3] and [18, Lemma 1.3].

Theorem 3.9. *Let $\mathcal{K}$ be a stable combinatorial model category. If Vopěnka's principle holds, then every semilocalizing subcategory of $\mathrm{Ho}(\mathcal{K})$ is singly generated and coreflective.*

Proof. First replace $\mathcal{K}$ with a Quillen equivalent stable simplicial combinatorial model category, which is possible according to Proposition 3.1. Let $\mathcal{C}$ be a semilocalizing subcategory of $\mathcal{T} = \mathrm{Ho}(\mathcal{K})$. Write it as the union of an ascending chain of full subcategories

$$\mathcal{C} = \bigcup_{i \in \mathrm{Ord}} \mathcal{C}_i,$$

indexed by the ordinals, where for each i there is an object $A_i \in \mathcal{C}$ such that $\mathcal{C}_i = \mathrm{sloc}(A_i)$. Then, by Theorem 3.7, each $\mathcal{C}_i$ is coreflective.

Consider the corresponding classes $\mathcal{S}_i = \gamma^{-1}(\mathcal{C}_i)$, where $\gamma\colon \mathcal{K} \to \mathrm{Ho}(\mathcal{K})$ is the canonical functor. These form an ascending chain of full subcategories of $\mathcal{K}$. Let $\mathcal{S} = \cup_{i \in \mathrm{Ord}} \mathcal{S}_i = \gamma^{-1}(\mathcal{C})$. By Corollary 3.5, there is a regular cardinal λ such that each $\mathcal{S}_i$ is closed under λ-filtered colimits, and so is $\mathcal{S}$.

Since $\mathcal{K}$ is locally presentable, Vopěnka's principle implies that $\mathcal{S}$ is accessible [2, Theorem 6.6 and Corollary 6.18]. Hence, there is a regular cardinal μ, which we may choose bigger than λ, and a set $\mathcal{X}$ of μ-presentable objects in $\mathcal{S}$ such that every object of $\mathcal{S}$ is a μ-filtered colimit of objects from $\mathcal{X}$.

Since $\mathcal{X}$ is a set, we have $\mathcal{X} \subseteq \mathcal{S}_k$ for some ordinal k. Hence, every object of $\mathcal{S}$ is a μ-filtered colimit of objects from $\mathcal{S}_k$. But the class $\mathcal{S}_k$ is closed under μ-filtered colimits, since every μ-filtered colimit is also λ-filtered. Therefore, $\mathcal{S}_k = \mathcal{S}$, that is, the chain $\{\mathcal{S}_i \mid i \in \mathrm{Ord}\}$ eventually stabilizes. Then $\{\mathcal{C}_i \mid i \in \mathrm{Ord}\}$ also stabilizes, since $\mathcal{C}_i = \gamma(\mathcal{S}_i)$ for all i. This proves that $\mathcal{C} = \mathcal{C}_k$ for some k, which is singly generated and coreflective. $\quad\square$

Under the assumptions of Theorem 3.9, every localizing subcategory $\mathcal{C}$ is also singly generated, since we may infer from Theorem 3.9 that $\mathcal{C} = \mathrm{sloc}(A)$ for some object A, and then $\mathcal{C} = \mathrm{loc}(A)$ as well.

It also follows that, under the assumptions of Theorem 3.9, all semilocalizing subcategories (and all localizing subcategories) are closed. If we assume, in addition, that $\mathcal{T}$ is tensor triangulated, and apply Theorem 3.9 to a localizing ideal, then the corresponding

coreflection C is a colocalization in the sense of [39]; that is, if X is such that $CX = 0$, then $C(F(E, X)) = 0$ for every object E in $\mathcal{T}$.

Hence, the question asked after [39, Lemma 3.6.4] of whether all localizing ideals are closed has an affirmative answer in tensor triangulated categories with combinatorial models, assuming Vopěnka's principle.

4. Nullity classes and cohomological Bousfield classes

It follows from Theorem 1.4, Theorem 2.4 and Theorem 3.9 that, if Vopěnka's principle holds, then in every triangulated category $\mathcal{T}$ with combinatorial models there is a bijective correspondence between localizing subcategories and colocalizing subcategories. This answers affirmatively [50, Problem 7.3] under the assumptions made here.

In fact, under the same assumptions, there is also a bijective correspondence between semilocalizing subcategories and semicolocalizing subcategories. Hence, we have:

Corollary 4.1. *Under Vopěnka's principle, every semilocalizing subcategory of a triangulated category with combinatorial models is part of a t-structure, and the same happens for every semicolocalizing subcategory.*

Proof. As stated in Theorem 1.7, every reflective semicolocalizing subcategory yields a t-structure, and so does every coreflective semilocalizing subcategory. Theorem 2.4 ensures reflectivity of all semicolocalizing subcategories and Theorem 3.9 ensures coreflectivity of all semilocalizing subcategories, under the assumptions made. $\square$

Another consequence of our results is the following.

Theorem 4.2. *Let $\mathcal{T}$ be a triangulated category with combinatorial models. Assuming Vopěnka's principle, every semicolocalizing subcategory of $\mathcal{T}$ is equal to E^{L} for some object E and every colocalizing subcategory is equal to $E^{\perp}$ for some E.*

Proof. Let $\mathcal{L}$ be a semicolocalizing subcategory of $\mathcal{T}$. Theorem 2.4 ensures that $\mathcal{L}$ is reflective and hence $\mathcal{L} = (^{\mathsf{J}}\mathcal{L})^{\mathsf{L}}$, by Proposition 1.5. Now consider $^{\mathsf{J}}\mathcal{L}$, which is a semilocalizing subcategory, hence singly generated by Theorem 3.9. That is, $^{\mathsf{J}}\mathcal{L} = \mathrm{sloc}(E)$ for some E. Consequently, $\mathcal{L} = (^{\mathsf{J}}\mathcal{L})^{\mathsf{L}} = \mathrm{sloc}(E)^{\mathsf{L}} = E^{\mathsf{L}}$ by (1.6), which proves our first claim. We argue in the same way for a colocalizing subcategory. $\square$

Semicolocalizing subcategories of the form E^{L} for some object E are called *nullity classes*, since E^{L} consists of objects X that are *E-null*, in the sense that $\mathcal{T}(\Sigma^k E, X) = 0$ for $k \geq 0$ (this terminology is consistent with [17] or [20], but slightly differs from that used in [56]). Thus, the following corollary is a rewording of Theorem 4.2.

Corollary 4.3. *Assuming Vopěnka's principle, every semicolocalizing subcategory of a triangulated category with combinatorial models is a nullity class.*

It was shown in [56] that there is a proper class of distinct nullity classes E^{L} in the derived category of $\mathbb{Z}$ or in the homotopy category of spectra. However, it is unknown if there is a proper class or only a set of distinct classes of the form $E^{\perp}$.

The same problem is open for classes of the form $^{\perp}E$. A localizing subcategory of the form $^{\perp}E$ for some object E is called a *cohomological Bousfield class*; cf. [37]. It follows from Corollary 2.5 that cohomological Bousfield classes of spectra are coreflective under Vopěnka's principle —this was first proved in [14], [16]. However, we do not know if every localizing subcategory of spectra is a cohomological Bousfield class. Indeed, we could not prove that colocalizing subcategories are singly generated, not even under Vopěnka's principle and in the presence of combinatorial models. As we next explain, there seems to be a reason for this.

4.1. **Torsion theories in abelian categories.** In an abelian category, the analogue of a semilocalizing subcategory is a full subcategory closed under coproducts, quotients and extensions (this is usually called a *torsion class*), and the analogue of a semicolocalizing subcategory is a full subcategory closed under products, subobjects and extensions (called a *torsion-free class*). In well-powered abelian categories, torsion classes are coreflective and torsion-free classes are reflective; see [19].

A torsion class closed under subobjects is called *hereditary*. These correspond to the localizing subcategories. Hereditary torsion classes of modules over a ring are singly generated and their orthogonal torsion-free classes are also singly generated; see [23]. In the non-hereditary case, the situation is more intriguing. On one hand, under Vopěnka's principle, every torsion class of abelian groups is singly generated. This was shown in [22] and [30] (we note that the proof of Theorem 3.9 can easily be adapted so as to hold for abelian groups, thus yielding another proof of this fact). On the other hand, there are torsion-free classes that are not singly generated in ZFC; for example, the class of abelian groups whose countable subgroups are free —see [21, Theorem 5.4]. This casts doubt on the fact that, in reasonably restricted triangulated categories, colocalizing or semicolocalizing subcategories are necessarily singly generated, even under large-cardinal assumptions.

References

[1] J. Adámek, H. Herrlich and G. E. Strecker, *Abstract and Concrete Categories. The Joy of Cats*, Pure and Applied Mathematics, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York. 1990. [Reprinted in TAC Reprints.]

[2] J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, London Math. Soc. Lecture Note Ser., vol. 189, Cambridge University Press, Cambridge, 1994.

[3] L. Alonso Tarrío, A. Jeremías López and M. J. Souto Salorio, Localization in categories of complexes and unbounded resolutions, *Canad. J. Math.* **52** (2000), 225–247.

[4] L. Alonso Tarrío, A. Jeremías López and M. J. Souto Salorio, Construction of *t*-structures and equivalences of derived categories, *Trans. Amer. Math. Soc.* **355** (2003), 2523–2543.

[5] P. Balmer, Tensor triangular geometry, in: *Proceedings of the International Congress of Mathematicians*, vol. II, Hyderabad, 2010, 85–112.

[6] A. Beligiannis and I. Reiten, *Homological and Homotopical Aspects of Torsion Theories*, Mem. Amer. Math. Soc., vol. 188, no. 993, Amer. Math. Soc., Providence, 2007.

[7] D. Benson, S. Iyengar and H. Krause, Local cohomology and support for triangulated categories, *Ann. Sci. Éc. Norm. Supér.* (4) **41** (2008), no. 4, 573–619.

[8] C. Berger and I. Moerdijk, On an extension of the notion of Reedy category, *Math. Z.* **269** (2011), 877–1004.

[9] D. Bourn and J.-M. Cordier, A general formulation of homotopy limits, *J. Pure Appl. Algebra* **29** (1983), 129–141.

[10] A. K. Bousfield, The Boolean algebra of spectra, *Comment. Math. Helv.* **54** (1979), 368–377.

[11] A. K. Bousfield, Correction to "The Boolean algebra of spectra", *Comment. Math. Helv.* **58** (1983), 599–600.

[12] A. K. Bousfield, Unstable localization and periodicity, in: *Algebraic Topology: New Trends in Localization and Periodicity (Sant Feliu, 1994)*, Progress in Math., vol. 136, Birkhäuser, Basel, 1996, 33–50.

[13] A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math., vol. 304, Springer-Verlag, Berlin, Heidelberg, 1972.

[14] C. Casacuberta and B. Chorny, The orthogonal subcategory problem in homotopy theory, in: *An Alpine Anthology of Homotopy Theory (Arolla, 2004)*, Contemp. Math., vol. 399, Amer. Math. Soc., Providence, 2006, 41–53.

[15] C. Casacuberta and A. Neeman, Brown representability does not come for free, *Math. Res. Lett.* **16** (2009), no. 1, 1–5.

[16] C. Casacuberta, D. Scevenels and J. H. Smith, Implications of large-cardinal principles in homotopical localizations, *Adv. Math.* **197** (2005), no. 1, 120–139.

[17] W. Chachólski, On the functors CW_A and P_A, *Duke Math. J.* **84** (1996), no. 3, 599–631.

[18] B. Chorny, Abstract cellularization as a cellularization with respect to a set of objects, in: *Categories in Algebra, Geometry and Mathematical Physics (Sydney, 2005)*, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, 2007, 165–170.

[19] S. E. Dickson, A torsion theory for abelian categories, *Trans. Amer. Math. Soc.* **121** (1966), 223–235.

[20] E. Dror Farjoun, *Cellular Spaces, Null Spaces and Homotopy Localization*, Lecture Notes in Math., vol. 1622, Springer-Verlag, Berlin, Heidelberg, 1996.

[21] M. Dugas and R. Göbel, On radicals and products, *Pacific J. Math.* **118** (1985), 79–104.

[22] M. Dugas and G. Herden, Arbitrary torsion classes of abelian groups, *Comm. Algebra* **11** (1983), no. 13, 1455–1472.

[23] M. Dugas and G. Herden, Arbitrary torsion classes and almost free abelian groups, *Israel J. Math.* **44** (1983), 322–334.

[24] D. Dugger, Universal homotopy theories, *Adv. Math.* **164** (2001), 144–176.

[25] D. Dugger, Combinatorial model categories have presentations, *Adv. Math.* **164** (2001), 177–201.

[26] D. Dugger, Spectral enrichments of model categories, *Homology, Homotopy Appl.* **8** (2006), 1–30.

[27] L. Fuchs, *Infinite Abelian Groups*, vol. 2, Academic Press, New York, 1973.

[28] P. Gabriel and F. Ulmer, *Lokal präsentierbare Kategorien*, Lecture Notes in Math., vol. 221, Springer-Verlag, Berlin, Heidelberg, 1971.

[29] N. Gambino, Weighted limits in simplicial homotopy theory, *J. Pure Appl. Algebra* **214** (2010), 1193–1199.

[30] R. Göbel and S. Shelah, Semirigid classes of cotorsion-free abelian groups, *J. Algebra* **93** (1985), no. 1, 136–150.

[31] P. G. Goerss and J. F. Jardine, *Simplicial Homotopy Theory*, Progress in Math., vol. 174, Birkhäuser, Basel, 1999.

[32] P. G. Goerss and K. Schemmerhorn, Model categories and simplicial methods, in: *Interactions Between Homotopy Theory and Algebra (Chicago, 2004)*, Contemp. Math., vol. 436, Amer. Math. Soc., Providence, 2007, 3–49.

[33] J. J. Gutiérrez, Cellularization of structures in stable homotopy categories, *Math. Proc. Cambridge Philos. Soc.* **153** (2012), 399–418.

[34] H. Herrlich and G. E. Strecker, *Category Theory*, Heldermann Verlag, Berlin, 1973.

[35] P. S. Hirschhorn, *Model Categories and Their Localizations*, Math. Surveys and Monographs, vol. 99, Amer. Math. Soc., Providence, 2003.

[36] J. Hollender and R. M. Vogt, Modules of topological spaces, applications to homotopy limits and E_∞ structures, *Arch. Math. (Basel)* **59** (1992), 115–129.

[37] M. Hovey, Cohomological Bousfield classes, *J. Pure Appl. Alg.* **103** (1995), 45–59.

[38] M. Hovey, *Model Categories*, Math. Surveys and Monographs, vol. 63, Amer. Math. Soc., Providence, 1999.

[39] M. Hovey, J. H. Palmieri and N. P. Strickland, *Axiomatic Stable Homotopy Theory*, Mem. Amer. Math. Soc., vol. 128, no. 610, Amer. Math. Soc., Providence, 1997.

[40] T. Jech, *Set Theory*, Academic Press, New York, 1978.

[41] B. Keller and P. Nicolás, Weight structures and simple dg modules for positive dg algebras, *Int. Math. Res. Not. IMRN* **2013** (2013), no. 5, 1028–1078.

[42] B. Keller and D. Vossieck, Aisles in derived categories, *Bull. Soc. Math. Belg.* Sér. A **40** (1988), no. 2, 239–253.

[43] H. Krause, Localization for triangulated categories, in: *Triangulated Categories*, London Math. Soc. Lecture Note Ser., vol. 375, Cambridge University Press, Cambridge, 2010, 161–235.

[44] J. Lurie, *Higher Topos Theory*, Annals of Math. Studies, vol. 170, Princeton University Press, Princeton, 2009.

[45] J. Lurie, Derived Algebraic Geometry I: Stable ∞-Categories, arXiv:math.CT/0608228.

[46] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Math., vol. 5, Springer-Verlag, New York, 1971.

[47] M. Makkai and R. Paré, *Accessible Categories: The Foundations of Categorical Model Theory*, Contemp. Math., vol. 104, Amer. Math. Soc., Providence, 1989.

[48] A. Neeman, *Triangulated Categories*, Annals of Math. Studies, vol. 148, Princeton University Press, Princeton, 2001.

[49] A. Neeman, Some adjoints in homotopy categories, *Ann. of Math. (2)* **171** (2010), 2143–2155.

[50] A. Neeman, Colocalizing subcategories of $\mathbf{D}(R)$, *J. reine angew. Math.* **653** (2011), 221–243.

[51] A. Przeździecki, An "almost" full embedding of the category of graphs into the category of abelian groups, arXiv:1104.5689v1.

[52] J. Rosický, Generalized Brown representability in homotopy categories, *Theory Appl. Categories* **14** (2005), 451–479.

[53] J. Rosický, On homotopy varieties, *Adv. Math.* **214** (2007), 525–550.

[54] S. Schwede and B. Shipley, Stable model categories are categories of modules, *Topology* **42** (2003), 103–153.

[55] M. Shulman, Homotopy limits and colimits and enriched homotopy theory, arXiv:math/0610194v3.

[56] D. Stanley, Invariants of t-structures and classification of nullity classes, *Adv. Math.* **224** (2010), 2662–2689.

[57] N. P. Strickland, Axiomatic stable homotopy, in: *Axiomatic, Enriched and Motivic Homotopy Theory (Cambridge, 2002)*, NATO Science Series II: Mathematics, Physics and Chemistry, vol. 131, Kluwer, Dordrecht, 2004, 69–98.

C. Casacuberta
Institut de Matemàtica
Universitat de Barcelona
Gran Via de les Corts Catalanes 585
08007 Barcelona, Spain
carles.casacuberta@ub.edu

J. J. Gutiérrez
Department of Algebra and Topology
Radboud Universiteit Nijmegen
Heyendaalseweg 135
6525 AJ Nijmegen, The Netherlands
j.gutierrez@math.ru.nl

J. Rosický
Department of Mathematics and Statistics
Faculty of Science
Masaryk University
Kotlářská 2
61137 Brno, Czech Republic
rosicky@math.muni.cz