Homotopy localization of groupoids

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Abstract

In order to study functorial changes caused by homotopy localizations on the fundamental group of unbased simplicial sets, it is convenient to use groupoids instead of groups, and therefore localizations of groupoids become useful. In this article we develop homotopy localization techniques in the model category of groupoids, with emphasis on the relationship with homotopy localizations of simplicial sets and also with discrete localizations of groups.

Introduction

When studying the effect of homotopy localizations of spaces on the fundamental group, it was observed in [7] that, for each map f and each path-connected space X, there is a natural group homomorphism $\pi_1(L_f X) \to L_\varphi \pi_1(X)$, where φ is induced by f on fundamental groups. However, this statement is only correct if a basepoint is chosen, or if one works with simplicial sets with a single vertex, since π_1 cannot be regarded as a functor otherwise. Thus, the extension of this result to unbased spaces or unbased simplicial sets requires that a theory of homotopy localizations of groupoids be developed.

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One of our specific motivations was the need to ensure the validity of results in [9] for arbitrary simplicial sets, not only for reduced ones.

The background to carry out such a project was already available in the literature, since homotopy localizations are known to exist in simplicial model categories satisfying certain assumptions [12], and the category of groupoids admits such a structure, as explained by Anderson [2] and Bousfield [4]. This model structure happens to be a common restriction of several known model structures on the category **Cat** of small categories. The weak equivalences of groupoids are the equivalences of categories, and the fibrations of groupoids are the morphisms with the "source lifting property", as described by Brown in [5]. If G and H are groupoids, the simplicial set $\mathrm{map}(G,H)$ is the nerve of the groupoid $\mathrm{Fun}(G,H)$ whose objects are functors $G \to H$ and whose morphisms are natural transformations.

A remarkable feature of homotopy functors on groupoids is that they are automatically *continuous*; that is, as shown in Section 2 of this article, for each functor L on groupoids sending weak equivalences to weak equivalences there is a natural map of simplicial sets $\mathrm{map}(G,H) \to \mathrm{map}(LG,LH)$ for all G, H, preserving composition and identity.

Another of our observations is that, if $f\colon G \to H$ is a group homomorphism and X is a group, then f induces a bijection $\mathrm{Hom}(H,X) \cong \mathrm{Hom}(G,X)$ if and only if f induces a weak equivalence $\mathrm{map}(H,X) \simeq \mathrm{map}(G,X)$. Thus, "homotopy f-local groups" are just f-local groups in the discrete sense, as defined in [7]. A slightly more general version of this result is given in Theorem 1.3 below. This observation provides, among other things, a convincing explanation of the (apparently unnatural) fact that discrete localizations of groups commute with finite direct products. This was proved by different authors using *ad hoc* arguments in various situations [8], [15], [18], [20]. In Section 3 we give a short proof of this fact by means of groupoids, using the natural isomorphism

$$\mathrm{Fun}(A \times B, H) \cong \mathrm{Fun}(A, \mathrm{Fun}(B,H)).$$

In [11] it was shown in a similar way that homotopy localizations of spaces commute with finite direct products.

In the last section we show that, if f is any map of simplicial sets and φ denotes the induced morphism of fundamental groupoids, then for every X we have

$$L_\varphi \pi X \cong L_\varphi \pi L_f X.$$

This is quite useful in practice, since it relates the fundamental groupoids πX and $\pi L_f X$ up to a certain homotopy localization in the model category of groupoids. It implies, for instance, that if X is 1-connected, then $\pi L_f X$ is annihilated by L_φ. It is still an open problem to decide if $\pi L_f X$ is necessarily contractible when X is 1-connected; see [17], [21].

1 Small categories and groupoids

A category is *small* if its objects and morphisms form sets. We denote by **Cat** the category whose objects are small categories and whose morphisms are functors between them. A *groupoid* is a small category in which all morphisms have inverses. We denote by **Gpd** the category of groupoids and view it as a full subcategory of **Cat**. In its turn, the category **Grp** of groups embeds into **Gpd** as a full subcategory, by viewing each group as a groupoid with only one object. We denote by $\{0\}$ the trivial group, and by $\mathbf{I}$ the groupoid with two objects 0, 1 and two nonidentity arrows $0 \to 1$, $1 \to 0$.

For a simplicial set X, we denote by πX the fundamental groupoid of X, whose set of objects is the set X_0 of vertices of X, and whose morphisms are equivalence classes of edge paths, defined as follows: a morphism from p to q is a finite sequence of composable arrows $y\colon d_1 y \to d_0 y$ for $y \in X_1$ and their formal inverses, starting at p and ending at q, subject to the relations $d_1 z = d_0 z \circ d_2 z$ for each $z \in X_2$ and $s_0 x = \mathrm{id}_x$ for each $x \in X_0$. The functor π is left adjoint to the nerve functor N from groupoids to simplicial sets, and the natural morphism of groupoids $\pi NG \to G$ is an isomorphism for every G. For each groupoid G, the nerve NG is a Kan simplicial set and its homotopy groups vanish except possibly in two dimensions, namely $\pi_0(NG)$ is naturally isomorphic to the set $\pi_0(G)$ of connected components of G, and, for each object x of G, the fundamental group $\pi_1(NG, x)$ is naturally isomorphic to the group $\pi_1(G, x)$ of automorphisms of x in G.

Several model category structures (in the sense of Quillen [19]) have been considered on **Cat** in the literature. In the model structure described by Thomason [22], a functor $f\colon C \to D$ is a weak equivalence if and only if the associated map $Nf\colon NC \to ND$ of nerves is a weak equivalence of simplicial sets, and f is a fibration if and only if $\mathrm{Ex}^2 Nf$ is a Kan fibration, where Ex denotes the Kan functor [14], which is right adjoint to the barycentric subdivision functor Sd on simplicial sets. There is another well-known model structure on **Cat**, which was treated with greater generality in [13]. Namely, weak equivalences are equivalences of categories, fibrations are functors having the right lifting property with respect to the inclusion $\{0\} \hookrightarrow \mathbf{I}$, and cofibrations are functors which are injective on the sets of objects.

It is remarkable that these two model structures on **Cat** restrict to the same model structure on the full subcategory **Gpd** of groupoids, which is in fact the model structure described in [2] and [4]. In order to verify this claim, and for other arguments in the article, we need to recall that a functor $f\colon C \to D$ is an equivalence of categories if and only if it is fully faithful and each object of D is isomorphic to some object in the image of f. If we use, in addition, that natural transformations of functors yield simplicial homotopies of maps on the nerves, we find that the following statements are equivalent for a morphism of groupoids $f\colon G \to H$.

(1) The map of nerves $Nf\colon NG \to NH$ is a weak equivalence.

(2) f induces a bijection $\pi_0(G) \cong \pi_0(H)$ and an isomorphism of groups $\pi_1(G, x) \cong \pi_1(H, f(x))$ for each object x of G.

(3) f is an equivalence of categories.

For brevity, we use the term *equivalence* to denote a morphism of groupoids satisfying any of these statements. These are the weak equivalences of the model structure on **Gpd** that we will use.

A *fibration* of groupoids is a morphism with the following properties.

Proposition 1.1. *For a morphism of groupoids $f\colon G \to H$, the following statements are equivalent:*

(1) *The morphism f has the right lifting property with respect to the inclusion $\{0\} \hookrightarrow \mathbf{I}$.*

(2) *The map Nf is a Kan fibration.*

(3) *The map $\mathrm{Ex}^i Nf$ is a Kan fibration for some $i \geq 0$.*

(4) *The map $\mathrm{Ex}^i Nf$ is a Kan fibration for all $i \geq 0$.*

Proof. Use the adjointness between $\mathrm{Ex}^i N$ and $\pi\mathrm{Sd}^i$ to transform the statement that $\mathrm{Ex}^i N f$ is a Kan fibration into the statement that f has the right lifting property with respect to $j_{n,k}\colon \pi\mathrm{Sd}^i\Lambda_k[n] \hookrightarrow \pi\mathrm{Sd}^i\Delta[n]$ for $n \geq 1$ and $0 \leq k \leq n$, where $\Lambda_k[n] \hookrightarrow \Delta[n]$ are the standard generating trivial cofibrations of simplicial sets. The morphism $j_{n,k}$ is an isomorphism for $i = 0$, $n \geq 2$, and it is the inclusion $\{0\} \hookrightarrow \mathbf{I}$ for $i = 0$, $n = 1$. It is an inclusion of contractible groupoids for all i, n, k. The right lifting property with respect to $j_{n,k}$ is equivalent to the right lifting property with respect to $\{0\} \hookrightarrow \mathbf{I}$ if $i \geq 1$. Hence, (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (1), and (1) $\Leftrightarrow$ (2). $\square$

Therefore, by part (1), a morphism of groupoids $f\colon G \to H$ is a fibration if and only if it has the *source lifting property*, i.e., for every object $x \in G$, each arrow β of H with source $f(x)$ may be lifted to an arrow α with source x, such that $f(\alpha) = \beta$. If the lifting is unique, then f is called a *covering morphism*; cf. [4], [5].

The *star* $\mathrm{St}_C(x)$ of a category C at an object x is the set of all arrows with source x. Using this language, the fibrations of groupoids are the *star-surjective* morphisms, that is, morphisms $f\colon G \to H$ such that the induced map $\mathrm{St}_C(x) \to \mathrm{St}_D(f(x))$ is surjective for all x. The covering morphisms are the *star-bijective* morphisms. The trivial fibrations between groupoids are the equivalences that are surjective on objects. A covering morphism is an equivalence if and only if it is bijective on objects (in fact, it is then an isomorphism, since an equivalence of categories is an isomorphism if and only if it is bijective on objects).

All groupoids are fibrant and cofibrant. Since the cofibrations of groupoids are the morphisms that are injective on the sets of objects, all group homomorphisms are cofibrations. However, a group homomorphism is a fibration if and only if it is an epimorphism; hence, the model structure on **Gpd** does not restrict to a model structure on **Grp**.

If C and D are small categories, then one defines a simplicial set $\mathrm{map}(C, D)$ as the nerve of the small category $\mathrm{Fun}(C, D)$ whose objects are the functors $C \to D$ and whose morphisms are the natural transformations. If G and H are groupoids, then $\mathrm{Fun}(G, H)$ is also a groupoid, since natural transformations are invertible. Therefore $\mathrm{map}(G, H)$ is fibrant. We shall denote by $\mathrm{Hom}(G, H)$ the set of morphisms $G \to H$, which is precisely the set of vertices of the simplicial set $\mathrm{map}(G, H)$.

The category of groupoids is a simplicial model category with these function complexes, tensored over simplicial sets as $G \otimes K = G \times \pi K$ and cotensored as $G^K = \mathrm{Fun}(\pi K, G)$; cf. [4]. This simplicial structure is also inherited from **Cat**.

For a groupoid G, let i_0 and i_1 denote the two inclusions of G into $\mathbf{I} \times G$. If f and g are morphisms from G to another groupoid H, then a *homotopy* from f to g is a morphism $\Delta\colon \mathbf{I} \times G \to H$ with $\Delta \circ i_0 = f$ and $\Delta \circ i_1 = g$. Such a morphism yields a natural transformation from f to g, since

$$\mathrm{Fun}(\mathbf{I} \times G, H) \cong \mathrm{Fun}(G, \mathrm{Fun}(\mathbf{I}, H)),$$

and a morphism $\eta\colon G \to \mathrm{Fun}(\mathbf{I}, H)$ is just a natural transformation between the morphisms $\mathrm{ev}_0 \circ \eta$ and $\mathrm{ev}_1 \circ \eta$. (Here ev_0 and ev_1 are the evaluation maps $\mathrm{Fun}(\mathbf{I}, H) \to H$.) More explicitly, a natural transformation η from f to g corresponds to the homotopy $\mathbf{I} \times G \to H$ sending $(0 \to 1, \alpha)$ to $g(\alpha) \circ \eta_x = \eta_y \circ f(\alpha)$ for every $\alpha\colon x \to y$ in G. Thus, for morphisms of groupoids, no distinction will be made between natural transformations and homotopies.

Theorem 1.2. *If $f\colon G \to H$ is a morphism of groupoids which is bijective on objects, then for every groupoid X the induced morphism*

$$f^*\colon \mathrm{Fun}(H, X) \to \mathrm{Fun}(G, X)$$

is a covering morphism.

PROOF. Since f is a cofibration, the morphism f^* is a fibration of groupoids, according to the axioms of a simplicial model category. Hence, f^* is star-surjective. In fact the assumption that f is bijective on objects implies directly that f^* is star-bijective, as the next argument shows. For a morphism $g\colon H \to X$, an element of the star $\mathrm{St}_{\mathrm{Fun}(H,X)}(g)$ is a morphism $\eta\colon H \to \mathrm{Fun}(\mathbf{I}, X)$ with $\mathrm{ev}_0 \circ \eta = g$, and is uniquely determined by the choice of a morphism $\eta_b \in \mathrm{St}_X(g(b))$ for each object b of H. Hence, given a natural transformation ξ in $\mathrm{St}_{\mathrm{Fun}(G,X)}(g \circ f)$, the natural transformation η defined as $\eta_b = \xi_{f^{-1}(b)}$ for all objects b of H is the unique element of $\mathrm{St}_{\mathrm{Fun}(H,X)}(g)$ with $f^*(\eta) = \xi$. □

As a special case, note that, if $f\colon G \to H$ is a group homomorphism and X is any group, then the covering morphism $\mathrm{Fun}(H, X) \to \mathrm{Fun}(G, X)$ simply sends each element $x \in X$ in the star of $\mathrm{Fun}(H, X)$, at any $H \to X$, to x itself.

Theorem 1.3. *Suppose that a morphism of groupoids $f: G \to H$ is bijective on objects. Then, for a groupoid X, the following statements are equivalent:*

(1) *f induces a bijection of sets* $\mathrm{Hom}(H, X) \cong \mathrm{Hom}(G, X)$*, that is, every morphism $G \to X$ can be factored through f in a unique way.*

(2) *f induces an equivalence of groupoids* $\mathrm{Fun}(H, X) \simeq \mathrm{Fun}(G, X)$.

(3) *f induces an isomorphism of groupoids* $\mathrm{Fun}(H, X) \cong \mathrm{Fun}(G, X)$.

(4) *f induces a weak equivalence of simplicial sets* $\mathrm{map}(H, X) \simeq \mathrm{map}(G, X)$.

(5) *f induces an isomorphism of simplicial sets* $\mathrm{map}(H, X) \cong \mathrm{map}(G, X)$.

PROOF. In order to prove that (1) $\Leftrightarrow$ (2), use the fact that a covering morphism is an equivalence if and only if it is bijective on objects. Now (1) and (2) together imply (3), since an equivalence of categories which is bijective on objects is an isomorphism, and clearly (3) implies both (1) and (2). Statements (4) and (2) are equivalent because the equivalences of groupoids are the morphisms inducing weak equivalences of nerves; and (5) and (3) are also equivalent, since $\pi NY \cong Y$ for all Y. $\square$

For two groups G, H, the simplicial set $\mathrm{map}(G, H)$ has a well-known homotopy type. Its only possibly nonzero homotopy groups are $\pi_0\mathrm{map}(G, H)$, which is isomorphic to the set $\mathrm{Rep}(G, H)$ of group homomorphisms $G \to H$ modulo conjugation in H, and $\pi_1(\mathrm{map}(G, H), \alpha)$ at each vertex $\alpha: G \to H$, which is the centralizer of the image of α in H, that is, the subgroup of H consisting of the elements x such that $x^{-1}\alpha(y)x = \alpha(y)$ for every $y \in G$. A homotopy between two group homomorphisms α, β from G to H can be identified with an element $x \in H$ such that $\beta(y) = x^{-1}\alpha(y)x$ for all $y \in G$; that is, two group homomorphisms are homotopic if and only if they are conjugate.

2 Homotopy idempotent functors

An *idempotent functor* on a category $\mathcal{C}$ is a functor $L: \mathcal{C} \to \mathcal{C}$ equipped with a natural transformation $\eta: \mathrm{Id} \to L$ such that $\eta_{LX} = L\eta_X$ for all X, and $L\eta_X: LX \to LLX$ is an isomorphism for all X. The pair (L, η) is more commonly called an *idempotent monad* on $\mathcal{C}$; see e.g. [8]. The natural transformation η will be omitted from the notation whenever possible and appropriate. Thus we speak of an idempotent functor L or (L, η) depending on the context.

Objects isomorphic to LX for some X are called *L-local*, and morphisms f such that Lf is an isomorphism are called *L-equivalences*. Thus, the natural map $\eta_X \colon X \to LX$ is an L-equivalence into an L-local object for every X. In fact, it is a terminal L-equivalence with domain X, and it is initial among morphisms from X into L-local objects. This is a "discrete" version of the following more general concept.

A *homotopy idempotent functor* on a model category $\mathcal{M}$ is a functor $L \colon \mathcal{M} \to \mathcal{M}$ sending weak equivalences to weak equivalences, taking fibrant values, and equipped with a natural transformation $\eta \colon \mathrm{Id} \to L$ such that $\eta_{LX} \simeq L\eta_X$ for all X, and $L\eta_X \colon LX \to LLX$ is a weak equivalence for all X, where $\simeq$ is the homotopy relation in $\mathcal{M}$. Thus L defines an idempotent monad on the homotopy category derived from $\mathcal{M}$. As above, the natural transformation η will sometimes be omitted from the notation.

Fibrant objects that are weakly equivalent to LX for some X are called *L-local*, and morphisms f such that Lf is a weak equivalence are called *L-equivalences*. If the model category structure on $\mathcal{M}$ is discrete, that is, the weak equivalences are the isomorphisms and all morphisms are fibrations and cofibrations, then homotopy idempotence is just ordinary idempotence.

Theorem 2.1. *A homotopy idempotent functor (L, η) on the category* **Gpd** *of groupoids is idempotent if η_X is bijective on objects for every X.*

Proof. By assumption, for every X, the morphisms η_{LX} and $L\eta_X$ are equivalences of groupoids. Since η_{LX} is bijective on objects, it is an isomorphism. Since η is a natural transformation, we have $L\eta_X \circ \eta_X = \eta_{LX} \circ \eta_X$, showing that $L\eta_X$ is also bijective on objects, and hence an isomorphism as well. As shown in [8, Proposition 1.1], if both η_{LX} and $L\eta_X$ are isomorphisms, then they coincide. □

The converse of Theorem 2.1 is obviously false, as the functor L that sends all groupoids to the trivial group is idempotent.

It is clear that every functor $L \colon \mathbf{Gpd} \to \mathbf{Gpd}$ preserves equivalences between groups, since these are just isomorphisms. However, it need not preserve equivalences between groupoids. For example, neither the functor that sends each groupoid G to the free product of its automorphism groups $\pi_1(G, x_i)$ at all objects x_i of G, nor the functor that sends each groupoid to the discrete groupoid with the same set of objects, preserve equivalences.

Suppose $L: \mathbf{Gpd} \to \mathbf{Gpd}$ is a functor that sends equivalences to equivalences. For a groupoid G, let $p: \mathbf{I} \times G \to G$ be the projection. Since p is an equivalence, Lp is also an equivalence by assumption, and therefore the morphism

$$(Lp)_*: \mathrm{Fun}(X, L(\mathbf{I} \times G)) \to \mathrm{Fun}(X, LG) \tag{2.1}$$

is an equivalence, hence fully faithful, for every X.

As a first use of this fact, take $X = LG$. Since $p \circ i_0$ and $p \circ i_1$ are both the identity of G, and therefore $Lp \circ Li_0$ and $Lp \circ Li_1$ are both the identity of LG, one infers using (2.1) that there is a unique homotopy

$$h_G: \mathbf{I} \times LG \to L(\mathbf{I} \times G)$$

from Li_0 to Li_1 such that $Lp \circ h_G$ is the constant homotopy at the identity. To show that h_G is natural in G, consider any morphism $f: F \to G$ and take $X = LF$ in (2.1). Then $L(\mathrm{id} \times f) \circ h_F$ and $h_G \circ (\mathrm{id} \times Lf)$ coincide, since both are homotopies from $L(i_0 \circ f)$ to $L(i_1 \circ f)$ that give the constant homotopy at Lf on applying the faithful functor $(Lp)_*$.

Theorem 2.2. *Let $L: \mathbf{Gpd} \to \mathbf{Gpd}$ be any functor sending equivalences to equivalences. Then L induces a natural morphism $l_{G,H}: \mathrm{Fun}(G, H) \to \mathrm{Fun}(LG, LH)$ for all G, H.*

PROOF. This morphism $l_{G,H}$ is defined on objects as $f \mapsto Lf$, and it is defined on arrows $\Delta: \mathbf{I} \times G \to H$ as $\Delta \mapsto L\Delta \circ h_G$. Naturality in H is clear, and naturality in G follows from the naturality of h_G. Checking that $l_{G,H}$ is a morphism of groupoids takes however more work. Let $\mathbf{J}$ be the groupoid with three objects 0, 1, 2 and two arrows between each pair of distinct objects. The projection $p: \mathbf{J} \times G \to G$ yields, as in (2.1), an equivalence

$$(Lp)_*: \mathrm{Fun}(LG, L(\mathbf{J} \times G)) \to \mathrm{Fun}(LG, LG). \tag{2.2}$$

Using three times the fact that (2.2) is an equivalence, we obtain a natural morphism

$$j_G: \mathbf{J} \times LG \to L(\mathbf{J} \times G)$$

consisting of homotopies $Li_0 \to Li_1$, $Li_1 \to Li_2$, and their composition, each of which is sent to the constant homotopy at the identity by $(Lp)_*$. A pair of composable homotopies Φ and Ψ between morphisms $G \to H$ is represented by a morphism $\Gamma\colon \mathbf{J} \times G \to H$, and the composite $L\Gamma \circ j_G$ demonstrates that $l_{G,H}(\Psi \circ \Phi) = l_{G,H}(\Psi) \circ l_{G,H}(\Phi)$. □

In other words, if we regard the category of groupoids as a 2-category, where the 2-cells are the natural transformations, then we have shown that each functor of groupoids that sends equivalences to equivalences extends to a 2-functor in a canonical way.

This is a remarkable property. It shows, for example, that the functor sending each group G to the free product $G * \mathbf{Z}$ does not extend to any functor on groupoids that preserves equivalences, since conjugate group homomorphisms $G \to H$ do not yield conjugate homomorphisms $G * \mathbf{Z} \to H * \mathbf{Z}$.

It also follows from Theorem 2.2 that, if a functor $L\colon \mathbf{Gpd} \to \mathbf{Gpd}$ preserves equivalences and sends the trivial group to itself, then L is automatically equipped with a natural transformation $\eta\colon \mathrm{Id} \to L$. This follows by taking $G = \{0\}$ in Theorem 2.2.

Proposition 2.3. *Let $L\colon \mathbf{Gpd} \to \mathbf{Gpd}$ be a functor sending equivalences to equivalences and equipped with a natural transformation $\eta\colon \mathrm{Id} \to L$. Then the following hold for all groupoids G and H:*

(1) $(\eta_G)^* \circ l_{G,H} = (\eta_H)_*$.

(2) $l_{G,H} \circ (\eta_G)^* = (L\eta_G)^* \circ l_{LG,H}$.

(3) $(\eta_H)_* \circ (\eta_G)^* = (\eta_G)^* \circ (\eta_H)_*$.

Proof. Statements (2) and (3) are checked easily, using the naturality of $l_{G,H}$ and the bifunctoriality of $\mathrm{Fun}(G, H)$. For statement (1) we have to prove that the composite

$$\mathrm{Fun}(G, H) \xrightarrow{l_{G,H}} \mathrm{Fun}(LG, LH) \xrightarrow{(\eta_G)^*} \mathrm{Fun}(G, LH)$$

equals $(\eta_H)_*$. For each object $f\colon G \to H$ of $\mathrm{Fun}(G, H)$, this amounts to the equality $(Lf) \circ \eta_G = \eta_H \circ f$, which is a consequence of the naturality of η. To check the same thing for morphisms, observe that $Lp \circ h_G \circ (\mathrm{id} \times \eta_G) = Lp \circ \eta_{\mathbf{I}\times G}$, hence $h_G \circ (\mathrm{id} \times \eta_G) = \eta_{\mathbf{I}\times G}$ by the faithfulness of (2.1). Now, for any morphism $\Delta\colon \mathbf{I} \times G \to H$ in $\mathrm{Fun}(G, H)$, we have $L\Delta \circ h_G \circ (\mathrm{id} \times \eta_G) = L\Delta \circ \eta_{\mathbf{I}\times G} = \eta_H \circ \Delta$, as required. □

The property stated in Theorem 2.2 is analogous to the "continuity" property discussed by Farjoun in [10]. Indeed, it yields, by taking nerves, a natural map

$$\mathrm{map}(G,H) \to \mathrm{map}(LG,LH)$$

for all groupoids G and H, preserving composition and identity. As in [10], the continuity property implies the following theorem.

Theorem 2.4. *Let (L,η) be a homotopy idempotent functor on the category of groupoids. Let X be L-local and G any groupoid. Then the map*

$$\mathrm{map}(LG,X) \to \mathrm{map}(G,X)$$

induced by η_G is a weak equivalence of simplicial sets.

PROOF. The morphism $\eta_X\colon X \to LX$ is an equivalence since X is L-local. Therefore, the induced morphism

$$(\eta_X)_*\colon \mathrm{Fun}(LG,X) \to \mathrm{Fun}(LG,LX)$$

is an equivalence. Let ξ be a homotopy inverse of $(\eta_X)_*$. Let us check that

$$\xi \circ l_{G,X}\colon \mathrm{Fun}(G,X) \to \mathrm{Fun}(LG,X)$$

is a homotopy inverse of $(\eta_G)^*$. Using Proposition 2.3 and the fact that $\eta_{LG} \simeq L\eta_G$, we have

$$\xi \circ l_{G,X} \circ (\eta_G)^* = \xi \circ (L\eta_G)^* \circ l_{LG,X} \simeq \xi \circ (\eta_{LG})^* \circ l_{LG,X} = \xi \circ (\eta_X)_* \simeq \mathrm{id},$$

and

$$(\eta_G)^* \circ \xi \circ l_{G,X} \simeq \xi \circ (\eta_X)_* \circ (\eta_G)^* \circ \xi \circ l_{G,X} =$$
$$\xi \circ (\eta_G)^* \circ (\eta_X)_* \circ \xi \circ l_{G,X} \simeq \xi \circ (\eta_G)^* \circ l_{G,X} = \xi \circ (\eta_X)_* \simeq \mathrm{id},$$

as needed. □

Corollary 2.5. *Let (L,η) be a homotopy idempotent functor on the category of groupoids. A morphism $f\colon G \to H$ is an L-equivalence if and only if $f^*\colon \mathrm{map}(H,X) \to \mathrm{map}(G,X)$ is a weak equivalence for all L-local groupoids X. Similarly, a groupoid X is L-local if and only if $f^*\colon \mathrm{map}(H,X) \to \mathrm{map}(G,X)$ is a weak equivalence for every L-equivalence $f\colon G \to H$.*

PROOF. Use the previous theorem and the commutative square

$$\begin{array}{ccc} \mathrm{map}(H,X) & \longrightarrow & \mathrm{map}(G,X) \\ \uparrow & & \uparrow \\ \mathrm{map}(LH,X) & \longrightarrow & \mathrm{map}(LG,X), \end{array}$$

as follows. If X is L-local, then, by Theorem 2.4, the vertical arrows are weak equivalences. If $G \to H$ is an L-equivalence, then it yields an equivalence $LG \simeq LH$ and therefore the bottom arrow is a weak equivalence, hence proving that the upper arrow is a weak equivalence as well. Conversely, if $f^*\colon \mathrm{map}(H,X) \to \mathrm{map}(G,X)$ is a weak equivalence for every L-local groupoid X, then by taking π_0 it follows that $\pi_0(f^*)\colon [H,X] \to [G,X]$ is bijective for every L-local groupoid X. Therefore, we have

$$[LH, LG] \cong [H, LG] \cong [G, LG]$$

and this yields a map $LH \to LG$ which is a homotopy inverse of Lf, hence showing that $f\colon G \to H$ is an L-equivalence. The second statement is proved in a similar way. □

In any simplicial model category, a morphism $f\colon G \to H$ of cofibrant objects and a fibrant object X are called *simplicially orthogonal* if $f^*\colon \mathrm{map}(H,X) \to \mathrm{map}(G,X)$ is a weak equivalence of simplicial sets. This is an enriched version of the usual orthogonality in categories. Recall e.g. from [1] that an object X and a morphism $f\colon A \to B$ in a category $\mathcal{C}$ are *orthogonal* if the function $\mathcal{C}(B,X) \to \mathcal{C}(A,X)$ induced by f is bijective.

Thus, Corollary 2.5 states that, for each homotopy idempotent functor L on groupoids, the L-equivalences and the L-local groupoids form a pair of simplicially orthogonal classes, each of which is precisely the complement of the other. Note that simplicially orthogonal classes are, a fortiori, orthogonal in the homotopy category, since $\pi_0\mathrm{map}(H,X) \cong [H,X]$ for all H and X.

A functor on the category of groupoids will be termed *homotopically trivial* if every non-empty groupoid in its image is contractible.

Theorem 2.6. *A homotopy idempotent functor L on the category of groupoids is homotopically trivial if and only if there is an L-equivalence that is not bijective on connected components.*

Proof. If an L-equivalence $G \to H$ is not bijective on connected components, then it has a retract of the form $p\colon \{0,1\} \to \{0\}$, $j\colon \{0\} \hookrightarrow \{0,1\}$, or $k\colon \varnothing \hookrightarrow \{0\}$. Since every retract of an L-equivalence is an L-equivalence, in the first case we have $p^*\colon X \simeq X \times X$ for every L-local groupoid X, which implies that X is contractible or empty. In the other cases, we obtain $j^*\colon X \times X \simeq X$ or $k^*\colon X \simeq \{0\}$, leading to similar conclusions.

Conversely, we suppose that all L-equivalences are bijective on components and exhibit a non-contractible non-empty L-local groupoid. It is enough to pick the discrete groupoid $\{0,1\}$; this is simplicially orthogonal to all morphisms $G \to H$ that are bijective on components, since $\mathrm{Fun}(G,\{0,1\})$ is a discrete groupoid with $2^{|\pi_0(G)|}$ objects, and similarly for H. □

Note that, for any homotopy idempotent functor L on groupoids, either $L(\varnothing) = \varnothing$ or $L(\varnothing)$ is contractible, in which case L is homotopically trivial.

The following is an important source of homotopy idempotent functors. For every morphism of groupoids $f\colon G \to H$, there is a homotopy idempotent functor L_f on **Gpd**, called *f-localization*, which can be constructed as in [3], [11] or [12], since the category **Gpd** is left proper, cofibrantly generated, and locally presentable (in fact, presentable). The L_f-local groupoids (called *f-local* for simplicity) are those X such that

$$f^*\colon \mathrm{map}(H,X) \to \mathrm{map}(G,X)$$

is a weak equivalence. Therefore, f itself is an L_f-equivalence. (We speak of *f-equivalences*, instead of L_f-equivalences, also for simplicity.) Thus, for each groupoid A, the f-localization morphism $A \to L_f A$ is an f-equivalence into an f-local groupoid.

Theorem 2.7. *A morphism of groupoids $f\colon G \to H$ is bijective on connected components if and only if every f-equivalence is bijective on connected components.*

Proof. One implication is easy, since f is itself an f-equivalence. For the converse, note that if f is bijective on connected components then $\{0,1\}$ is f-local (as in the proof of Theorem 2.6) and hence f-localization is not homotopically trivial. The result then follows from Theorem 2.6. □

Hence, for every morphism f of groupoids, L_f preserves the set π_0 of connected components, unless it is homotopically trivial. However, L_f need not preserve the set of objects in general. Thus, the f-localization of a group need not be a group, although it is always homotopy equivalent to a group since it is necessarily a connected groupoid.

3 Localizing groups

Idempotent functors on groups have been considered by several authors, also in the context of homotopy localizations of spaces. For a group homomorphism $\varphi\colon A \to B$, a group X was called *φ-local* in [7] if the induced map of sets $\mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$ is a bijection. A group homomorphism $f\colon G \to H$ is a *φ-equivalence* if $f^*\colon \mathrm{Hom}(H, X) \to \mathrm{Hom}(G, X)$ is bijective for every φ-local group X. A *φ-localization* of a group G is a φ-equivalence $G \to L_\varphi G$ into a φ-local group. The existence of such localizations can be proved by standard arguments, as in [1] or in [3]. These localizations are idempotent functors on the category of groups.

Theorem 3.1. *Let L be any idempotent functor on the category of groups. Then, for a group homomorphism $f\colon G \to H$, the following statements are equivalent:*

(1) *f induces an isomorphism of groups $LG \cong LH$.*

(2) *f induces a bijection of sets $\mathrm{Hom}(H, LX) \cong \mathrm{Hom}(G, LX)$ for every X.*

(3) *f induces an equivalence of groupoids $\mathrm{Fun}(H, LX) \simeq \mathrm{Fun}(G, LX)$ for every X.*

(4) *f induces an isomorphism of groupoids $\mathrm{Fun}(H, LX) \cong \mathrm{Fun}(G, LX)$ for every X.*

(5) *f induces a weak equivalence $\mathrm{map}(H, LX) \simeq \mathrm{map}(G, LX)$ for every X.*

(6) *f induces an isomorphism $\mathrm{map}(H, LX) \cong \mathrm{map}(G, LX)$ for every X.*

Proof. The equivalence of (1) and (2) amounts to the well-known orthogonality between L-equivalences and L-local objects when L is an idempotent functor in any category; see e.g. the survey article [8]. The other equivalences follow from Theorem 1.3. □

Hence, by regarding the category of groups as a full subcategory of the category of groupoids, we find that the concepts of orthogonality and simplicially enriched orthogonality coincide on groups. This observation was one of the starting points of this article.

These two concepts do not coincide on groupoids; for example, let $f\colon \{0\} \to \{0,1\}$ be the inclusion. Then the groupoids that are orthogonal to f are precisely those with only one object, i.e., the groups. However, by Theorem 2.6, the groupoids that are simplicially orthogonal to f are the contractible groupoids.

Let G, H be groups and (L, η) an idempotent functor on **Grp**. Then $LG \times LH$ is L-local, since every inverse limit of L-local groups is L-local. Using the fact that $\eta_{G\times H}\colon G \times H \to L(G \times H)$ is initial among homomorphisms into L-local groups, we find a unique group homomorphism

$$\theta_{G,H}\colon L(G \times H) \to LG \times LH$$

such that $\theta_{G,H} \circ \eta_{G\times H} = \eta_G \times \eta_H$.

The following result was proved independently by several authors in special cases [8], [15], [18], [20]. Its proof is shorter and most natural using groupoids. This is a worthwhile achievement of the use of a simplicial enrichment in the study of group localizations.

Theorem 3.2. *If L is any idempotent functor on groups, then the natural homomorphism $\theta_{G,H}\colon L(G \times H) \to LG \times LH$ is an isomorphism for all groups G, H.*

Proof. It suffices to show that $\theta_{G,H}$ is an L-equivalence. This is inferred, using Theorem 3.1, from the following isomorphisms of groupoids, where X is any L-local group:

$$\begin{aligned} &\mathrm{Fun}(L(G \times H), X) \cong \mathrm{Fun}(G \times H, X) \cong \mathrm{Fun}(G, \mathrm{Fun}(H, X)) \cong \\ &\quad \mathrm{Fun}(G, \mathrm{Fun}(LH, X)) \cong \mathrm{Fun}(G \times LH, X) \cong \mathrm{Fun}(LH, \mathrm{Fun}(G, X)) \cong \\ &\quad \mathrm{Fun}(LH, \mathrm{Fun}(LG, X)) \cong \mathrm{Fun}(LG \times LH, X), \end{aligned}$$

as claimed. □

Exactly the same argument shows that, if L is a homotopy idempotent functor on the category **Gpd** of groupoids, then there is a natural equivalence

$$L(G \times H) \to LG \times LH$$

for all groupoids G and H. Hence, homotopy idempotent functors on groupoids preserve finite products, up to homotopy.

Let $f\colon A \to B$ be any morphism of groupoids that is bijective on connected components. We next describe L_fG for each groupoid G in terms of discrete group localizations, using the fact that each groupoid is equivalent to a disjoint union of groups.

Specifically, given a groupoid G, choose an object v_i at each connected component C_i of G, and, for each object v of C_i, choose an arrow a_v from v_i to v, with a_{v_i} equal to the identity. Let K_i be the subgroupoid of C_i generated by all these arrows. Thus, K_i has only one arrow $a_w a_v^{-1}$ from any v to another w. Then the morphism $\pi_1(G, v_i) \times K_i \to C_i$ which is defined as $(v_i, v) \mapsto v$ on objects and sends each arrow $(x, a_w a_v^{-1})$ to $a_w x a_v^{-1}$ is an isomorphism of groupoids, for all i. Hence, G is isomorphic to the disjoint union $\cup_i(\pi_1(G, v_i) \times K_i)$, where each K_i is contractible and i runs through the set $\pi_0(G)$. Using this notation, we prove the next result.

Theorem 3.3. *Let $f\colon A \to B$ be any morphism of groupoids that is bijective on connected components. Choose an object v_i in each connected component of A, and let f_i be the group homomorphism from $A_i = \pi_1(A, v_i)$ to $B_i = \pi_1(B, f(v_i))$ induced by f. Let Φ be the free product of the homomorphisms f_i. Then, for each groupoid G, the f-localization morphism $\eta_G\colon G \to L_fG$ is bijective on connected components and induces a Φ-localization of the group $\pi_1(G, v)$ at every object v.*

Proof. The fact that $\eta_G\colon G \to L_fG$ is bijective on connected components is implied by Theorem 2.7. Now fix any object v of G and consider the group homomorphism $\pi_1(G, v) \to \pi_1(L_fG, \eta_G(v))$ induced by η_G. Our aim is to prove that it is a Φ-localization.

First we need to show that a group H is f-local as a groupoid if and only if it is Φ-local as a group. Thus, let H be any group. If we write $A \cong \cup_i(A_i \times X_i)$ and $B \cong \cup_i(B_i \times Y_i)$ where all X_i and Y_i are contractible, we have

$$\operatorname{map}(B, H) \cong \operatorname{map}(\cup_i(B_i \times Y_i), H) \cong \prod_i \operatorname{map}(B_i \times Y_i, H) \simeq \prod_i \operatorname{map}(B_i, H),$$

and similarly with A. Therefore, $\operatorname{map}(B, H) \to \operatorname{map}(A, H)$ is a weak equivalence if and only if $\operatorname{map}(B_i, H) \to \operatorname{map}(A_i, H)$ is a weak equivalence for all i, since homotopy groups commute with products. This shows that, indeed, H is f-local as a groupoid if and only if it is f_i-local as a group for all i, by Theorem 3.1, or equivalently Φ-local, as Φ is the coproduct of the group homomorphisms f_i.

Now let C be the connected component of G which contains v, and D the connected component of $L_f G$ which contains $\eta_G(v)$. Hence, $D \cong \pi_1(L_f G, \eta_G(v)) \times K$ where K is a contractible groupoid. Since each connected component of $L_f G$ is f-local and K is contractible, we infer that $\pi_1(L_f G, \eta_G(v))$ is f-local as a groupoid, and therefore it is Φ-local as a group.

Finally, observe that the restriction $C \to D$ is a retract of $\eta_G \colon G \to L_f G$ and thus it is also an f-equivalence. Therefore, if H is any Φ-local group, hence f-local as a groupoid, we infer that η_G induces

$$\mathrm{map}(\pi_1(L_f G, \eta_G(v)), H) \simeq \mathrm{map}(D, H) \simeq \mathrm{map}(C, H) \simeq \mathrm{map}(\pi_1(G, v), H).$$

This shows that $\pi_1(G, v) \to \pi_1(L_f G, \eta_G(v))$ is a Φ-equivalence, as needed. □

Hence, localizing a groupoid G with respect to a morphism of groupoids f can be expressed, up to homotopy, as a localization of each member of a set of groups (one representative of each connected component of G) with respect to a group homomorphism (the free product of a set of representatives of f at connected components).

Example 3.4. Let f be the disjoint union of the homomorphisms $f_p \colon \mathbf{Z}/p \to \{0\}$ where p takes values in a set of primes P. Then, if $\eta_G \colon G \to L_f G$ denotes the f-localization of a groupoid G and v is any vertex of G, the group $\pi_1(L_f G, \eta_G(v))$ is obtained from $\pi_1(G, v)$ by factoring out the smallest normal subgroup for which the quotient is P-torsion-free.

4 Interaction with the fundamental groupoid

Every homotopy idempotent functor L on the category of simplicial sets gives rise to a distinguished class of groupoids, namely the class of groupoids G whose nerve NG is L-local. If $L = L_f$ for some map f (which is always the case if we assume the validity of Vopěnka's principle in set theory, according to [9]), then, as we next show, the corresponding distinguished class of groupoids is precisely the class of φ-local groupoids, where $\varphi = \pi f$ is the morphism induced by f on fundamental groupoids. This and other results of the present article were used in Section 6 of [9].

Proposition 4.1. *Let $f \colon X \to Y$ be any map of simplicial sets and let $\varphi = \pi f$. Then a groupoid G is φ-local if and only if NG is f-local.*

Proof. By definition, NG is f-local if and only if the map of simplicial sets induced by f,

$$\mathrm{map}(Y, NG) \to \mathrm{map}(X, NG),$$

is a weak equivalence, where "map" denotes the simplicial function complex. But the space $\mathrm{map}(X, NG)$ is a 1-type by [6], whose representing groupoid is $\mathrm{Fun}(\pi X, G)$, and the same happens with $\mathrm{map}(Y, NG)$. □

Corollary 4.2. *Let f be any map of simplicial sets, and let $\varphi = \pi f$. If a map $g\colon X \to Y$ is an f-equivalence of simplicial sets, then the morphism $\pi g\colon \pi X \to \pi Y$ is a φ-equivalence of groupoids.* □

This result extends [7, Proposition 3.3] from groups to groupoids. We record the following immediate consequences.

Corollary 4.3. *Let f be any map of simplicial sets, and let $\varphi = \pi f$. Then the natural morphism $\pi X \to \pi L_f X$ is a φ-equivalence for all X.*

Proof. This follows from Corollary 4.2, since the localization map $\eta_X\colon X \to L_f X$ is an f-equivalence for all X. □

Thus, we obtain a natural morphism of groupoids

$$L_\varphi \pi X \to \pi L_f X, \tag{4.3}$$

which is a φ-equivalence and hence yields an isomorphism

$$L_\varphi \pi X \cong L_\varphi \pi L_f X, \tag{4.4}$$

for all maps f of simplicial sets and every X. This is in fact a special case of a more general phenomenon relating localizations in model categories by means of adjunctions, which will be discussed in more detail elsewhere.

It follows from Theorem 2.6 and Corollary 4.3 that f-localizations of simplicial sets preserve connectivity. This is a well-known fact; see e.g. [21]. However, it is not yet known whether f-localizations preserve 1-connectivity or not. Note that, if πX is contractible, then it follows from (4.4) that $\pi L_f X$ is annihilated by L_φ. Thus, the open question is whether $\pi L_f X$ is necessarily contractible when X is 1-connected.

References

[1] J. Adámek and J. Rosický, Locally Presentable and Accessible Categories, London Math. Soc. Lecture Note Ser. vol. 189, Cambridge University Press, Cambridge, 1994.

[2] D. W. Anderson, Fibrations and geometric realizations, Bull. Amer. Math. Soc. **84** (1978), no. 5, 765–788.

[3] A. K. Bousfield, Constructions of factorization systems in categories, J. Pure Appl. Algebra **9** (1976/77), no. 2, 207–220.

[4] A. K. Bousfield, Homotopy spectral sequences and obstructions, Israel J. Math. **66** (1989), no. 1-3, 54–104.

[5] R. Brown, Fibrations of groupoids, J. Algebra **15** (1970), 103–132.

[6] R. Brown, P. J. Higgins, Tensor products and homotopies for ω-groupoids and crossed complexes, J. Pure Appl. Algebra **47** (1987), no. 1, 1–33.

[7] C. Casacuberta, Anderson localization from a modern point of view, in: The Čech Centennial; a Conference on Homotopy Theory (Boston, 1993), Contemp. Math. vol. 181, Amer. Math. Soc., Providence, 1995, 35–44.

[8] C. Casacuberta, On structures preserved by idempotent transformations of groups and homotopy types, in: Crystallographic Groups and their Generalizations (Kortijk, 1999), Contemp. Math. vol. 262, Amer. Math. Soc., Providence, 2000, 39–68.

[9] C. Casacuberta, D. Scevenels, J. H. Smith, Implications of large-cardinal principles in homotopical localization, Adv. Math. (to appear).

[10] E. Dror Farjoun, Higher homotopies of natural constructions, J. Pure Appl. Algebra **108** (1996), no. 1, 23–34.

[11] E. Dror Farjoun, Cellular Spaces, Null Spaces and Homotopy Localization, Lecture Notes in Math. vol. 1622, Springer-Verlag, Berlin Heidelberg New York, 1996.

[12] P. S. Hirschhorn, Model Categories and their Localizations, Math. Surveys Monographs vol. 99, Amer. Math. Soc., Providence, 2003.

[13] A. Joyal, M. Tierney, Strong stacks and classifying spaces, in: Category Theory (Como, 1990), Lecture Notes in Math. vol. 1488, Springer-Verlag, Berlin Heidelberg New York, 1991, 213–236.

[14] D. M. Kan, On c.s.s. complexes, Amer. J. Math. **79** (1957), 449–476.

[15] A. Libman, Localization of groups, G-modules and chain complexes, Masters' thesis, The Hebrew University of Jerusalem, 1997.

[16] S. Mac Lane, Categories for the Working Mathematician, Graduate Texts in Math. vol. 5, Springer-Verlag, New York, 1971.

[17] J. A. Neisendorfer, Localization and connected covers of finite complexes, in: The Čech Centennial: A Conference on Homotopy Theory (Boston, 1993), Contemp. Math. vol. 181, Amer. Math. Soc., Providence, 1995, 385–389.

[18] N. E. O'Sullivan, The genus and localization of groups, Ph.D. thesis, Queen Mary and Westfield College, London, 1996.

[19] D. G. Quillen, Homotopical Algebra, Lecture Notes in Math. vol. 43, Springer-Verlag, Berlin Heidelberg New York, 1967.

[20] P. Ribenboim, Torsion et localisation de groupes arbitraires, in: Séminaire d'Algèbre Paul Dubreil, Lecture Notes in Math. vol. 740, Springer-Verlag, Berlin Heidelberg New York, 1978, 444–456.

[21] J.-Y. Tai, On f-localization functors and connectivity, in: Stable and Unstable Homotopy (Toronto, 1996), Fields Inst. Commun. vol. 19, Amer. Math. Soc., Providence, 1998, 285–298.

[22] R. W. Thomason, Cat as a closed model category, Cahiers Topologie Géom. Différentielle **21** (1980), no. 3, 305–324.

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