ON THE EXTENDED GENUS OF FINITELY GENERATED ABELIAN GROUPS

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0. Introduction

In [9], Guido Mislin introduced the idea of the *genus* of a finitely generated nilpotent group N. This was defined as the set of isomorphism classes of finitely generated nilpotent groups M such that, for all primes p, the p-localizations of M and N are isomorphic,

$$M_p \cong N_p, \text{ all primes } p. \qquad (0.1)$$

Interesting examples have been given, by Milnor, Mislin, Hilton and others [4, 7, 9], of non-isomorphic groups M, N satisfying (0.1). However, it is easy to prove that no such examples are possible if N (and hence also M) is abelian. On the other hand, it would be reasonable to expect interesting examples, even in the abelian case, if only N is required to be finitely generated. Indeed, such examples, in the special case N = Z, are analyzed in [5]. Thus we are led to formulate the concept of the *extended genus* of N, written EG(N), where we no longer require that the groups in question be finitely generated — but we still require them to be nilpotent.

In this paper we are concerned primarily with the case when N is a given finitely generated abelian group A. In Section 1 we discuss the general case, but we soon confine attention to this special situation and prove that one may then in fact restrict oneself to the case $A = \mathbf{Z}^k$, the free abelian group of rank k. We say that an abelian group B is A-*like* if its isomorphism class is in EG(A). Then, if $A = \mathbf{Z}^k$, the group B is a torsionfree group of rank k, and such abelian groups have been extensively studied (see [1, 2, 3, 10]). However, the requirement that B be $\mathbf{Z}^k$-like imposes strong restrictions on B; for example, B cannot contain a non-zero element which, for some prime p, is divisible by arbitrarily high powers of p. Thus we are able to bring to bear on the study of B methods which do not seem to be applicable to the broad class of torsionfree abelian groups of finite rank. In particular, we obtain, in Section 2, *representations* of such a group B by means of a sequence M_* of matrices in $GL_k(\mathbf{Q})$, one for each prime p, and we are able to relate properties of B to properties of the sequence M_*.

In Section 3 we show how to impose an important restriction on the representing sequences M_* — without in any way restricting the $\mathbf{Z}^k$-like group B — and thus obtain a better insight into the group B itself. In particular, if the representing sequence is thus restricted — or, as we say, *reduced* — then we obtain an embedding $\mathbf{Z}^k \subseteq B \subseteq \mathbf{Q}^k$ and a convenient set of generators for B, so embedded. This enables us to give an effective criterion for when $B \cong \mathbf{Z}^k$.

This, and other, applications of the reduced representations R_* are given in Section 4. In particular, we obtain necessary and sufficient conditions for B to be *completely decomposable* and *almost completely decomposable* (see [1]). We also calculate Ext (B, **Z**) which may be said to measure the extent to which B departs from freeness.

It is to be expected that, in those parts of homotopy theory susceptible to the methods of localization, the advantages of assuming that the groups entering the discussion — principally, homotopy and homology groups — are finitely generated are shared if we merely assume that the groups are *like* finitely generated groups. This optimistic expectation has already been borne out in [6], where **Z** is replaced by an arbitrary **Z**-like group B, called a group of *pseudo-integers*, and thus the role of the circle (eg, in the study of circle bundles) is replaced by that of the 'pseudo-circle' K(B, 1) (in the study of K(B, 1)-bundles). We hope to return to this aspect in a subsequent paper.

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1. A-LIKE ABELIAN GROUPS

We may consider a fixed nilpotent group N and describe a nilpotent group M as N-*like* if $M_p \cong N_p$ for all primes p. We may further consider the set of isomorphism classes of N-like groups and call this the *extended genus* of N, written EG(N). The justification for this terminology is that the (Mislin) genus is confined to finitely generated nilpotent groups, and we certainly wish in this article to remove the restriction that our groups be finitely generated. By this generalization we render the concept of extended genus significant even when N is abelian.

We have already discussed the case $N = \mathbf{Z}$ in [5]; a $\mathbf{Z}$-like group is there called a group of *pseudo-integers*. Note that a nilpotent group which is $\mathbf{Z}$-like is necessarily abelian. This follows from the proposition (compare [7]):

Proposition 1.1. — *If* M, N *are nilpotent groups and* M *is* N-*like, then* nil M = nil N.

Proof. — We know that, for nilpotent groups, $\text{nil}\, N_p \leqslant \text{nil}\, N$. Thus $\prod_p N_p$ is nilpotent and $\text{nil} \prod_p N_p \leqslant \text{nil}\, N$. On the other hand N embeds in $\prod_p N_p$, so that $\text{nil}\, N \leqslant \prod_p N_p$. Thus $\text{nil}\, N = \text{nil} \prod_p N_p$. However, if M is N-like then $\prod_p M_p \cong \prod_p N_p$; thus nil M = nil N.

Remark. — Of course, we are not claiming that a group M such that $M_p \cong N_p$ for all p, with N nilpotent, is itself necessarily nilpotent. There are examples of non-trivial groups M such that M_p is trivial for all primes p. It is to be understood that our discussion is confined to the category of nilpotent groups. Indeed, we will shortly confine it to the category of abelian groups.

The following proposition shows that the extended genus of a torsion nilpotent group is trivial.

Proposition 1.2. — *If* M, N *are nilpotent groups and* M *is* N-*like, and if* N *is a torsion group, then* $M \cong N$.

Proof. — Since $M_0 \cong N_0$ and N_0 is trivial, M_0 is trivial, so M is a torsion group. Now for a torsion nilpotent group N, we know that N is the restricted direct product of its p-torsion subgroups; and the p-torsion subgroup of N is N_p. Thus $M = \prod_{\text{res}} M_p$, $N = \prod_{\text{res}} N_p$ and $M_p \cong N_p$, so that $M \cong N$.

We now show that we may reduce the study of EG(N) to the case of torsionfree nilpotent groups, together with a group extension problem. We write TN for the torsion subgroup of N and FN = N/TN.

Proposition 1.3. — *Let* M, N *be nilpotent groups and let* M *be* N-*like. Then* $TM \cong TN$ *and* FM *is* FN-*like*.

Proof. — We know that $(TN)_p = T(N_p)$; for the exact sequence $TN \rightarrowtail N \twoheadrightarrow FN$ p-localizes to the exact sequence $(TN)_p \rightarrowtail N_p \twoheadrightarrow (FN)_p$

and $(TN)_p$ is a torsion group while $(FN)_p$ is torsionfree. Thus we may suppress parentheses and write, unambiguously, TN_p, FN_p for $(TN)_p$ $(= T(N_p))$, $(FN)_p$ $(= F(N_p))$, respectively.

We have, for each p, an isomorphism $\theta_p : M_p \cong N_p$. Then θ_p restricts to an isomorphism $\theta'_p : TM_p \cong TN_p$, and induces an isomorphism θ''_p : $FM_p \cong FN_p$. This shows that TM is TN-like and FM is FN-like. The proposition now follows from Proposition 1.2.

We can solve the extension problem in an important special case. We confine ourselves now to abelian groups and we call the abelian group A *almost torsionfree* if $A_p = 0$ for almost all primes p ; of course, a finitely generated abelian group is almost torsionfree.

THEOREM 1.4. — *Let* B, A *be abelian groups and let* B *be* A*-like. If* A *is almost torsionfree, then there exists an isomorphism* θ : $\mathrm{Ext}(FB, TB) \cong \mathrm{Ext}(FA, TA)$ *such that* $\theta[B] = [A]$. (*Here* [A] *is the equivalence class of the extension* $TA \rightarrowtail A \twoheadrightarrow FA$.)

We first quote a lemma whose proof may be found in [7].

LEMMA 1.5. — *If* P *is a family of primes and if* K, L *are abelian groups with* L P*-local, then the* P*-localization* $e : K \to K_P$ *induces* e^* : $\mathrm{Ext}(K_P, L) \cong \mathrm{Ext}(K, L)$.

REMARK. — This lemma plays an important role in the proof of the fact that, if K is only assumed nilpotent, then e^*: $H^n(K_p ; L) \cong H^n(K ; L)$.

PROOF OF THEOREM 1.4. — Since A is almost torsionfree,

$$TA = \bigoplus_p TA_p = \prod_p TA_p.$$

Thus

$$\mathrm{Ext}(FA,TA) = \mathrm{Ext}(FA,\prod_p TA_p) = \prod_p \mathrm{Ext}(FA,TA_p) \xleftarrow{\prod e_p^*} \prod_p \mathrm{Ext}(FA_p, TA_p).$$

Moreover, under the isomorphism above, [A] corresponds to the element $([A_p])$. Now since $TB \cong TA$, B is also almost torsionfree, so $\mathrm{Ext}(FB, TB) \cong \prod_p \mathrm{Ext}(FB_p, TB_p)$ under an isomorphism sending [B] to $([B_p])$. We have (see the proof of Proposition 1.3) a commutative diagram, for each p,

$$\begin{array}{ccccc} TB_p & \rightarrowtail & B_p & \twoheadrightarrow & FB_p \\ \downarrow \theta'_p & & \downarrow \theta_p & & \downarrow \theta''_p \\ TA_p & \rightarrowtail & A_p & \twoheadrightarrow & FA_p \end{array}$$

It follows that $(\theta_p''^{-1}, \theta_p')$ induces an isomorphism $\tilde{\theta}_p$:

$$\text{Ext}\,(\text{FB}_p, \text{TB}_p) \cong \text{Ext}\,(\text{FA}_p, \text{TA}_p)$$

such that $\tilde{\theta}_p[\text{B}_p] = [\text{A}_p]$. Thus $\tilde{\theta} = \prod_p \tilde{\theta}_p$ is an isomorphism

$$\tilde{\theta} : \prod_p \text{Ext}\,(\text{FB}_p, \text{TB}_p) \cong \prod_p \text{Ext}\,(\text{FA}_p, \text{TA}_p)$$

such that $\tilde{\theta}([\text{B}_p]) = ([\text{A}_p])$. It follows that $\tilde{\theta}$ induces an isomorphism θ :

$$\text{Ext}\,(\text{FB}, \text{TB}) \cong \text{Ext}\,(\text{FA}, \text{TA})$$

under which $\theta[\text{B}] = [\text{A}]$.

Corollary 1.6. — *Let* B, A *be abelian groups and let* B *be* A-*like. If* A *is finitely generated, then*

$$\text{B} \cong \text{TB} \oplus \text{FB},$$

TB $\cong$ TA *and* FB *is* FA-like. *Moreover, any group* T $\oplus$ F, *where* T $\cong$ TA *and* F *is* FA-*like, is* A-*like.*

Proof. — A is almost torsionfree and Ext (FA, TA) = 0 since FA is free.

Thus to study EG(A), where A is finitely generated, it suffices to consider the case A = $\mathbf{Z}^k$, the free abelian group of rank k. This then is the focus of our attention in the remainder of this paper.

2. Sequential representation of a $\mathbf{Z}^k$-like-group

Throughout the rest of this paper k will be a fixed positive integer and B will denote a given $\mathbf{Z}^k$-like group, that is, an abelian group such that

$$\text{B}_p \cong \mathbf{Z}_p^k, \text{ for all primes } p. \tag{2.1}$$

Such a group B is torsionfree of rank k ; hence it can be embedded in $\mathbf{Q}^k$. However, we prefer not to assume from the outset that it is so embedded, since we will be interested more in the isomorphism class of B rather than in B itself. The collection of isomorphism classes of groups B satisfying (2.1) we call the *extended genus* of $\mathbf{Z}^k$ and write EG($\mathbf{Z}^k$).

The (easy) case $k = 1$ was fully analysed in [5]. There we saw that if, indeed, we assume, as we may, that $\mathbf{Z} \subseteq \text{B} \subseteq \mathbf{Q}$, then we may represent B by the sequence of non-negative integers $(n_1, n_2, \ldots, n_i, \ldots)$ in the following sense. We suppose the primes enumerated as $p_1, p_2, \ldots,$

p_i, ..., and then B is generated by the set of rationals $\left\{\frac{1}{p_i^{n_i}}, i = 1, 2, \dots\right\}$. Moreover, if also B′ is represented by the sequence $(n'_1, n'_2, \dots, n'_i, \dots)$, then $B \cong B'$ if and only if the corresponding sequences are *almost equal*, that is, if and only if $n_i = n'_i$ for almost all i. It turns out to be more consistent with the generalization to arbitrary $k \geqslant 1$ to regard the representing sequence for B as actually consisting of the rationals $\frac{1}{p_i^{n_i}}$ and this is the point of view we will adopt.

Thus, to generalize the representing sequence of a **Z**-like group, we start by choosing isomorphisms

$$f_p : B_p \cong \mathbf{Z}_p^k, \text{ for each prime } p\, ; f_0 : B_0 \cong \mathbf{Q}^k. \tag{2.2}$$

We will often write (2.2), for the sake of brevity,

$$\{f_p, p \geqslant 0\}. \tag{2.3}$$

Now B_p is naturally embedded in B_0, so $f_0 f_p^{-1}$ is a monomorphism,

$$f_0 f_p^{-1} : \mathbf{Z}_p^k \rightarrowtail \mathbf{Q}k. \tag{2.4}$$

We take the canonical basis $\{e_1, e_2, \dots, e_k\}$ for $\mathbf{Z}^k$, $\mathbf{Z}_p^k$, $\mathbf{Q}^k$; with respect to this basis, $f_0 f_p^{-1}$ is represented by a matrix M_p ; explicitly,

$$f_0 f_p^{-1}(e_j) = \sum_i a_{jp}^i e_i, \qquad M_p = (a_{jp}^i),\ a_{jp}^i \in \mathbf{Q}. \tag{2.5}$$

Since $f_0 f_p^{-1}$ is a monomorphism, $M_p \in GL_k(\mathbf{Q})$; since we will never alter the bases of $\mathbf{Z}^k$, $\mathbf{Z}_p^k$, $\mathbf{Q}^k$, we may even write $M_p : \mathbf{Z}_p^k \rightarrowtail \mathbf{Q}^k$, instead of (2.4).

Definition 2.1. — We write M_* for the sequence of matrices $\{M_{p_i}\}$ and call it the *sequential representation* (or, briefly, *representation*) of B associated with (2.3). We may regard M_* as an element of $\prod_p GL_k(\mathbf{Q})$.

Notice that this definition is evidently consistent with our terminology in the case $k = 1$. For if the representing sequence in that case consisted of the rationals $\frac{1}{p_i^{n_i}}$, then we may take $f_0 = Id$, f_{p_i} = multiplication by $p_i^{n_i}$ so that M_{p_i} is the (1×1)-matrix $\frac{1}{p_i^{n_i}}$.

Suppose now that

$$\{f'_p, p \geqslant 0\} \tag{2.3′}$$

is another set of isomorphisms $f'_p : B_p \cong \mathbf{Z}_p^k$, $f'_0 : B_0 \cong \mathbf{Q}^k$. Then if M'_* is the sequence of matrices associated with (2.3′), let $N \in GL_k(\mathbf{Q})$ be the matrix of $f_0'f_0^{-1}: \mathbf{Q}^k \cong \mathbf{Q}^k$ and $C_p \in GL_k(\mathbf{Z}_p)$ the matrix of $f_p f_p'^{-1}$: $\mathbf{Q}_p^k \cong \mathbf{Z}_p^k$. Since $f_0'f_p'^{-1} = (f_0'f_0^{-1})(f_0 f_p^{-1})(f_p f_p'^{-1})$, we have

$$M_p' = NM_pC_p, \text{ for all } p,\ N \in GL_k(\mathbf{Q}),\ C_p \in GL_k(\mathbf{Z}_p). \qquad (2.6)$$

Conversely, if M_* is a representation of B and M_*' is related to M_* by (2.6), then M_*' is also a representation of B; we simply define $f_p' = C_p^{-1}f_p : B_p \cong \mathbf{Z}_p^k$, $f_0' = Nf_0 : B_0 \cong \mathbf{Q}^k$.

Let us therefore declare two sequences of matrices M_* and M_*' to be *equivalent*, and write $M_* \sim M_*'$, if they are related as in (2.6). The equivalence classes may be regarded as constituting the set of double cosets

$$\Lambda = GL_k(\mathbf{Q}) \setminus \prod_p GL_k(\mathbf{Q}) / \prod_p GL_k(\mathbf{Z}_p), \qquad (2.7)$$

where $GL_k(\mathbf{Q})$ is embedded diagonally in $\prod_p GL_k(\mathbf{Q})$.

Now suppose that $\phi : B' \cong B$; and let M_* be the representation of B associated with $\{f_p, p \geqslant 0\}$. It is then plain that M_* is the representation of B' associated with $\{f_p\phi_p, p \geqslant 0\}$. The argument thus far establishes that the association of M_* with B sets up a function

$$\Phi : EG(\mathbf{Z}^k) \to \Lambda \qquad (2.8)$$

THEOREM 2.1. — *The function $\Phi : EG(\mathbf{Z}^k) \to \Lambda$ is injective.*

PROOF. — The injectivity of Φ follows from the following proposition which enables us to recover the isomorphism class of B from a representation M_* of B.

PROPOSITION 2.2. — *If M_* is a representation of B, then* $B \cong \bigcap_p \operatorname{Im} M_p$.

PROOF OF PROPOSITION. — We have $\operatorname{Im} M_p = f_0(B_p) = (f_0B)_p$. Now, for any torsionfree abelian [1] group A, where we may regard any localization A_p as embedded in A_0, we have

$$\bigcap_p A_p = A. \qquad (2.9)$$

[1] It would suffice that A be nilpotent; see [7].

Thus $\bigcap_p \operatorname{Im} M_p = f_0 B$. This shows that f_0 restricts to an isomorphism $B \cong \bigcap_p \operatorname{Im} M_p$.

Proposition 2.2 immediately leads to the conclusion of the proof of Theorem 2.1. For if B, B′ are $\mathbf{Z}^k$-like groups yielding equivalent representations M_*, M'_* then M_* represents B and B′, so $B \cong B' \cong \bigcap_p \operatorname{Im} M_p$. We will write H for $\bigcap_p \operatorname{Im} M_p$.

On the other hand, the function Φ of (2.8) is certainly not surjective. Indeed, Proposition 2.2 immediately suggests the following criterion. We consider a sequence M_* of matrices $M_p \in GL_k(\mathbf{Q})$.

THEOREM 2.3. — *The equivalence class of* M_* *is in the image of* Φ *if and only if* $H = \bigcap_p \operatorname{Im} M_p$ *has rank* k.

PROOF. — If M_* is a representation of B, then $\bigcap_p \operatorname{Im} M_p = H \cong B$, so H has rank k. Assume conversely that H has rank k. We will show that H is $\mathbf{Z}^k$-like and that M_* is a representation of H. The first objective is achieved by the following lemma.

LEMMA 2.4. — *The following statements are equivalent*

(i) rank $H = k$;
(ii) $\operatorname{Im} M_p = H_p$, *for all* p ;
(iii) $\operatorname{Im} M_p = H_p$, *for some* p.

PROOF OF LEMMA. — Obviously (ii) $\Rightarrow$ (iii). Now $M_p : \mathbf{Z}_p^k \to \mathbf{Q}^k$ is monomorphic, so (iii) implies that H_p has rank k for some p. But rank H = rank H_p so (iii) $\Rightarrow$ (i). It remains to show that (i) $\Rightarrow$ (ii). Consider the inclusion $j : H \subseteq \operatorname{Im} M_p$. Since $\operatorname{Im} M_p$ is p-local, it suffices to show that j is p-surjective.

Since rank $H = k$, there exists a $\mathbf{Q}$-basis $\{h_1, h_2, \ldots, h_k\}$ for $\mathbf{Q}^k$ consisting of elements of H. Let $x \in \operatorname{Im} M_p$ and set $x = \Sigma \lambda_i h_i$, $\lambda_i \in \mathbf{Q}$. Write $\lambda_i = \dfrac{a_i}{p^{m_i} b_i}$, where $p \nmid b_i$ and $b_i > 0$. Set $n = b_1 b_2 \ldots b_k$, so that n is a p'-number. Then

$$nx = \sum (n\lambda_i) h_i \quad \text{and} \quad n\lambda_i \in \mathbf{Z}\left[\frac{1}{p}\right] \subseteq \mathbf{Z}_q,\ q \neq p.$$

Thus $nx \in \operatorname{Im} M_q$, since $H \subseteq \operatorname{Im} M_q$ and $\operatorname{Im} M_q$ is q-local. But, of course, $nx \in \operatorname{Im} M_p$, so that $nx \in H$ and hence j is p-surjective. This completes the proof of the lemma.

We now complete the proof of the theorem by observing that, since $M_p : Z_p^k \cong H_p$, we may take $f_p = M_p^{-1} : H_p \cong Z_p^k, f_0 = \mathrm{Id} : H_0 = \mathbf{Q}^k$ and then the representation of H associated with $\{f_p, p \geqslant 0\}$ is $M*$.

We say that $M*$ is *realizable* if it is the representation of some group B. We have identified, through Theorem 2.3, the subset of Λ consisting of equivalence classes of realizable sequences. We give an example to show that not all sequences are realizable.

EXAMPLE 2.1. — Let $k = 1$ and $M_p = p$, for all p. If $\frac{a}{b}$ were a reduced fraction in $H = \bigcap_p \mathrm{Im}\, M_p$, then $p \mid a$, for all p, which is absurd. Thus $H = 0$ and $M*$ is not realizable. This accords with the fact (see [5]) that every group of pseudo-integers has a representation $\left\{\frac{1}{p_i^{n_i}}, n_i \geqslant 0\right\}$. Thus we would need $\frac{1}{p^n} = NM_pC_p$, for all p and this would imply that, for almost all p, M_p would be expressible as $\frac{a}{p^n b}$, $p \nmid a$, $p \nmid b$.

The criterion of realizability given by Theorem 2.3 is not very effective in practice. A more useful criterion, as we shall later demonstrate, is provided by the following theorem.

THEOREM 2.5. — *The sequence* $M*$ *is realizable if and only if* M_p^{-1} *has entries in* $\mathbf{Z}_p$ *for almost all* p.

PROOF. — In fact, the following statements are all equivalent (recall that $H = \bigcap_p \mathrm{Im}\, M_p$) :

(i) $M*$ is realizable (i.e., the equivalence class of $M*$ is in the image of Φ (2.8));

(ii) rank $H = k$;

(iii) H contains a $\mathbf{Q}$-basis $\{h_1, h_2, ..., h_k\}$ for $\mathbf{Q}^k$;

(iv) there is a $\mathbf{Q}$-basis $\{h_1, h_2, ..., h_k\}$ for $\mathbf{Q}^k$ and, for each p and each i, an element $c_{ip} \in \mathbf{Z}_p^k$ such that $h_i = M_p c_{ip}$;

(v) there exists a matrix $J \in GL_k(\mathbf{Q})$ such that $M_p^{-1}J$ has entries in $\mathbf{Z}_p$ for all p;

(vi) M_p^{-1} has entries in $\mathbf{Z}_p$ for almost all p;

(vii) there exists an integer $n \geqslant 1$ such that nM_p^{-1} has entries in $\mathbf{Z}_p$ for all p.

For Theorem 2.3 asserts that (i) $\Leftrightarrow$ (ii); the implications (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) are clear; likewise the implications (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (v).

In fact, we will also use criterion (vii) above in an essential way in the sequel.

REMARK. — Had we been content to consider subgroups of $\mathbf{Q}^k$ which are $\mathbf{Z}^k$-like, we could have eliminated the factor N in the equivalence relation (2.6). We would thus have set up a one-one correspondence between the set of such subgroups and a subset of the coset space $\prod_p \mathrm{GL}_k(\mathbf{Q}) \,/\, \prod_p \mathrm{GL}_k(\mathbf{Z}_p)$; moreover the subset would have been determined by the criteria of Theorems 2.3 or 2.5. However, this formulation would, of course, have been inadequate to handle the isomorphism problem. Nevertheless, we should remember that we may always take $f_0 = \mathrm{Id}$ in (2.3) when a specific subgroup of $\mathbf{Q}^k$ is in question.

Our final observation in this section shows us how to recognize, given two $\mathbf{Z}^k$-like groups B and B′, that there exists a monomorphism $j: \mathrm{B}' \rightarrowtail \mathrm{B}$.

THEOREM 2.6. — *Let* M_*, M'_* *be sequences representing* B, B′ *and associated with* $\{f_p, p \geqslant 0\}$, $\{f_p', p \geqslant 0\}$ *respectively. Then there exists* $j: \mathrm{B}' \rightarrowtail \mathrm{B}$ *with* $f_0' = f_0 j_0$ *if and only if* $\mathrm{M}_p^{-1}\mathrm{M}_p'$ *has entries in* $\mathbf{Z}_p$ *for each* p.

PROOF. — Suppose j exists. Then there exists, for each p, a monomorphism $\gamma_p : \mathbf{Z}_p^k \rightarrowtail \mathbf{Z}_p^k$ such that the diagram

$$\begin{array}{ccccccc}
\mathbf{Z}_p^k & \overset{f_p}{\twoheadleftarrow} & \mathrm{B}_p' & \hookrightarrow & \mathrm{B}_0' & \overset{f_0}{\rightarrowtail\!\!\!\!\twoheadrightarrow} & \mathbf{Q}^k \\
\downarrow \gamma_p & & \downarrow j_p & & \downarrow j_0 & & \| \\
\mathbf{Z}_p^k & \overset{f_p}{\twoheadleftarrow} & \mathrm{B}_p & \hookrightarrow & \mathrm{B}_0 & \overset{f_0}{\rightarrowtail\!\!\!\!\twoheadrightarrow} & \mathbf{Q}^k
\end{array}$$

commutes. Then $f_0 f_p^{-1}\gamma_p = f_0' f_p'^{-1}$, so that the matrix of γ_p is $\mathrm{M}_p^{-1}\mathrm{M}_p'$. This ensures that $\mathrm{M}_p^{-1}\mathrm{M}_p'$ has entries in $\mathbf{Z}_p$.

Conversely, if $\mathrm{M}_p^{-1}\mathrm{M}_p'$ has entries in $\mathbf{Z}_p$, it determines a monomorphism $\gamma_p : \mathbf{Z}_p^k \rightarrowtail \mathbf{Z}_p^k$ and hence a monomorphism $j(p) : \mathrm{B}_p' \rightarrowtail \mathrm{B}_p$ such that the diagram

$$\begin{array}{ccccccc} \mathbf{Z}_p^k & \overset{f_p}{\twoheadleftarrow\!\!\prec} & B'_p & \hookrightarrow & B'_0 & \overset{f'_0}{\rightarrowtail\!\!\!\!\twoheadrightarrow} & \mathbf{Q}^k \\ \downarrow \gamma_p & & \downarrow j(p) & & & & \| \\ \mathbf{Z}_p^k & \overset{f_p}{\twoheadleftarrow\!\!\prec} & B_p & \hookrightarrow & B_0 & \overset{f_0}{\rightarrowtail\!\!\!\!\twoheadrightarrow} & \mathbf{Q}^k \end{array} \quad \text{commutes.}$$

Set $j(0) = f_0^{-1}f_0' : B_0' \rightarrowtail\!\!\!\twoheadrightarrow > B_0$. Then the diagram

$$\begin{array}{ccc} B'_p & \hookrightarrow & B'_0 \\ \downarrow j(p) & & \downarrow j(0) \\ B_p & \hookrightarrow & B_0 \end{array}$$

commutes for each p. We may then invoke (2.9) to infer the existence of a monomorphism $j : B' \rightarrowtail B$ with $j_p = j(p)$, $p \geqslant 0$.

We remark that if j exists, then j_0 is an isomorphism, so we may always choose f_0, f_0' so that $f_0' = f_0 j_0$. In particular, if j is an inclusion, we take $f_0 = f_0'$.

3. Reduced representations

To obtain finer results on $\mathbf{Z}^k$-like groups, it is convenient to pick out, among the representations M_* of a given group B, certain special representations, which we call *reduced* representations and which are characterized by generalizations of important properties of the sequences associated with groups of pseudo-integers (the case $k = 1$). We will now make the key definition.

Definition 3.1. — A sequence $R_* \in \prod_p GL_k(\mathbf{Q})$ is *reduced* if

(i) $R_p \in GL_k\left(\mathbf{Z}\left[\frac{1}{p}\right]\right)$ for each p; and

(ii) R_p^{-1} has entries in $\mathbf{Z}$ for each p.

Notice that a reduced sequence is certainly realizable, by Theorem 2.5. We prove the converse :

Theorem 3.1. — *Every equivalence class of realizable sequences contains a reduced sequence.*

We first need a lemma.

LEMMA 3.2. — *Let* $\{P, Q\}$ *be an arbitrary partitioning of the set of primes. Then any matrix* $M \in GL_k(\mathbf{Q})$ *is expressible as* $M = RS$, *where* $R \in GL_k(\mathbf{Z}_Q)$, $S \in GL_k(\mathbf{Z}_P)$.

PROOF. — Choose a positive integer n so that nM has integer entries. We may then find matrices $U, V \in SL_k(\mathbf{Z})$ so that $D = U(nM)V$ is diagonal, say $D = \bigoplus_{i=1}^{k} (n_i)$. Write $n = rs$, $n_i = r_i s_i$, where r, r_i are P-numbers and s, s_i are Q-numbers, and set

$$R = \frac{1}{r} U^{-1} [\bigoplus_i (r_i)], \qquad S = \frac{1}{s} [\bigoplus_i (s_i)] V^{-1}.$$

Plainly $R \in GL_k(\mathbf{Z}_Q)$, $S \in GL_k(\mathbf{Z}_p)$ and $RS = M$.

REMARKS. — 1. Clearly we can generalize the lemma to any *finite* partitioning of the set of primes. We can even generalize it to an infinite partitioning, if suitably interpreted.

2. It is obvious, but important for the sequel, that if M were originally diagonal, we may assume R and S diagonal.

We now proceed to

PROOF OF THEOREM 3.1. — Let M_* be a realizable sequence. Then, according to Theorem 2.5 (formulation (vii)) we know that there exists $m \geqslant 1$ such that mM_p^{-1} has entries in $\mathbf{Z}_p$ for all p. Consider the matrix $m^{-1}M_p \in GL_k(\mathbf{Q})$. By Lemma 3.2. we may write

$$m^{-1}M_p = R_pS_p, \quad R_p \in GL_k\left(\mathbf{Z}\left[\frac{1}{p}\right]\right), \quad S_p \in GL_k(\mathbf{Z}_p).$$

Then, according to (2.6), $M_* \sim R_*$. Moreover,

$$R_p^{-1} = S_p(mM_p^{-1})$$

and thus has entries in $\mathbf{Z}_p$. Since $R_p^{-1} \in GL_k\left(\mathbf{Z}\left[\frac{1}{p}\right]\right)$, this implies that R_p^{-1} has entries in $\mathbf{Z}$.

REMARK. — We follow up Remark 2 by pointing out that if, for some p, M_p is diagonal then we may take R_p to be diagonal.

We will give a number of applications of reduced sequences in the next section. These are based on the facility which reduced sequences give us actually to describe a $\mathbf{Z}^k$-like group in terms of a reduced representation. We express the result as follows.

THEOREM 3.3. — *Let the group* B *have the reduced representation* R_*, *where* $R_p = (a^i_{jp})$, *for all p. Then* B *is isomorphic to* $H = \bigcap_p \operatorname{Im} R_p$ *and* H *is generated by the set*

$$\left\{ \sum_{i=1}^{k} a^i_{jp} e_i, \quad j = 1, 2, \ldots, k \,;\, all\ p \right\}. \tag{3.1}$$

PROOF. — We have only to show that, if K is the group generated by the elements (3.1.), then, for each prime p, K_p is the image of $g_p : \mathbf{Z}^k_p \to \mathbf{Q}^k$, given by $g_p e_j = R_p e_j$. Since $g_p(\mathbf{Z}^k) \subseteq K$ it follows that $g_p(\mathbf{Z}^k_p) \subseteq K_p$. Thus it remains to show that $K_p \subseteq g_p(\mathbf{Z}^k_p)$ or, equivalently, that $K \subseteq g_p(\mathbf{Z}^k_p)$. We proceed as follows :

(i) Obviously $\sum_{i=1}^{k} a^i_{jp} e_i \in \operatorname{Im} g_p, \quad j = 1, 2, \ldots, k$;

(ii) $e_i \in \operatorname{Im} g_p$, $i = 1, 2, \ldots, k$, because R_p^{-1} has integer entries ;

(iii) if q is a prime different from p, then $a^i_{jq} \in \mathbf{Z}\left[\frac{1}{q}\right] \subseteq \mathbf{Z}_p$,

so $\sum_{i=1}^{k} a^i_{jq} e_i \in \operatorname{Im} g_p$, which is p-local.

Thus $K_p = \operatorname{Im} g_p$, and the theorem is proved.

REMARKS. — 1. This result generalizes the fact that a group B of pseudo-integers, represented by the sequence $\left\{\frac{1}{p_i^{n_i}}\right\}$, is generated by the rationals $\left\{\frac{1}{p_i^{n_i}}\right\}$.

2. Since $e_1, e_2, \ldots, e_k \in H$, we see that we have an embedding $\mathbf{Z}^k \subseteq H \subseteq \mathbf{Q}^k$, that is, each $\mathbf{Z}^k$-like group B may be so embedded as a suitable H. This also generalizes an elementary remark in the case $k = 1$.

Let us now give an example.

EXAMPLE 3.1. — Let $k = 2$ and

$$M_p = \begin{pmatrix} 1 & \frac{1}{p} \\ 0 & \frac{1}{p^2} \end{pmatrix}$$

Then
$$M_p^{-1} = \begin{pmatrix} 1 & -p \\ 0 & p^2 \end{pmatrix},$$
so M_* is reduced and represents the subgroup of $\mathbf{Q}^2$ generated by the elements $\left\{e_1, \frac{1}{p}e_1 + \frac{1}{p^2}e_2, \text{ all } p\right\}$.

The following may be viewed as a counterexample.

EXAMPLE 3.2. — Let $k = 1$ and $M_p = \dfrac{1}{p+1}$. Then M_* is realizable but not reduced. Indeed, it is plain that $\operatorname{Im} M_p = \mathbf{Z}_p$, so that $\bigcap_p \operatorname{Im} M_p = \mathbf{Z}$. On the other hand, the group generated by
$$\left\{\frac{1}{p+1}, \text{ all } p\right\}$$
is certainly not finitely generated, so that the conclusion of Theorem 3.3 fails.

We now proceed to give several applications of the idea of reduced representations R_*, and of Theorem 3.3. As will be seen, reduced representations are considerably more useful than arbitrary (realizable) representations.

4. Applications of Reduced Representations

A. *Groups isomorphic to* $\mathbf{Z}^k$.

If B is $\mathbf{Z}^k$-like then we know that $B \cong \mathbf{Z}^k$ if and only if B is finitely generated. This enables us to prove

THEOREM 4.1. — *Let* B *be* $\mathbf{Z}^k$*-like ; then* $B \cong \mathbf{Z}^k$ *if and only if, for all representations* M_* *of* B, M_p *has entries in* $\mathbf{Z}_p$ *for almost all* p.

PROOF. — Suppose $B \cong \mathbf{Z}^k$. Then $M_* \sim I_*$, where $I_p = I$ for all p. Thus $M_p = NC_p$, $N \in GL_k(\mathbf{Q})$, $C_p \in GL_k(\mathbf{Z}_p)$, so that M_p has entries in $\mathbf{Z}_p$ for almost all p.

Conversely, suppose B is $\mathbf{Z}^k$-like and that, for any representation M_* of B, M_p has entries in Z_p for almost all p. Choose a reduced representation R_* of B. Then R_p has entries in $\mathbf{Z}_p$ for $p \notin \Sigma$, where Σ is a finite set of primes. But $R_p \in GL_k\left(\mathbf{Z}\left[\frac{1}{p}\right]\right)$ so R_p has entries in $\mathbf{Z}$ for $p \notin \Sigma$.

The conclusion of Theorem 3.3 now implies that $B \cong H$, where

$$H = \{e_1, e_2, ..., e_k, \Sigma a^i_{jp} e_i ; j = 1, 2, ..., k ; p \in \Sigma\}.$$

Thus H, and hence B, is finitely generated, so that $B \cong \mathbf{Z}^k$.

REMARK. — It is, in any case, easy to see that if any representation $M*$ of B has the property that M_p has entries in $\mathbf{Z}_p$ for almost all p, then all representations of B have this property.

B. *Completely decomposable and almost completely decomposable groups.*

We say that a torsionfree group B of rank k is *completely decomposable* (cd) if

$$B = \bigoplus_{i=1}^{k} B_i, \tag{4.1}$$

where B_i has rank 1. Of course, if B is $\mathbf{Z}^k$-like and (4.1) holds then each B_i is a group of pseudo-integers. We further say that B is *almost completely decomposable* (acd) if there exists $q \geqslant 1$ such that

$$qB \subseteq \bigoplus_{i=1}^{k} B_i \subseteq B. \tag{4.2}$$

Once again, (4.2) implies that each B_i is a group of pseudo-integers, in the case that B is $\mathbf{Z}^k$-like. The following example shows that a $\mathbf{Z}^k$-like group may fail to be acd.

EXAMPLE 4.1. — Let $A_1, A_2, ..., A_k$ be groups of pseudo-integers such that

(i) $\mathrm{Hom}\,(A_j, A_{j'}) = 0$ if $j \neq j'$; and

(ii) $\dfrac{1}{p_{2j}} \notin A_j$, for all i, j.

Let B be the subgroup of the vector space $\mathbf{Q}<x_1, x_2, ..., x_k>$ generated by $A_1x_1, A_2x_2, ..., A_kx_k$ and the elements $\dfrac{x_1 + x_2 + ... + x_k}{p_{2j}}$. It then follows, by an easy extension of the argument given in [1 ; Example 2.2. ; p. 21] that B is not acd. On the other hand, it is also easy to see that B is $\mathbf{Z}^k$-like. For the basis $\{x_1, x_2, ..., x_k\}$ provides an obvious embedding

$$i : \mathbf{Z}^k \rightarrowtail B ;$$

and it is plain that

$$i_p : \mathbf{Z}_p^k \cong B_p \text{ if } p = p_{2i+1},$$
$$\mathbf{Z}_p^k \overset{i_p}{\rightarrowtail} B_p \twoheadrightarrow \mathbf{Z}/p \text{ if } p = p_{2i},$$

so that, in any case, $\mathbf{Z}_p^k \cong B_p$.

We may, however, use our sequential representations to characterize those $\mathbf{Z}^k$-like groups which are cd (acd).

PROPOSITION 4.2. — *Let* B *be* $\mathbf{Z}^k$*-like and let* $l + m = k$. *Then* $B = B' \oplus B''$, *where* B' *is* $\mathbf{Z}^l$*-like,* B'' *is* $\mathbf{Z}^m$*-like, if and only if* B *admits a representation* M_* *where* $M_p = M_p' \oplus M_p''$, $M_p' \in GL_l(\mathbf{Q})$, $M_p'' \in GL_m(\mathbf{Q})$, *for all* p.

PROOF. — Let $B = B' \oplus B''$. Then we may choose isomorphisms $f_p : B_p \cong \mathbf{Z}_p^k$, $p \geqslant 0$ of the form $f_p' \oplus f_p''$, where $f_p' : B_p' \cong \mathbf{Z}_p^l$, $f_p'' : B_p'' \cong \mathbf{Z}_p^m$. This gives a representation M_* of the desired form.

Conversely, if B admits a representation of the given form, then $B \cong H = \bigcap_p \operatorname{Im} M_p = \bigcap_p (\operatorname{Im} M_p' \oplus \operatorname{Im} M_p'') = \bigcap_p \operatorname{Im} M_p' \oplus \bigcap_p \operatorname{Im} M_p''$. Now rank $\bigcap_p \operatorname{Im} M_p' \leqslant l$, rank $\bigcap_p \operatorname{Im} M_p'' \leqslant m$, rank $H = k = l + m$. Thus rank $\bigcap_p \operatorname{Im} M_p' = l$, rank $\bigcap_p \operatorname{Im} M_p'' = m$, so that M_*', M_*'' are realizable by $H' = \bigcap_p \operatorname{Im} M_p'$, $H'' = \bigcap_p \operatorname{Im} M_p''$ (see the proof of Theorem 2.3). Since $H = H' \oplus H''$, and H' is $\mathbf{Z}^l$-like, H'' is $\mathbf{Z}^m$-like, the proposition is proved.

An easy extension of Proposition 4.2. yields

THEOREM 4.3. — *Let* B *be* $\mathbf{Z}^k$*-like. Then* B *is* cd *if and only if* B *admits a representation* D_* *where each* D_p *is a diagonal matrix.*

The corresponding result for acd groups is the following.

THEOREM 4.4. — *Let* B *be* $\mathbf{Z}^k$*-like. Then* B *is* acd *if and only if* B *admits a representation* M_* *where* M_p *is diagonal for almost all* p.

PROOF. — Suppose (4.2) holds. Choose, according to Theorem 4.3, a family of isomorphisms $\{f_p \,;\, p \geqslant 0\}$, yielding a diagonal representation D_* of $\bigoplus_{i=1}^k B_i$. Choose a family of isomorphisms $g_p : B_p \cong \mathbf{Z}_p^k$, $p \geqslant 0$, such that $g_0 = f_0$ (this is possible since $(qB)_0 = \bigoplus_{i=1}^k B_{i0} = B_0$).

Since $\frac{1}{q}: qB \cong B$, $g_p\left(\frac{1}{q}\right)_p : (qB)_p \cong \mathbf{Z}_p^k$. Thus we may choose isomorphisms $h_p : (qB)_p \cong \mathbf{Z}_p^k$, $p \geqslant 0$, with

$$h_p = g_p\left(\frac{1}{q}\right)_p, \ p \neq 0 \,;\ h_0 = f_0.$$

If the isomorphisms g_p, $p \geqslant 0$, yield the representation M'_* of B, then it is plain that the isomorphisms h_p, $p \geqslant 0$, yield the representation qM'_* of qB; and by Theorem 2.6, the matrices $M_p'^{-1}D_p$ and $qD_p^{-1}M_p'$ have entries in $\mathbf{Z}_p$ for all p.

Now define

$$M_p = \begin{cases} D_p & \text{if } p \nmid q \\ M'_p & \text{if } p \mid q. \end{cases} \tag{4.3}$$

Since $D_p^{-1}M'_p$ has entries in $\mathbf{Z}_p$ if $p \nmid q$ it follows that $M_p'^{-1}D_p \in GL_k(\mathbf{Z}_p)$ if $p \nmid q$. Thus, by (4.3), $M_* \sim M'_*$ and M_p is diagonal for almost all p.

Conversely, suppose B admits a representation M_*, which is almost diagonal. By the Remark following the proof of Theorem 3.1., we may suppose this representation reduced. If $M_p = (a^i_{jp})$ then, by Theorem 3.3, $B \cong H$, where H is generated by the set

$$\left\{ \sum_{i=1}^{k} a^i_{jp} e_i,\ j = 1, 2, \ldots, k\,;\ \text{all } p \right\}.$$

Let Σ be the finite set of primes p such that M_p fails to be diagonal. Then it is obvious that H is generated by certain subgroups $B_1, B_2, \ldots, B_k$, together with the finite set of elements

$$\left\{ \sum_{i=1}^{k} a^i_{jp} e_i,\ j = 1, 2, \ldots, k\,;\ p \in \Sigma \right\},$$

where B_i is a group of pseudo-integers containing e_i, $i = 1, 2, \ldots, k$. If q is the lcm of the denominators of the entries a^i_{jp}, $i, j = 1, 2, \ldots, k$, $p \in \Sigma$, then $qH \subseteq B_1 \oplus B_2 \oplus \ldots \oplus B_k$, so that

$$qH \subseteq B_1 \oplus B_2 \oplus \ldots \oplus B_k \subseteq H,$$

and H, and hence B, is acd.

Remark. — We may incorporate Theorem 4.3 into Theorem 4.4 by observing that if B is acd, then M_* may be chosen to be diagonal except at the prime divisors of q. Conversely, if Σ (in the proof of Theorem 4.4) is empty, then plainly B is cd.

We give an example of a group provable by Theorem 4.4 not to be acd ; in fact, we revert to an earlier example.

EXAMPLE 4.2. — The (reduced) sequence M_*, where

$$M_p = \begin{pmatrix} 1 & \frac{1}{p} \\ 0 & \frac{1}{p^2} \end{pmatrix}$$

represents a group B which is not acd. Let us suppose that there exist matrices $N \in GL_2(\mathbf{Q})$, $C_p \in GL_2(\mathbf{Z}_p)$ such that $D_p = NM_pC_p$ is diagonal for almost all p. If D_p is diagonal for $p \in \Pi$, set

$$N = \begin{pmatrix} x & z \\ y & t \end{pmatrix}, \quad D_p^{-1} = \begin{pmatrix} a_p & 0 \\ 0 & b_p \end{pmatrix}, \quad p \in \Pi.$$

Then

$$C_p^{-1} = \begin{pmatrix} a_p x & \frac{1}{p^2} a_p(px + z) \\ b_p y & \frac{1}{p^2} b_p(py + t) \end{pmatrix} \in GL_2(\mathbf{Z}_p), \; p \in \Pi \qquad (4.4)$$

We prove that (4.4) must fail. We note first that, if $p \in \Pi$, then

$$\det C_p^{-1} = \frac{1}{p^2} a_p b_p(xt - yz) \text{ is a unit in } \mathbf{Z}_p.$$

Let v_p be the usual p-valuation in $\mathbf{Q}$. Since there exist infinitely many p such that $v_p(xt - yz) = 0$, it follows that there exist infinitely many p such that $v_p(a_p) + v_p(b_p) = 2$; we confine attention to these p, which constitute an infinite subset Π_0 of Π.

Now $z \neq 0$ or $t \neq 0$; by symmetry we may assume $z \neq 0$. Then there exists an infinite subset Π_1 of Π_0 such that $v_p(px + z) = 0$ if $p \in \Pi_1$; and there exists an infinite subset Π_2 of Π_1 such that $v_p(py + t) \leqslant 1$ if $p \in \Pi_2$.

Let $p \in \Pi_2$. Since $\frac{1}{p^2} a_p(px + z) \in \mathbf{Z}_p$, $v_p(a_p) \geqslant 2$. Hence $v_p(b_p) \leqslant 0$, so that $v_p\left(\frac{1}{p^2} b_p(py + t)\right) \leqslant -1$, contradicting $\frac{1}{p^2} b_p(py + t) \in \mathbf{Z}_p$. This establishes our claim that B is not acd.

C. Ext (B,**Z**) *and* Hom (B, **Q**/**Z**).

We first observe that if B is $\mathbf{Z}^k$-like, then we can write

$$B \cong B' \oplus \mathbf{Z}^l, \tag{4.5}$$

where $l \leqslant k$ and B′ *does not admit a* **Z**-*summand*. Moreover, l and the isomorphism class of B′ are uniquely determined by B, since we plainly have

PROPOSITION 4.5. — Hom (B,**Z**) = 0 ⇔ B *does not admit a* **Z**-*summand.*

PROOF. — Obviously, Hom (B, **Z**) ≠ 0 if B admits a **Z**-summand. Conversely, if there exists $\phi \neq 0 : B \to \mathbf{Z}$, then $\mathrm{Im}\,\phi \cong \mathbf{Z}$, so we have a short exact sequence $\ker \phi \rightarrowtail B \twoheadrightarrow \mathbf{Z}$, which must split.

It follows that, if (4.5) holds, then Hom (B, **Z**) ≅ $\mathbf{Z}^l$, so that l is uniquely determined by B. It now follows from [11 ; Theorem 7] that B′ is determined up to isomorphism by (4.5).

Thus we see that, in studying Ext (B, **Z**) and Hom (B, **Q**/**Z**) for a given $\mathbf{Z}^k$-like group B, $k \geqslant 1$, we may assume Hom (B, **Z**) = 0. *Thus we will make this assumption in this subsection*, and emphasize that our assumption involves no significant loss of generality. We first prove

PROPOSITION 4.6. — *If* Hom (B,**Z**) = 0, *then*

$$\mathrm{Hom}\,(B, \mathbf{Q}/\mathbf{Z}) \cong \mathbf{Q}^k \oplus \mathrm{Ext}\,(B,\mathbf{Z})$$

PROOF. — We have the exact sequence

$$0 \to \mathrm{Hom}\,(B, \mathbf{Z}) \to \mathrm{Hom}\,(B, \mathbf{Q}) \to \mathrm{Hom}\,(B, \mathbf{Q}/\mathbf{Z}) \to \mathrm{Ext}\,(B, \mathbf{Z}) \to 0.$$

However, Hom (B, **Z**) = 0 and Hom (B,**Q**) ≅ $\mathbf{Q}^k$, since any homomorphism $B \to \mathbf{Q}$ has a unique extension $\mathbf{Q}^k \to \mathbf{Q}$. Since $\mathbf{Q}^k$ is injective, the exact sequence

$$0 \to \mathbf{Q}^k \to \mathrm{Hom}\,(B, \mathbf{Q}/\mathbf{Z}) \to \mathrm{Ext}\,(B, \mathbf{Z}) \to 0$$

splits.

We now prove

PROPOSITION 4.7. — *If* Hom (B, **Z**) = 0, *then*

$$\mathrm{Ext}\,(B, \mathbf{Z}) \cong V \oplus T\,(\mathrm{Ext}\,(B, \mathbf{Z})),$$

where V *is a* **Q**-*vector space of uncountable rank and* TA *is the torsion subgroup of* A.

PROOF. — Since B is torsionfree, Ext (B, Z) is divisible. Thus T (Ext (B, Z)) splits off and

$$\mathrm{Ext}\,(\mathrm{B}, \mathbf{Z}) \cong \mathrm{F}(\mathrm{Ext}\,(\mathrm{B}, \mathbf{Z})) \oplus \mathrm{T}(\mathrm{Ext}(\mathrm{B}, \mathbf{Z})),$$

where $\mathrm{FA} = \mathrm{A}/\mathrm{TA}$; moreover, F(Ext (B, Z)) is torsionfree and divisible, and hence a **Q**-vector space.

We may now argue as in the proof of the theorem of Stein-Serre [8] that, if B is not free,

$$\mathrm{card\ Hom}\,(\mathrm{B}, \mathbf{Q}) = \aleph_0,$$

$$\mathrm{card\ Hom}\,(\mathrm{B}, \mathbf{Q}/\mathbf{Z}) = \underline{c},$$

so that

$$\mathrm{card\ Ext}\,(\mathrm{B},\mathbf{Z}) = \underline{c}.$$

Now we will see below (Theorem 4.8) that, since, by Proposition 4.6,

$$\mathrm{T}(\mathrm{Ext}\,(\mathrm{B}, \mathbf{Z})) \cong \mathrm{T}(\mathrm{Hom}\,(\mathrm{B}, \mathbf{Q}/\mathbf{Z})),$$

$$\mathrm{card\ T}(\mathrm{Ext}\,(\mathrm{B},\mathbf{Z})) = \mathrm{card\ T}(\mathrm{Hom}\,(\mathrm{B}, \mathbf{Q}/\mathbf{Z})) = \aleph_0.$$

Thus card F(Ext (B Z)) = $\underline{c}$, and Proposition 4.7 is proved.

Our study of Ext (B, Z) and Hom (B, Q/Z) is completed by

THEOREM 4.8. — *If* B *is* $\mathbf{Z}^k$*-like, then*

$$\mathrm{T}(\mathrm{Hom}\,(\mathrm{B}, \mathbf{Q}/\mathbf{Z})) \cong (\mathbf{Q}/\mathbf{Z})^k.$$

(Note that we do not require Hom (B, Z) = 0 *in this theorem.)*

PROOF. — We choose a reduced sequential representation R_* for B as in Theorem 3.3. We replace B by its isomorph H, so that B is generated by the elements $\sum_i a^i_{jp}e_i$, $j = 1, 2, \ldots, k$, all p. We use R_* to define, for each prime p, an isomorphism

$$\Psi_p : \mathrm{T}_p(\mathrm{Hom}\,(\mathrm{B}, \mathbf{Q}/\mathbf{Z})) \cong (\mathbf{Z}/p^\infty)^k,$$

by the rule

$$\Psi_p(f) = \{f(\Sigma a^i_{jp}e_i)\}_j \tag{4.6}$$

Notice that, since $p^n f = 0$ for some n, it follows that $p^n f(\Sigma a^i_{jp}e_i) = 0$, so that $\Psi_p(f) \in (\mathbf{Z}/p^\infty)^k$. We prove that Ψ_p is bijective. To show it injective, let $\Psi_p(f) = 0$. Then $f(\Sigma a^i_{jp}e_i) = 0$, $j = 1, 2, \ldots, k$, so that $f(e_i) = 0$, $i = 1, 2, \ldots, k$. Let q be a prime different from p, and let m be chosen so that $q^m\mathrm{R}_q$ has entries in **Z**—recall that R_q has entries in

$\mathbf{Z}\left[\frac{1}{q}\right]$. Then $q^m f(\Sigma a^i_{jq} e_i) = 0$. But, for some n, $p^n f = 0$, so $p^n f(\Sigma a^i_{jq} e_i) = 0$. Since p, q are distinct primes we infer that $f(\Sigma a^i_{jq} e_i) = 0$. Thus $f = 0$ and hence Ψ_p is injective.

To show Ψ_p surjective, let $(x_1, x_2, ..., x_k) \in (\mathbf{Z}/p^\infty)^k$, and let n be such that $p^n x_j = 0$, $j = 1, 2, ..., k$. Define

$$g : \langle \Sigma a^i_{jp} e_i,\ j = 1, 2, ..., k \rangle \to \mathbf{Z}/p^\infty$$

by

$$g : (\Sigma a^i_{jp} e_i) = x_j,$$

note that g does define a homomorphism because the elements $\Sigma a^i_{jp} e_i$, $j = 1, 2, ..., k$, are linearly independent. Extend g to $f : \mathrm{B} \to \mathbf{Z}/p^\infty$. We must show that $f \in \mathrm{T}_p(\mathrm{Hom}\,(\mathrm{B}, \mathbf{Q}/\mathbf{Z}))$. Now $p^n f(\Sigma a^i_{jp} e_i) = 0$. Thus $p^n f(e_i) = 0$, $i = 1, 2, ..., k$. Let q be a prime distinct from p. With m chosen as above, we infer that $q^m p^n f(\Sigma a^i_{jq} e) = 0$. But q^m determines an automorphism of $\mathbf{Z}/p^\infty$, so that $p^n f(\Sigma a^i_{jq} e_i) = 0$. Thus $p^n f = 0$ and $f \in \mathrm{T}_p(\mathrm{Hom}\,(\mathrm{B}, \mathbf{Q}/\mathbf{Z}))$. Obviously $\Psi_p(f) = (x_1, x_2, ..., x_k)$. Finally, the isomorphisms Ψ_p fit together to produce the required isomorphism

$$\Psi : \mathrm{T}(\mathrm{Hom}(\mathrm{B}, \mathbf{Q}/\mathbf{Z})) \cong (\mathbf{Q}/\mathbf{Z})^k.$$

REMARK. — Of course, it is clear a priori that g admits a *unique* extension f.

We may now give a complete description of Ext (B, **Z**) for any $\mathbf{Z}^k$-like group B. First, write B, uniquely, as $\mathrm{B}' \oplus \mathbf{Z}^l$ as in (4.5), where $l \leqslant k$ and $\mathrm{Hom}\,(\mathrm{B}', \mathbf{Z}) = 0$. If $l = k$, then $\mathrm{B} = \mathbf{Z}^k$ and $\mathrm{Ext}\,(\mathrm{B}, \mathbf{Z}) = 0$. Otherwise, let $l + m = k$, so that B′ is $\mathbf{Z}^m$-like, $m \geqslant 1$. Then, by Propositions 4.6, 4.7 and Theorem 4.8, $\mathrm{Ext}\,(\mathrm{B}, \mathbf{Z}) \cong \mathrm{Ext}\,(\mathrm{B}', \mathbf{Z}) = \mathrm{V} \oplus \mathrm{T}(\mathrm{Ext}\,(\mathrm{B}', \mathbf{Z}) \cong \mathrm{V} \oplus \mathrm{T}(\mathrm{Hom}\,(\mathrm{B}', \mathbf{Q}/\mathbf{Z})) \cong \mathrm{V} \oplus (\mathbf{Q}/\mathbf{Z})^m$; thus

$$\mathrm{Ext}\,(\mathrm{B}, \mathbf{Z}) \cong \mathrm{V} \oplus (\mathbf{Q}/\mathbf{Z})^m, \tag{4.7}$$

where V is a **Q**-vector space of uncountable rank ; on the other hand, again assuming $\mathrm{B} = \mathrm{B}' \oplus \mathbf{Z}^l$, with $\mathrm{Hom}\,(\mathrm{B}', \mathbf{Z}) = 0$ and $l + m = k$, we find $\mathrm{Hom}\,(\mathrm{B}, \mathbf{Q}/\mathbf{Z}) = (\mathbf{Q}/\mathbf{Z})^k$ if $l = k$ (so that $\mathrm{B} = \mathbf{Z}^k$) ; and, otherwise,

$$\mathrm{Hom}\,(\mathrm{B}, \mathbf{Q}/\mathbf{Z}) \cong \mathrm{V} \oplus (\mathbf{Q}/\mathbf{Z})^k. \tag{4.8}$$

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