On nilpotent groups which are finitely generated at every prime

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0 Introduction

Certain theorems of algebraic topology, as of abelian or nilpotent group theory, depend for their validity on the assumption that groups which enter into their statement are finitely generated. Consider, for example, the following theorem on fibre spaces:

Theorem 0.1 *Let $F \xrightarrow{i} E \xrightarrow{g} B$ be a fibration with F, E connected and B simply-connected, and let*

$$\begin{array}{ccccc} F & \xrightarrow{i} & E & \xrightarrow{g} & B \\ \downarrow f_F & & \downarrow f_E & & \downarrow f_B \\ F & \xrightarrow{i} & E & \xrightarrow{g} & B \end{array}$$

be a self-map of the fibration. Then if $(f_B)_* : H_*(B) \cong H_*(B)$ and if $(f_E)_* : i_*H_*(F) \cong i_*H_*(F)$, then $(f_F)_* : H_*(F) \cong H_*(F)$ and $(f_E)_*: H_*(E) \cong H_*(E)$.

As pointed out by J. M. Cohen ([4]), this theorem is false in general but becomes true if one adds the condition that the spaces F, E, B have finitely generated homology groups.

We have recently noticed that theorems like Theorem 0.1 can be rendered valid by imposing a condition much weaker than that of finite generation on the abelian (or, more generally, nilpotent) groups involved; namely, one assumes that such an abelian group is *finitely generated at every prime*.

To explain this notion, consider a P-local nilpotent group N, where P is a family of primes. We say that a collection of elements $\{x_i\}$ in N *generate* the P-local subgroup M of N if M is the smallest P-local subgroup of N containing the elements x_i. We then say that N is *finitely generated* (fg) *as a P-local group* if there exists a finite set $\{x_1, x_2, \ldots, x_k\}$ of elements of N which generate it in the above sense. We further say that the nilpotent group G is *finitely generated at every prime* (fgp) if G_p is a finitely generated p-local group for all primes p. Of course, fg nilpotent groups are fgp, but the converse is false. Indeed, as shown in [5], there are even uncountably many isomorphism classes of abelian groups B such that $B_p \cong \mathbf{Z}_p$ for all primes p.

Among the fgp nilpotent groups are to be found the groups B which are N-like for a given fg nilpotent group N, in the sense that $B_p \cong N_p$ for all primes p; we would also say that B is in the *extended genus* of N, written $B \in EG(N)$. We should, more accurately, describe the isomorphism class of B as belonging to $EG(N)$.

It was shown in [2] that the study of the extended genera $EG(A)$, where A is a given fg abelian group, reduces to the study of $EG(\mathbf{Z}^k)$, where $\mathbf{Z}^k$ is free abelian of rank k. Of course, there are abelian groups B which are fgp without being A-like for some fg abelian group A. Indeed, one readily proves:

Theorem 0.2 *Let B be a fgp abelian group. Then B is A-like for some fg abelian group A if and only if B is almost torsionfree, that is, $TB_p = 0$ for almost all primes p.*

In Section **1** we study criteria for a torsionfree abelian group of rank k to be $\mathbf{Z}^k$-like. Here we would like to mention the criterion of *coatomicity*, drawn to our attention by H. Zöschinger, and the notion of *relative height*, which is a kind of relative valuation on the elements of $\mathbf{Q}^k$ in a given embedding

$$\mathbf{Z}^k \cong A \subseteq B \subseteq \mathbf{Q}^k \tag{0.1}$$

According to [8], an R-module M is *coatomic* if each proper submodule of M is contained in a maximal submodule of M; or, equivalently, if each non-trivial quotient module of M has maximal submodules. In the case in which $R = \mathbf{Z}$, so that we are dealing with abelian groups, it is plain that K is a maximal proper subgroup of H if and only if H/K is simple, that is, if and only if $H/K = \mathbf{Z}/p$ for some prime p. It turns out, not surprisingly, that a torsionfree abelian group of rank k is $\mathbf{Z}^k$-like if and only if it is coatomic (see Theorem 1.1).

Now given $0 \neq x \in \mathbf{Q}^k$ in (0.1), there is the familiar p-valuation, or *p-height*, of x rel B which we designate $h_p^B(x)$. We relativize this notion by introducing the relative p-valuation $h_p^{B,A}(x)$, defined by

$$h_p^{B,A}(x) = h_p^B(x) - h_p^A(x). \tag{0.2}$$

It is then plain that

$$h_p^{B,A}(x) = h_p^{B,A}(px)$$

so that it becomes meaningful to talk, in this relative sense, of the non-zero elements of B having *bounded* relative p-heights $h_p^{B,A}$. Indeed, we show that a torsionfree abelian group B of rank k is $\mathbf{Z}^k$-like if and only if, for each prime p, there exists a (non-negative) integer $N(p)$ such that

$$h_p^{B,A}(x) \leq N(p) \quad \text{for all } 0 \neq x \in B$$

(see Theorem 1.4).

In Section **2** we study fgp nilpotent groups in general; of course, our results there have relevance for $\mathbf{Z}^k$-like groups. We draw particular attention to the following unexpected phenomenon (see Corollary 2.3).

Theorem 0.3 *Let $\varphi\colon G \to G$ be an endomorphism of the nilpotent group G, let $x \in G$, and let $H_\varphi(x)$ be the orbit group of x under φ, that is,*

$$H_\varphi(x) = \langle x, \varphi x, \ldots, \varphi^n x, \ldots \rangle.$$

Then if $H_\varphi(x)$ is fgp, it is fg.

Thus, in particular, we see that, while there are uncountably many isomorphism classes of $\mathbf{Z}^k$-like groups, the only such isomorphism class which can occur as an orbit group for an endomorphism of a nilpotent group is that of $\mathbf{Z}^k$ itself. The study of those groups occurring as orbit groups thus becomes a subject of interest. Abelian orbit groups are easy to characterize: they are precisely the cyclic modules over the polynomial ring $\mathbf{Z}[x]$. This fact provides them with an obvious ring structure. Theorem 0.3 shows that nilpotent orbit groups are also strongly restricted, but we do not have an explicit characterization of them.

Theorem 0.3 is related to the principal group-theoretical notions discussed in [6], [3]. One says that φ is *finitary* if $H_\varphi(x)$ is fg for all $x \in G$; and that the automorphism φ is *special* if φ^{-1} is finitary — or, equivalently, if $x \in \varphi H_\varphi(x)$ for all $x \in G$. Thus φ is a *pseudo-identity* ([6]) if and only if φ is finitary and special[1]. Theorem 0.3 has the obvious consequence, in an obvious terminology, that an endomorphism of a nilpotent group G is finitary (special, a pseudo-identity) if it is finitary (special, a pseudo-identity) at every prime. The converse also holds. For, given $y \in G_p$, we may write $y^n = e_p x$ for some p'-number n and some $x \in G$, so that (see (2.3) and (2.4)) $H_{\varphi_p}(y) = H_\varphi(x)_p$. Hence, if φ is finitary, then φ_p is finitary for every p.

The first-named author gratefully acknowledges support from the Comissió Interdepartamental de Recerca i Innovació Tecnològica, CIRIT.

1 Characterizations of $\mathbf{Z}^k$-like groups

Let B be a subgroup of $\mathbf{Q}^k$ of rank k, let $\{x_1, x_2, \ldots, x_k\}$ be a maximal linearly independent set in B, and let $A = \langle x_1, x_2, \ldots, x_k \rangle$. We may recognize whether B is $\mathbf{Z}^k$-like by any of the criteria embodied in the following theorem.

Theorem 1.1 *If B, A are as above, then the following statements are equivalent:*

(i) *B is $\mathbf{Z}^k$-like;*

(ii) *B_p is fg for each prime p;*

(iii) *$(B/A)_p$ is finite for each prime p;*

[1]The idea of separating the notion of pseudo-identity into its two "constituent" attributes of being *finitary* and *special* is due to Francesco Caserta.

(iv) $(B/A)_p$ *has finite exponent for each prime* p*;*

(v) B *is coatomic;*

(vi) B *has no non-trivial divisible quotient.*

PROOF. Since B_p is a torsionfree rank k $\mathbf{Z}_p$-module, it is clear that $B_p \cong \mathbf{Z}_p^k$ if and only if B_p is fg. Thus (i) $\Leftrightarrow$ (ii). Since B/A is a torsion group and A is fg, it is clear that $(B/A)_p$ is finite if and only if B_p is fg. Thus (ii) $\Leftrightarrow$ (iii).

Now B is coatomic if and only if every non-zero quotient H has maximal subgroups. But

$$\begin{aligned} & H \quad \text{has maximal subgroups} \quad \Leftrightarrow \\ \Leftrightarrow \quad & H \quad \text{has a } \mathbf{Z}/p \text{ quotient for some } p \quad \Leftrightarrow \\ \Leftrightarrow \quad & H \quad \text{is not divisible.} \end{aligned} \tag{1.1}$$

For if H is not divisible then $pH \neq H$ for some prime p and then H/pH is a non-trivial $\mathbf{Z}/p$-vector space. Thus B is coatomic if and only if no non-trivial quotient is divisible, so (v) $\Leftrightarrow$ (vi). Plainly (iii) $\Rightarrow$ (iv).

To complete the argument, observe that, if $T = B/A$, then

(a) if T_p is infinite then T_p has a $\mathbf{Z}/p^\infty$ quotient; and

(b) if T_p has finite exponent there is no surjection $B_p \twoheadrightarrow \mathbf{Z}/p^\infty$.

To see (a), we note that $T_p \subseteq (\mathbf{Q}^k/A)_p = (\mathbf{Z}/p^\infty)^k$. If T_p is infinite, some projection $\pi_i : T_p \to \mathbf{Z}/p^\infty$, $1 \leq i \leq k$, has infinite range; but then $\pi_i : T_p \twoheadrightarrow \mathbf{Z}/p^\infty$. To see (b), we note that, if T_p has exponent p^n and if $\kappa\colon B_p \twoheadrightarrow \mathbf{Z}/p^\infty$, then $\kappa|_{A_p}\colon A_p \twoheadrightarrow \mathbf{Z}/p^\infty$. For if $y \in \mathbf{Z}/p^\infty$, let $p^n z = y$, $z = \kappa x$, $x \in B_p$. Then $y = \kappa(p^n x)$ and $p^n x \in A_p$. But, of course, there can be no surjection $A_p \twoheadrightarrow \mathbf{Z}/p^\infty$.

Now if B has a non-trivial divisible quotient, then B_p has a $\mathbf{Z}/p^\infty$ quotient for some p. This contradicts (ii), so (ii) $\Rightarrow$ (vi). If T_p is infinite then T, and hence B, has a $\mathbf{Z}/p^\infty$ quotient, by (a) above, so (vi) fails. Thus (vi) $\Rightarrow$ (iii). Finally if (vi) fails, then, as above, B_p has a $\mathbf{Z}/p^\infty$ quotient for some p, so (iv) fails, by (b) above. Thus (iv) $\Rightarrow$ (vi) and Theorem 1.1 is proved. $\square$

Remark. Conditions (ii), (v), (vi) are plainly intrinsic to B. On the other hand, conditions (iii) and (iv) depend on the choice of maximal linearly independent set in B — or, at least, on the group A they generate. Although the group B/A does depend on this choice, it is not difficult to see directly that the qualitative properties described in (iii) and (iv) do not.

We now proceed to give a very concrete characterization of $\mathbf{Z}^k$-like groups in terms of the p-heights of their elements. If B is, as before, a rank k subgroup of $\mathbf{Q}^k$, we may assign a *p-valuation* (see [1]) to the non-zero elements of $\mathbf{Q}^k$ by defining

$$h_p^B(x) = \sup\{r \mid \exists y \in B_p \quad \text{with} \quad p^r y = x\}, \quad 0 \neq x \in \mathbf{Q}^k. \tag{1.2}$$

Thus $h_p^B(x)$ is an integer or infinity. Plainly $h_p^B(x) \geq 0$ if $x \in B_p$, and $h_p^B(x) = h_p^{B_p}(x)$. Also we observe that

$$h_p^B(px) = h_p^B(x) + 1, \quad 0 \neq x \in \mathbf{Q}^k. \tag{1.3}$$

Now it is clear from (1.2) that if $B_p \cong \mathbf{Z}_p^k$ then no element of $\mathbf{Q}^k$ can have an infinite p-valuation h_p^B. On the other hand, an example due to I. Kaplansky ([7], Theorem 19, p. 46) shows that there is a rank 2 subgroup of $\mathbf{Q}^2$ which contains no element with infinite p-valuation but which has $\mathbf{Z}/p^\infty$ as a quotient. Thus we need a refinement of the notion of p-valuation in order to characterize $\mathbf{Z}^k$-like groups.

The intuitive idea is that, in $\mathbf{Z}^k$-like groups, the p-heights of elements are *bounded*. To give precision to this idea, we introduce a relativization of the notion of p-valuation, in the following definition.

Definition 1.1 Let A, B, with $A \subseteq B$, be two rank k subgroups of $\mathbf{Q}^k$. Then, for each $x \in \mathbf{Q}^k$ such that $h_p^A(x) < \infty$, we introduce a *relative p-valuation* by defining

$$h_p^{B,A}(x) = h_p^B(x) - h_p^A(x).$$

If $x \in B$ we call $h_p^{B,A}(x)$ the *p-height of x rel A*.

Notice that if $A_p \cong \mathbf{Z}_p^k$ then $h_p^A(x) < \infty$ for all non-zero x in $\mathbf{Q}^k$, so that $h_p^{B,A}(x)$ is always defined. We collect some elementary properties of $h_p^{B,A}$ in the following portmanteau lemma.

Lemma 1.2 (a) $h_p^{B,A}(x) \geq 0$ *whenever it is defined;*

(b) $h_p^{B,A} = h_p^{B_p,A} = h_p^{B_p,A_p}$*;*

(c) $h_p^{B,A}(x) = h_p^{B,A}(px)$.

Proof. (a) follows from the inclusion $A_p \subseteq B_p$; (b) is obvious; (c) follows from (1.3). $\square$

We now assign to A its meaning in Theorem 1.1; thus $\{x_1, x_2, \ldots, x_k\}$ is a linearly independent set in B, and $A = \langle x_1, x_2, \ldots, x_k \rangle$. We set $T = B/A$ as before. We now prove

Theorem 1.3 $h_p^{B,A}(x) \leq N$ *for all* $0 \neq x \in B$ *if and only if* $p^N B_p \subseteq A_p$.

Proof. Suppose $h_p^{B,A}(x) \leq N$ for all $0 \neq x \in B$. If $0 \neq y \in B_p$, there exists a p'-number m such that $my \in B$. Then $h_p^{B,A}(my) \leq N$ so that $h_p^A(my) \geq -N$. Thus $p^N(my) \in A_p$, so that $p^N y \in A_p$, and it follows that $p^N B_p \subseteq A_p$.

Conversely, suppose that $p^N B_p \subseteq A_p$. If there existed $x \in B$ with $h_p^{B,A}(x) \geq N+1$, then

$$h_p^B(x) - h_p^A(x) \geq N+1,$$

$$\text{so} \quad N - h_p^B(x) \leq -(h_p^A(x)+1).$$

Now $p^{-h_p^B(x)}x \in B_p$, so $p^{N-h_p^B(x)}x \in A_p$. Thus $p^{-(h_p^A(x)+1)}x \in A_p$, contradicting the definition of h_p^A. $\square$

We are now able to prove our main theorem.

Theorem 1.4 *Let B, A be as in Theorem 1.1. Then B is $\mathbf{Z}^k$-like if and only if, for each prime p, there exists $N(p)$ such that $h_p^{B,A}(x) \leq N(p)$ for all $0 \neq x \in B$.*

Proof. We know from Theorem 1.1 that B is $\mathbf{Z}^k$-like if and only if T_p has finite exponent for each prime p. But T_p has finite exponent for each p if and only if there exists, for each p, a positive integer $N(p)$ such that $p^{N(p)} B_p \subseteq A_p$. We now apply Theorem 1.3 to complete the proof. $\square$

It is interesting, in the light of Theorem 1.4, to see what relative heights $h_p^{B,A}(x)$ can occur when B is $\mathbf{Z}^k$-like. Let us suppose that $N(p)$ is actually chosen so that $p^{N(p)}$ is the exponent of T_p. We set

$$M = N(p)$$

$$m = \min\{h_p^B(x_1), \ldots, h_p^B(x_k)\}$$

where $A = \langle x_1, x_2, \ldots, x_k\rangle$ and prove, for B a $\mathbf{Z}^k$-like group,

Theorem 1.5 *There exists $x \in B$ with $h_p^{B,A}(x) = h$ if and only if $m \leq h \leq M$.*

(It is plain from Theorem 1.4 and the choice of M that $m \leq M$).

Proof. Let $y \in B_p$ be chosen so that its image in T_p has exponent p^M. Then y is not divisible by p in B_p; moreover we may choose y to be in B. Thus $h_p^B(y) = 0$, $h_p^A(y) = -M$, and $h_p^{B,A}(y) = M$. Furthermore, Theorem 1.4 tells us that, for all $0 \neq x \in B$, $h_p^{B,A}(x) \leq M$.

We next show that, for all $0 \neq x \in B$, $h_p^{B,A}(x) \geq m$. Let $x = \sum_{i=1}^k \lambda_i x_i$, $\lambda_i \in \mathbf{Q}$. Now $p^{-h_p^A(x)}x \in A_p$, so that $p^{-h_p^A(x)}\lambda_i \in \mathbf{Z}_p$, all i. Also $p^{-m}x_i \in B_p$, all i, so

$$p^{-h_p^A(x)-m}x \in B_p, \quad \text{whence} \quad h_p^B(x) \geq h_p^A(x) + m,$$

$$\text{or} \qquad h_p^{B,A}(x) \geq m,$$

as required.

It only remains to prove that, if $m < h < M$, then there exists $x \in B$ with $h_p^{B,A}(x) = h$. We have elements y, z such that $h_p^{B,A}(z) = m$, $h_p^{B,A}(y) = M$. Moreover $p^M y = y'$ with y' in A_p and we may replace y' by ny', for a suitable p'-number n, to obtain finally an element $\bar{y}$ in A with $h_p^{B,A}(\bar{y}) = M$, $h_p^A(\bar{y}) = 0$.

Renaming $\bar{y}$, we thus have elements y, z, actually in A, such that

$$h_p^{B,A}(z) = m, \qquad h_p^A(z) = 0;$$

$$h_p^{B,A}(y) = M, \qquad h_p^A(y) = 0.$$

Set

$$x = p^{h-m}z + y.$$

Then $x \in A$ and $x \neq 0$ since $h_p^A(y) = 0$ and $h > m$. On the other hand,

$$p^{-h}x = p^{-m}z + p^{-h}y \in B_p,$$

since $h_p^B(z) = m$, $h_p^B(y) > h$;

$$p^{-h-1}x = p^{-m-1}z + p^{-h-1}y \notin B_p,$$

since $p^{-m-1}z \notin B_p$, but $h_p^B(y) \geq h+1$, so $p^{-h-1}y \in B_p$;

$$\text{and} \quad p^{-1}x = p^{h-m-1}z + p^{-1}y \notin A_p,$$

since $h \geq m+1$, so $p^{h-m-1}z \in A$, but $p^{-1}y \notin A_p$. Thus $h_p^B(x) = h$, $h_p^A(x) = 0$, $h_p^{B,A}(x) = h$, and the theorem is proved. $\square$

Remark. It is easy to compute m and M from a reduced representation of B (see [2]). Indeed, given such a representation R_* one sees that M is the largest power of p occurring in the denominator of an entry in R_p, written as a reduced fraction. Of course, we assume here that $A = \langle e_1, e_2, \ldots, e_k \rangle$. For example, let $k = 2$ and

$$R_p = \begin{pmatrix} p^{-1} & p^{-4} \\ 0 & p^{-3} \end{pmatrix}, \qquad \text{all } p.$$

Thus $B = \langle p^{-1}e_1, p^{-4}e_1 + p^{-3}e_2, \quad \text{all } p \rangle$. We see that $h_p^B(e_1) = 1$, $h_p^B(e_2) = 0$, so $m = 0$; and $M = 4$, for all p.

On the other hand, B is *not* characterized by m and M. For if

$$R_p' = \begin{pmatrix} p^{-2} & p^{-4} \\ 0 & p^{-2} \end{pmatrix}, \qquad \text{all } p$$

we again have $m = 0$, $M = 4$ for all p; but if $B' = \langle p^{-2}e_1, p^{-4}e_1 + p^{-2}e_2, \quad \text{all } p \rangle$ then $B \not\cong B'$. This can be seen for example as follows. Assume that there exists a matrix $N \in GL_2(\mathbf{Q})$ such that $R_p^{-1}NR_p' \in GL_2(\mathbf{Z}_p)$ for all primes p (see [2]). Then writing

$$R_p^{-1}NR_p' = \begin{pmatrix} p & -1 \\ 0 & p^3 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} p^{-2} & p^{-4} \\ 0 & p^{-2} \end{pmatrix}$$

and evaluating this product we deduce that in particular $p^{-1}z+pt$ and $p^{-1}x-p^{-2}z$ belong to $\mathbf{Z}_p$ for all primes p. But x, y, z, t are rational numbers which do not depend on p. If $t=0$ then $p^{-1}z\in\mathbf{Z}_p$ for all p, which forces $z=0$ and hence contradicts the fact $\det N\neq 0$. If $t\neq 0$ and $z\neq 0$ then we can choose a prime p such that both t and z have p-valuation 0. Then $z+p^2t$ must also have p-valuation 0 and this contradicts the fact $p^{-1}z+pt\in\mathbf{Z}_p$. Hence $z=0$. But then $p^{-1}x\in\mathbf{Z}_p$ for all primes p, which gives $x=0$. This is again absurd.

2 Properties of fgp nilpotent groups

Recall that we are using the notation *fgp* for those nilpotent groups which are finitely generated at every prime. Our main aim in this section is to describe certain properties which fgp nilpotent groups have in common with fg nilpotent groups. Of course, $\mathbf{Z}^k$-like groups form an important subclass of the class of fgp groups.

If $\varphi\colon G\to G$ is an endomorphism of the nilpotent group G and if $x\in G$, we write $H_\varphi(x)$ for the *orbit group* of x under φ; thus

$$H_\varphi(x)=\langle x,\varphi x,\ldots,\varphi^n x,\ldots\rangle. \tag{2.1}$$

If $e_p : G\to G_p$ is the p-localization, then we understand by $H_{\varphi_p}(y)$, $y\in G_p$, the p-local subgroup of G_p generated by the φ_p-orbit of y, so we may write

$$H_{\varphi_p}(y)=\langle y,\varphi_p y,\ldots,\varphi_p^n y,\ldots\rangle_p. \tag{2.2}$$

Notice that

$$H_\varphi(x)_p=H_{\varphi_p}(e_p x) \tag{2.3}$$

and that

$$H_{\varphi_p}(y)=H_{\varphi_p}(y^n)\qquad\text{for any } p'\text{-number } n. \tag{2.4}$$

We now introduce a key definition.

Definition 2.1 Let $\varphi\colon G\to G$ be an endomorphism of the nilpotent group G. We say that (G,φ) *has property P* if, for all $x\in G$,

$$H_\varphi(x)\quad\text{fg}\quad\Leftrightarrow\quad H_{\varphi_p}(e_p x)\quad\text{fg for all primes } p. \tag{2.5}$$

We further say that G *has property* P if (G,φ) has property P for all $\varphi\colon G\to G$.

Notice (i) that $H_{\varphi_p}(e_px)$ is to be understood as a p-local group and is to be understood as fg *in this sense*; and (ii) that the implication '$\Rightarrow$' is essentially trivial and need not detain us in the sequel.

We will show that *every* nilpotent group has property P. We first need a technical lemma.

Lemma 2.1 *Let*

$$\begin{array}{ccccc} G' & \rightarrowtail & G & \overset{\kappa}{\twoheadrightarrow} & G'' \\ \downarrow\varphi' & & \downarrow\varphi & & \downarrow\varphi'' \\ G' & \rightarrowtail & G & \overset{\kappa}{\twoheadrightarrow} & G'' \end{array}$$

be an endomorphism of a short exact sequence of nilpotent groups. Then if (G',φ') *and* (G'',φ'') *have property* P*,* (G,φ) *also has property* P*.*

Proof. We suppose that (G',φ') and (G'',φ'') have property P; and we assume that $x\in G$ with $H_{\varphi_p}(e_px)$ fg for each prime p. Then $H_{\varphi''_p}(e''_p\kappa x)=\kappa_pH_{\varphi_p}(e_px)$ is fg for each p, so that $H_{\varphi''}(\kappa x)$ is fg. Thus there exists n such that

$$\begin{aligned} &(\varphi'')^n\kappa x\in\langle\kappa x,\varphi''\kappa x,\ldots,(\varphi'')^{n-1}\kappa x\rangle \\ \text{or}\qquad &\kappa\varphi^nx\in\langle\kappa x,\kappa\varphi x,\ldots,\kappa\varphi^{n-1}x\rangle. \end{aligned}$$

This implies that $\varphi^nx=yz$, where $y\in\langle x,\varphi x,\ldots,\varphi^{n-1}x\rangle$ and $z\in G'$. But $z=y^{-1}\varphi^nx$, so that

$$z\in G'\cap\langle x,\varphi x,\ldots,\varphi^nx\rangle.$$

Since $z\in H_\varphi(x)$, $H_{\varphi'}(z)=H_\varphi(z)\subseteq H_\varphi(x)$. Thus $H_{\varphi'_p}(e'_pz)\subseteq H_{\varphi_p}(e_px)$; but a p-local subgroup of a fg p-local nilpotent group is fg, so that $H_{\varphi'_p}(e'_pz)$ is fg for each p. It follows that $H_{\varphi'}(z)$ is fg, so that there exists m such that

$$\varphi^mz\in\langle z,\varphi z,\ldots,\varphi^{m-1}z\rangle. \tag{2.6}$$

Now

$$\varphi^{m+n}x=(\varphi^my)(\varphi^mz) \tag{2.7}$$

and

$$\varphi^my\in\langle\varphi^mx,\varphi^{m+1}x,\ldots,\varphi^{m+n-1}x\rangle, \tag{2.8}$$

$$\varphi^i z \in \langle \varphi^i x, \varphi^{i+1} x, \dots, \varphi^{i+n} x \rangle. \tag{2.9}$$

From (2.6) and (2.9) we see that

$$\varphi^m z \in \langle x, \varphi x, \dots, \varphi^{m+n-1} x \rangle,$$

so that (2.7) and (2.8) tell us that

$$\varphi^{m+n} x \in \langle x, \varphi x, \dots, \varphi^{m+n-1} x \rangle.$$

This shows that (G, φ) has property P so the lemma is proved. $\square$

Theorem 2.2 *Every nilpotent group G has property P.*

PROOF. We first set $G = A$, a torsionfree abelian group, and suppose that $\varphi : A \to A$ is an endomorphism such that, for some $x \in A$, $H_{\varphi_p}(e_p x)$ is fg for each prime p. Since we may assume $A \subseteq A_p \subseteq A_0$ for each p, it is unnecessary to write '$e_p x$' and we use the simplified notation $H_{\varphi_p}(x)$.

Fix a prime q and suppose that

$$\varphi_q^n x \in \langle x, \varphi_q x, \dots, \varphi_q^{n-1} x \rangle_q.$$

This implies that

$$\varphi^n x = \sum_{i=0}^{n-1} \lambda_i \varphi^i x, \quad \lambda_i \in \mathbf{Z}_q, \quad 0 \le i \le n-1. \tag{2.10}$$

But then $\lambda_i \in \mathbf{Z}_p$, $0 \le i \le n-1$, for almost all primes p. Thus there exists a *finite* set of primes S such that

$$\varphi_p^n x \in \langle x, \varphi_p x, \dots, \varphi_p^{n-1} x \rangle_p \qquad \text{unless } p \in S. \tag{2.11}$$

Enumerate the primes in S as $p_1, p_2, \dots, p_s$. For each i, there exists n_i such that

$$\varphi_{p_i}^{n_i} x \in \langle x, \varphi_{p_i} x, \dots, \varphi_{p_i}^{n_i - 1} x \rangle_{p_i}, \quad i = 1, 2, \dots, s.$$

Let $N = \max(n, n_1, n_2, \dots, n_s)$. Then

$$\varphi_p^N x \in \langle x, \varphi_p x, \dots, \varphi_p^{N-1} x \rangle_p, \quad \text{for all primes } p. \tag{2.12}$$

Now consider the subgroup $\langle x, \varphi x, \ldots, \varphi^{N-1}x \rangle$ of A. Its p-localization is $\langle x, \varphi_p x, \ldots, \varphi_p^{N-1}x \rangle_p$ and (2.12) tells us that $\varphi^N x$ belongs to each of these p-localizations. Since any torsionfree nilpotent group is the intersection of its p-localizations, we infer that

$$\varphi^N x \in \langle x, \varphi x, \ldots, \varphi^{N-1}x \rangle.$$

This shows that $H_\varphi(x)$ is fg, so our theorem is proved in this special case.

Next we set $G = T$, a torsion abelian group, and suppose that $\varphi\colon T \to T$ is an endomorphism such that, for some $x \in T$, $H_{\varphi_p}(e_p x)$ is fg for each prime p. Consider the *finite* set of primes S such that p divides $|x|$ if $p \in S$. Enumerate the primes in S as $p_1, p_2, \ldots, p_s$. If $p \notin S$, then $H_{\varphi_p}(e_p x) = 0$. For each i there exists n_i such that

$$\varphi_{p_i}^{n_i}(e_{p_i}x) \in \langle e_{p_i}x, \varphi_{p_i}e_{p_i}x, \ldots, \varphi_{p_i}^{n_i-1}e_{p_i}x \rangle_{p_i}, \quad i = 1, 2, \ldots, s.$$

Let $N = \max(n_1, n_2, \ldots, n_s)$. Then

$$\varphi_p^N(e_p x) \in \langle e_p x, \varphi_p e_p x, \ldots, \varphi_p^{N-1}e_p x \rangle_p, \quad \text{all primes } p. \tag{2.13}$$

We again consider the subgroup $U = \langle x, \varphi x, \ldots, \varphi^{N-1}x \rangle$ of T and observe from (2.13) that, since $\varphi_p^N e_p x = e_p \varphi^N x$, it follows that $\varphi^N x$ is an element of T such that $e_p \varphi^N x \in U_p$ for all primes p. This implies that $\varphi^N x \in U$ or

$$\varphi^N x \in \langle x, \varphi x, \ldots, \varphi^{N-1}x \rangle.$$

This shows that $H_\varphi(x)$ is fg, so our theorem is also proved in this special case.

We now allow G to be any abelian group. Let T be its torsion subgroup with quotient $A = G/T$. If $\varphi\colon G \to G$ is an arbitrary endomorphism, then φ induces a commutative diagram

$$\begin{array}{ccccc} T & \rightarrowtail & G & \twoheadrightarrow & A \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi''} \\ T & \rightarrowtail & G & \twoheadrightarrow & A. \end{array}$$

But we have proved that (T, φ') and (A, φ'') have property P, so that (G, φ) has property P, by Lemma 2.1. Since φ was an arbitrary endomorphism of G, it follows that G has property P.

We now complete the proof by induction on $c = \mathrm{nil}\, G$, the result having been proved if $c = 1$. If $\mathrm{nil}\, G = c$, let Γ be the smallest non-trivial term of the lower central series of G, so that $\mathrm{nil}\,(G/\Gamma) = c - 1$. If $\varphi\colon G \to G$, then φ induces a commutative diagram

$$\begin{array}{ccccc} \Gamma & \rightarrowtail & G & \twoheadrightarrow & G/\Gamma \\ \downarrow \varphi' & & \downarrow \varphi & & \downarrow \varphi'' \\ \Gamma & \rightarrowtail & G & \twoheadrightarrow & G/\Gamma. \end{array}$$

Since Γ is commutative, (Γ, φ') has property P; by our inductive hypothesis $(G/\Gamma, \varphi'')$ has property P. Thus (G, φ) has property P, by Lemma 2.1. Since φ was an arbitrary endomorphism of G, we conclude that G has property P, so that the theorem is proved. $\square$

Corollary 2.3 *If $\varphi\colon G \to G$ is an endomorphism of the nilpotent group G and if, for some $x \in G$, $H_\varphi(x)$ is fgp, then $H_\varphi(x)$ is fg. In particular if $H_\varphi(x)$ is A-like for some fg abelian group A, then $H_\varphi(x) \cong A$.*

Corollary 2.4 *If $\varphi\colon G \to G$ is an endomorphism of a fgp nilpotent group and if $H_\varphi(x)$ is an orbit group for φ, then $H_\varphi(x)$ is fg. That is, every endomorphism of a fgp nilpotent group is finitary* ([6]).

Corollary 2.5 *If $\varphi\colon G \to G$ is an automorphism of a fgp nilpotent group, then φ is a pseudo-identity* ([6]).

For both φ and φ^{-1} are finitary; and, for any automorphism φ, $x \in \varphi H_\varphi(x)$ if and only if φ^{-1} is finitary.

Remark. The second part of Corollary 2.3 is of especial interest to us if $A = \mathbf{Z}^k$, for it then tells us that orbit groups are groups of a very special kind. They can only be $\mathbf{Z}^k$-like if they are actually isomorphic to $\mathbf{Z}^k$. In fact, the precise structure of abelian orbit groups is easy to describe explicitly.

Theorem 2.6 *An abelian group A is an orbit group if and only if there exists an ideal α of the polynomial ring $\mathbf{Z}[x]$ such that the underlying abelian group in the ring $\mathbf{Z}[x]/\alpha$ is isomorphic to A.*

Proof. Any endomorphism $\varphi\colon A \to A$ induces a $\mathbf{Z}[x]$-module structure on A by the evident rule

$$x \cdot a = \varphi(a), \quad a \in A. \tag{2.14}$$

Moreover, it is plain that (2.14) sets up a one-one correspondence between the set $\mathrm{End}(A)$ and the collection of $\mathbf{Z}[x]$-module structures on A. Now A is an orbit group for some endomorphism of an abelian group containing A if and only if there is an endomorphism φ of A itself such that the associated $\mathbf{Z}[x]$-module structure given by (2.14) is cyclic. This last condition is plainly equivalent to the existence of a $\mathbf{Z}[x]$-module epimorphism

$$f\colon \mathbf{Z}[x] \twoheadrightarrow A,$$

in which, of course, $\alpha = \ker f$ is a submodule of $\mathbf{Z}[x]$, i.e. an ideal. $\square$

This simple structure suggests an alternative argument for showing that each abelian group G has property P. Namely, if H is an orbit group for some endomorphism of G, then, by Theorem 2.6, H is isomorphic to the underlying abelian group in some ring of the form $\mathbf{Z}[x]/\alpha$, and hence H_p is isomorphic to the underlying abelian group in the ring

$$\mathbf{Z}_p[x]/(\alpha \otimes \mathbf{Z}_p). \tag{2.15}$$

Now it is plain that

Lemma 2.7 $\mathbf{Z}_p[x]/(\alpha \otimes \mathbf{Z}_p)$ *is finitely generated as* $\mathbf{Z}_p$*-module if and only if the ideal* $\alpha \otimes \mathbf{Z}_p$ *contains a monic polynomial.*

Moreover it is readily seen that, if each $\alpha \otimes \mathbf{Z}_p$ contains a monic polynomial, then α must itself contain a monic polynomial, and hence $\mathbf{Z}[x]/\alpha$ is fg as abelian group. This says precisely that if H is fgp then it is fg, as required.

Another consequence of Theorem 2.6 is that every abelian orbit group carries a ring structure induced by the multiplication in $\mathbf{Z}[x]$. It is thus of some interest to observe that every fg abelian group A is indeed an orbit group, for we can always write it in the form

$$A \cong \mathbf{Z}e_1 \oplus \ldots \oplus \mathbf{Z}e_k \oplus (\mathbf{Z}/d_1)e_{k+1} \oplus \ldots \oplus (\mathbf{Z}/d_m)e_{k+m}$$

with $d_m|d_{m-1}|\dots|d_2|d_1$. Then the shift endomorphism $\varphi(e_i) = e_{i+1}$, $i = 1, \dots, k+m-1$, $\varphi(e_{k+m}) = 0$, is well defined and $A = H_\varphi(e_1)$. This provides A with a ring structure, namely that of $\mathbf{Z}[x]$ divided by the ideal

$$\alpha = (d_1 x^k, \dots, d_m x^{k+m-1}, x^{k+m}).$$

Note that, although obviously every orbit group is countably generated, Corollary 2.3 shows that the converse is false, even in the abelian case.

We finally consider Hopficity. It is well-known that fg nilpotent groups are Hopfian. Moreover, essentially the same argument shows that this fact holds also for fg nilpotent p-local groups. For it is easily seen to hold for fg abelian p-local groups and one then argues by induction on nilpotency class, using the fact that the terms of the lower central series of a p-local nilpotent group are themselves p-local.

Let us call a nilpotent group G *fully Hopfian* if G_p is Hopfian for all primes p. The above remark tells us the following:

Theorem 2.8 *If G is a fgp nilpotent group, then G is fully Hopfian.*

It is easy to see that a fully Hopfian nilpotent group is always Hopfian. However, the converse is false, as Example 2.11 below shows. We first establish the facts needed for this example.

Lemma 2.9 *Let G_i, $i = 1, 2, \dots$, be a sequence of groups such that*

$$\mathrm{Hom}(G_i, G_j) = 0 \quad \textit{if } i > j,$$

and let $\prod_i G_i$ be their restricted direct product. If $\varphi \colon \prod_i G_i \to \prod_i G_i$, then φ maps $\prod_{i \geq k} G_i$ to itself.

Proof. Suppose $1 \neq x \in \prod_{i \geq k} G_i$ and $\varphi x \notin \prod_{i \geq k} G_i$. Then there exists $j < k$ such that $\pi_j \varphi : \prod_{i \geq k} G_i \to G_j$ satisfies $\pi_j \varphi x \neq 1$. Let $x = (x_i)$, $i \geq k$. Then $\pi_j \varphi x = \prod_i \pi_j \varphi x_i$ and, for some i, $\pi_j \varphi x_i \neq 1$. This means that $\pi_j \varphi \iota_i \colon G_i \to G_j$ is *not* the constant homomorphism, although $j < i$. □

Theorem 2.10 *With the notation and hypothesis of the lemma, if each G_i is Hopfian, then $\prod_i G_i$ is Hopfian.*

Proof. We first show, by induction on the number of factors, that $\prod_{i=1}^N G_i$ is Hopfian. For we know that a given $\varphi\colon \prod_{i=1}^N G_i \to \prod_{i=1}^N G_i$ sends $\prod_{i=2}^N G_i$ to $\prod_{i=2}^N G_i$. Thus we have

$$\begin{array}{ccccc} \prod_{i=2}^N G_i & \rightarrowtail & \prod_{i=1}^N G_i & \twoheadrightarrow & G_1 \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi''} \\ \prod_{i=2}^N G_i & \rightarrowtail & \prod_{i=1}^N G_i & \twoheadrightarrow & G_1. \end{array}$$

Now suppose φ surjective. Then φ'' is surjective and hence, G_1 being Hopfian, an automorphism. Thus φ' is surjective and hence, by our inductive hypothesis, an automorphism. It now follows that φ is itself an automorphism, so the inductive step is established.

We now prove the theorem. By the lemma each $\varphi\colon \prod_i G_i \to \prod_i G_i$ sends $\prod_{i\geq k} G_i$ to $\prod_{i\geq k} G_i$. Thus we have, for each $k \geq 1$,

$$\begin{array}{ccccc} \prod_{i\geq k} G_i & \rightarrowtail & \prod_i G_i & \twoheadrightarrow & \prod_{i=1}^{k-1} G_i \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi''} \\ \prod_{i\geq k} G_i & \rightarrowtail & \prod_i G_i & \twoheadrightarrow & \prod_{i=1}^{k-1} G_i. \end{array}$$

Now suppose φ surjective. Then φ'' is surjective and hence, by what we have just proved, an automorphism. Thus φ' is surjective. If φ were not an automorphism, there would exist $x \neq 1$ in $\prod_i G_i$ with $\varphi x = 1$. Now fix k to be the smallest index such that the component x_k of x is non-trivial. Then $x \in \prod_{i\geq k} G_i$ and we have

$$\begin{array}{ccccc} \prod_{i\geq k+1} G_i & \rightarrowtail & \prod_{i\geq k} G_i & \twoheadrightarrow & G_k \\ \downarrow & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi_k} \\ \prod_{i\geq k+1} G_i & \rightarrowtail & \prod_{i\geq k} G_i & \twoheadrightarrow & G_k. \end{array}$$

Since φ' is surjective, so is φ_k. But $x_k \neq 1$ and $\varphi_k x_k = 1$, contradicting the Hopficity of G_k. $\square$

Example 2.11 We denote by $S(i)$ the subgroup of $\mathbf{Q}$ generated by the set $\{p^{-i}, \quad \text{all } p\}$. This group is clearly $\mathbf{Z}$-like for each $i \geq 0$ and hence fully Hopfian by Theorem 2.8. If $G_i = S(i)$ then the conditions of Theorem 2.10 are satisfied. Thus if $G = \bigoplus_i S(i)$, then G is Hopfian; but G is plainly not fully Hopfian, since no G_p is Hopfian. Of course, one may construct many such examples using $\mathbf{Z}^k$-like groups for the constituent G_i.

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