On Postnikov pieces of finite dimension

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Abstract

We prove that, if X is a connected CW-complex of finite dimension with only a finite number of nonzero Postnikov invariants, then the homotopy groups $\pi_n(X)$ are rational vector spaces for $n \geq 2$ and they vanish for all n sufficiently large. Moreover, the fundamental group $\pi_1(X)$ is torsion-free and all its abelian subgroups have finite rank. Our argument relies on Miller's solution of a conjecture due to Sullivan.

0 Introduction

It is well known that if a simply connected finite CW-complex is not contractible, then it has infinitely many nonzero homotopy groups. This was proved by Serre in [15]. In fact, he proved that if X is a simply connected space of finite type with nontrivial mod 2 homology and such that $H_n(X; \mathbf{Z}/2) = 0$ for n sufficiently large, then there are infinitely many values of i such that $\pi_i(X)$ contains a subgroup isomorphic to $\mathbf{Z}$ or $\mathbf{Z}/2$ (see Théorème 10 in p. 217 of [15]).

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In the same article, Serre conjectured that, under the same assumptions, $\pi_i(X)$ contains a copy of $\mathbf{Z}/2$ for infinitely many values of i. This conjecture was proved by McGibbon and Neisendorfer in [10] thirty years after. In fact, they proved Serre's conjecture for all primes p and without requiring that X be of finite type. The basic ingredient in their article was Miller's solution of the Sullivan conjecture [11]. An alternative argument, still relying on Miller's theorem, was given by Neisendorfer in [12], using localization with respect to the constant map from an Eilenberg–Mac Lane space $K(\mathbf{Z}/p, 1)$ to a point.

The results of Serre and McGibbon–Neisendorfer have subsequently been improved by several authors. For simply connected spaces of finite mod 2 type, the assumption that $H_n(X; \mathbf{Z}/2)$ be zero for almost all n was weakened by assuming only that the cohomology $H^*(X; \mathbf{Z}/2)$ be locally finite as a module over the mod 2 Steenrod algebra in [7], or that the reduced cohomology $\tilde{H}^*(X; \mathbf{Z}/2)$ be nilpotent in [8], or that X be an n-cone in [5]. The hypothesis that X be simply connected was relaxed in [13] and in [18]. More recently, Grodal has proved in [6] that if $\pi_1(X)$ is a finite p-group, X has finite mod p type, and the module of indecomposables of $H^*(X; \mathbf{Z}/p)$ is locally finite, then either X is mod p equivalent to the classifying space of a p-compact toral group or $\pi_i(X)$ contains p-torsion for infinitely many values of i. This generalizes a result of Dwyer and Wilkerson in [4].

In the present article, we consider CW-complexes of finite dimension with finitely many nonzero homotopy groups, without any a priori restriction on the fundamental group. Plenty of examples come to mind, such as wedges of circles, compact surfaces with infinite fundamental group, rationalizations of spheres or complex projective spaces, and finite products of any of these. However, we do not know any example of a finite CW-complex with finitely many nonzero homotopy groups which is not a $K(G, 1)$, and the results of this paper suggest that it is unlikely that there exist any.

Our study originated from the observation that, if a space X has finite dimension and only a finite number of nonzero Postnikov invariants, then the 1-connected cover of X is a rational Postnikov piece. In order to prove this claim, which is stated as Theorem 1.1 below, one can either resort to the main result in [10], or give an alternative proof using techniques of homotopical localization. Both arguments, however, rely on the solution of the Sullivan conjecture.

Of course, not all groups can be fundamental groups of finite-dimensional Postnikov pieces. As we explain below, such a group is necessarily torsion-free and cannot contain any abelian subgroup of infinite rank.

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1 Statement and proof of results

All spaces considered in this paper are connected CW-complexes. Thus, we say that a space X has dimension d if it has cells up to this dimension only. We say that a space X is a Postnikov piece if the homotopy groups $\pi_i(X)$ vanish for almost all values of i. A space X is called a GEM if it is homotopy equivalent to a (weak, possibly infinite) product of Eilenberg–Mac Lane spaces $K(A_n, n)$ where $n \geq 1$ and each A_n is abelian.

Let us recall the definition of Postnikov invariants of (not necessarily simple) spaces. If X is any connected space, then for each $n \geq 1$ there is a homotopy fibration

$$X\langle n\rangle \to X \to P_n X$$

where $P_n X$ is a Postnikov piece, $X\langle n\rangle$ is n-connected and the map $X\langle n\rangle \to X$ induces isomorphisms of homotopy groups at dimensions higher than n. For $n \geq 2$, the homotopy fibration

$$K(\pi_n(X),\, n) \to P_n X \to P_{n-1} X$$

is classified, in the sense of [14], by a certain map

$$k_{n+1}\colon P_{n-1}X \to L(\pi_n(X),\, n+1),$$

where we use the notation $L(A, m)$ for the total space of a split fibre bundle

$$K(A, m) \to L(A, m) \to K(\mathrm{Aut}(A),\, 1),$$

in which the fundamental group of the base acts on the fibre by cellular homeomorphisms. Thus we may view k_{n+1} as a cohomology class in

$$H^{n+1}(P_{n-1}X;\, \pi_n(X)),$$

where twisted coefficients are intended. These cohomology classes are called Postnikov invariants of X.

The Postnikov invariants of the 1-connected cover $X\langle 1\rangle$ are the images of the Postnikov invariants of X under the obvious homomorphisms

$$H^{n+1}(P_{n-1}X;\, \pi_n(X)) \to H^{n+1}(P_{n-1}X\langle 1\rangle;\, \pi_n(X)).$$

If $n \geq 1$, then the condition that the Postnikov invariants k_m of a space X vanish for $m \geq n+3$ is equivalent to the condition that the n-connected cover $X\langle n\rangle$ be a GEM. This is also true for $n = 0$ if the space X is simple; i.e., if $\pi_1(X)$ is abelian and its action on the higher homotopy groups of X is trivial.

Theorem 1.1 *If X is a CW-complex of finite dimension with only a finite number of nonzero Postnikov invariants, then the following hold:*

(a) *X is a Postnikov piece.*

(b) *The homotopy groups $\pi_n(X)$ are $\mathbf{Q}$-vector spaces for $n \geq 2$.*

(c) *If G is any subgroup of $\pi_1(X)$, then the mod p homology groups $H_n(G; \mathbf{Z}/p)$ vanish for all primes p if n is greater than the dimension of X.*

Proof. Suppose that the Postnikov invariants k_i of X vanish for $i \geq m+2$, where $m \geq 1$. Then the Postnikov invariants of the 1-connected cover $X\langle 1\rangle$ of X also vanish for $i \geq m+2$. Hence, $X\langle 1\rangle$ decomposes up to homotopy as a product

$$X\langle 1\rangle \simeq X\langle m\rangle \times P_m X\langle 1\rangle$$

and the m-connected cover $X\langle m\rangle$ is a GEM. If X has dimension d, then $X\langle 1\rangle$ has dimension d as well. Therefore, the integral homology groups $H_n(X\langle 1\rangle; \mathbf{Z})$ vanish for

$n > d$. If the space $X\langle 1\rangle$ is not a Postnikov piece, then there is a positive integer $n > d$ and an abelian group A such that $K(A,n)$ is a retract of $X\langle 1\rangle$. Thus, the identity map of $K(A,n)$ factors through $X\langle 1\rangle$, yielding a contradiction in homology. This shows that $X\langle 1\rangle$ is a Postnikov piece and hence so is X.

Now it follows from Theorem 1 in [10] that $H_n(X\langle 1\rangle; \mathbf{Z}/p) = 0$ for all $n > 0$ and every prime p. This implies that the integral homology groups $H_n(X\langle 1\rangle; \mathbf{Z})$ are $\mathbf{Q}$-vector spaces for $n \geq 2$. Since $X\langle 1\rangle$ is simply connected, the homotopy groups $\pi_n(X)$ are also $\mathbf{Q}$-vector spaces for $n \geq 2$.

Next, consider the homotopy fibration

$$X\langle 1\rangle \to X \to K(\pi, 1), \tag{1.1}$$

where π denotes the fundamental group of X. Since the homology groups $H_n(X\langle 1\rangle; \mathbf{Z}/p)$ vanish for all primes p and $n > 0$, the Serre spectral sequence associated with (1.1) for homology with mod p coefficients collapses, yielding isomorphisms

$$H_n(X; \mathbf{Z}/p) \cong H_n(\pi; \mathbf{Z}/p) \qquad \text{for all primes } p \text{ and all } n.$$

If G is any subgroup of π, then there is a covering space $Y \to X$ with $\pi_1(Y) \cong G$. Then there is also a homotopy fibration

$$X\langle 1\rangle \to Y \to K(G, 1),$$

from which we infer that $H_n(Y; \mathbf{Z}/p) \cong H_n(G; \mathbf{Z}/p)$ for all primes p and all n. Since Y is a CW-complex of dimension d, we have shown that every subgroup of π has bounded mod p homology for all primes p. $\sharp$

Part (c) in Theorem 1.1 implies, of course, that $\pi_1(X)$ is torsion-free. In fact it says much more than this. For instance, it implies that $\pi_1(X)$ cannot contain any abelian subgroup of infinite rank.

Theorem 1.2 *For an abelian group A, the following assertions are equivalent:*

- *A is isomorphic to the fundamental group of a Postnikov piece of finite dimension.*

- *A is a torsion-free abelian group of finite rank.*
- *There is a $K(A,1)$ of finite dimension.*

PROOF. Suppose that $A \cong \pi_1(X)$ where X is a Postnikov piece of dimension d. Then A is torsion-free and hence it embeds into $A \otimes \mathbf{Q}$. Suppose that the dimension of $A \otimes \mathbf{Q}$ over $\mathbf{Q}$ is greater than d. Then A contains a subgroup isomorphic to $\mathbf{Z}^{d+1}$. Since $H_{d+1}(\mathbf{Z}^{d+1}; \mathbf{Z}/p)$ is nonzero, this contradicts Theorem 1.1 and hence we have proved that the rank of A is at most d.

If A has finite rank (say, r), then A embeds into $\mathbf{Q}^r$. If $r = 1$, then there is a telescope of circles which is a $K(A,1)$ of dimension 2. If $r \geq 2$, consider the cellular chain complex of the universal cover of a $K(\mathbf{Q}^r, 1)$ of dimension $2r$. This is a free resolution of $\mathbf{Z}$ as a trivial $\mathbf{Z}[A]$-module, showing that the cohomology of $K(A,1)$ vanishes above dimension $2r$ with arbitrary coefficients. Hence, there is a $K(A,1)$ of finite dimension; see e.g. Theorem E in [17]. $\sharp$

Groups G for which there is a $K(G,1)$ of finite dimension have been extensively studied; see e.g. [1]. However, as pointed out in the Introduction, we do not know any example of a finite CW-complex with finitely many homotopy groups which is not a $K(G,1)$. Theorem 1.1 tells us that, in order to decide whether such spaces exist or not, the following question should be investigated.

Question 1.3 Can the higher homotopy groups of a finite CW-complex X be nonzero rational vector spaces?

Of course, the answer is negative under suitable assumptions on X; for instance, if X is nilpotent and finite, then the higher homotopy groups of X are finitely generated abelian groups, so they cannot be nonzero rational vector spaces. However, if X is finite but not nilpotent, then the higher homotopy groups of X need not be finitely generated, not even as modules over the integral group ring of the fundamental group $\pi_1(X)$; see [16]. This striking fact suggests that Question 1.3 might be more difficult than one would naively expect.

If we assume that X has dimension 2, then the answer to Question 1.3 is also negative, since $H_2(X\langle 1\rangle; \mathbf{Z})$ would then be a free abelian group and a rational vector space, hence zero, and $X\langle 1\rangle$ would be contractible. For the validity of this argument it is not needed that X be finite; thus, it follows from Theorem 1.1 that every Postnikov piece X of dimension 2 is a $K(G, 1)$. Motivated by these considerations and by Theorem 1.2, we also prompt the following question.

Question 1.4 Suppose that a group G is isomorphic to the fundamental group of a Postnikov piece of finite dimension. Is there a $K(G, 1)$ of finite dimension?

2 An alternative argument

In the proof of Theorem 1.1, we have resorted to Theorem 1 in [10] at the key step. In this section we point out that there is another argument based on localization techniques, inspired by Neisendorfer's article [12].

Thus, let X be a Postnikov piece of finite dimension. We give an alternative argument to prove that the 1-connected cover $X\langle 1\rangle$ is a rational space.

Let L denote localization, in the sense of [3], with respect to the constant map $f\colon K(\mathbf{Z}/p,\, 1) \to *$, where p is any fixed prime. Then, since $X\langle 1\rangle$ has finite dimension, the based function space

$$\mathrm{map}_*(K(\mathbf{Z}/p,\, 1),\, X\langle 1\rangle)$$

is weakly contractible by [11], so that $X\langle 1\rangle$ is f-local; i.e., the localization map

$$X\langle 1\rangle \to LX\langle 1\rangle$$

is a homotopy equivalence. Now, for every abelian group A and $n \geq 2$ there is a homotopy fibration

$$F \to K(A, n) \to K(A \otimes \mathbf{Z}[1/p],\, n)$$

where F is a p-torsion space. Then it follows, as in § 7 of [2], that LF is contractible and hence, using Theorem 1.H.1 in [3], we have

$$LK(A, n) \simeq LK(A \otimes \mathbf{Z}[1/p],\, n) \simeq K(A \otimes \mathbf{Z}[1/p],\, n),$$

so that the p-completion $\widehat{LK(A,n)}_p$ is contractible. But if Y is any simply connected space, then $\widehat{(LY)}_p$ is homotopy equivalent to L_gY if g is the constant map from the wedge $K(\mathbf{Z}/p,1)\vee M(\mathbf{Z}[1/p],1)$ to a point, where M denotes a Moore space. (As observed by McGibbon in [9], this follows directly from Lemma 1.3 in [12].) Thus, from the fact that $X\langle 1\rangle$ is a simply connected Postnikov piece of finite dimension we infer inductively that the p-completion of $X\langle 1\rangle$ is contractible for all primes p and hence $X\langle 1\rangle$ is a rational space.

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