LOCALIZING WITH RESPECT TO SELF MAPS OF THE CIRCLE

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Abstract. We describe a general procedure to construct idempotent functors on the pointed homotopy category of connected CW-complexes, some of which extend P-localization of nilpotent spaces, at a set of primes P. We focus our attention on one such functor, whose local objects are CW-complexes X for which the p-th power map on the loop space ΩX is a self homotopy equivalence if $p \notin P$. We study its algebraic properties, its behaviour on certain spaces, and its relation with other functors such as Bousfield's homology localization, Bousfield-Kan completion and Quillen's plus-construction.

0. Introduction

Localization methods at a set of primes P have been broadly used in group theory and homotopy theory during the last two decades. These methods have been applied primarily in the category of nilpotent groups and in the pointed homotopy category of nilpotent CW-complexes; see [1], [7], [16].

If one seeks to extend P-localization to arbitrary CW-complexes, then there is in fact more than one choice available. Among the various choices, the ones which have been described in the literature are contained in the work of Bousfield ([4], [5]) and Bousfield-Kan ([7]).

In this paper, we study extensions of P-localization of nilpotent CW-complexes to all CW-complexes from two points of view:

(1) We consider the family of all possible such extensions and provide a uniform construction of some of them, namely of those which are *spliceable*, in the sense explained below.

(2) Among such spliceable functors, there is one which is most intimately tied to P-localization of groups in the sense of [16], [22]. We explain its basic properties.

To explain (1), we observe that the functors in our discussion are interrelated by natural transformations. In this way they form a partially ordered system containing a unique maximal element and a unique minimal element. The minimal element of this system turns out to be Bousfield's $H_*(\ ;\mathbb{Z}_P)$-localization ([4]), meaning that $H_*(\ ;\mathbb{Z}_P)$-localization factors uniquely through every extension of P-localization of nilpotent spaces to all spaces.

The $H_*(\ ;\mathbb{Z}_P)$-localization functor is characterized by the class of maps which become homotopy equivalences after localization, namely all maps which induce an isomorphism in singular homology with $\mathbb{Z}_P$ coefficients. The functors which we describe render invertible all maps inducing at the fundamental group a homomorphism from a specially selected class and an isomorphism in homology with certain twisted coefficients. The construction of these functors involves two processes. Firstly, homology localization in the sense of Bousfield in categories of the following type: an object consists of a space together with a twisted coefficient system, and a morphism is a map compatible with the coefficient systems; see Section 5. Secondly, "splicing" such functors in order to obtain idempotent functors in the homotopy category of CW-complexes; see Section 6.

While splicing affords a uniform point of view of P-localization through the eyes of homology, we emphasize that some of our functors have independent descriptions and constructions which are more direct and geometrically inspired. One nontrivial result seen with the aid of such a double description of a localizing functor is that homology and cohomology equivalences with certain P-local twisted coefficients coincide (Corollary 7.3).

We now turn to the special P-localizing functor $X \to X_P$ mentioned in (2) above. Recall that a group G is called P-*local* if for all primes $n \notin P$ the power function $x \mapsto x^n$ is a bijection—or, equivalently, a homotopy equivalence, regarding G as a discrete space. The distinguishing feature of our special localizing functor is that its local spaces have loop spaces on which the power function $\omega \mapsto \omega^n$, $n \notin P$, is a homotopy equivalence; i.e. P-local spaces are classifying spaces of (A_∞-associative) P-local H-groups. They are also characterized by the property that $\pi_1(X)$ is a P-local group acting P-locally on the higher homotopy groups $\pi_i(X)$, i.e. each element of the form
$$1 + x + \cdots + x^{n-1} \in \mathbb{Z}[\pi_1(X)], \quad n \notin P, \text{ acts as an isomorphism on } \pi_i(X), \quad i \geq 2; \text{ cf.}$$
[19].

To deal with this kind of actions, we use the localization $P[G]$ of the integral group ring $\mathbb{Z}G$ of a P-local group G with respect to the elements $1+x+\cdots+x^{n-1}$, $x \in G$, $n \notin P$, in the ring-theoretical sense ([13]). Abelian groups on which G acts P-locally

are then identified as $P[G]$-modules. This is done in Section 2, where we also establish some foundational facts concerning $P[G]$-modules.

In Section 3, we provide an explicit construction of the functor $X \to X_P$ in countably many steps, and proceed to collect its basic properties. In Section 4, we prove that this functor extends P-localization of nilpotent spaces. Finally, in Section 8, we discuss its interrelation with other extensions of P-localization of nilpotent spaces and determine its effect on special spaces and maps, in particular on a wedge of circles.

1. Categorical preliminaries

In order to explain the interrelation between various localizing functors known from the literature and new ones obtained in this paper, we invoke the general facts concerning localizing functors and orthogonal pairs described in [11], [12]. For the convenience of the reader, we collect here what is needed. Let $\mathcal{C}$ denote a given category.

Definition 1.1. An object X and a morphism $f : A \to B$ in $\mathcal{C}$ are said to be *orthogonal* if the function $f^* : \mathrm{Mor}\,(B, X) \longrightarrow \mathrm{Mor}\,(A, X)$ is bijective.

For a class of morphisms $\mathcal{M}$, $\mathcal{M}^\perp$ denotes the class of objects which are orthogonal to each $f \in \mathcal{M}$. Similarly, for a class of objects $\mathcal{O}$, $\mathcal{O}^\perp$ denotes the class of morphisms which are orthogonal to each object $X \in \mathcal{O}$.

Definition 1.2. An *orthogonal pair* $(\mathcal{S}, \mathcal{D})$ in $\mathcal{C}$ consists of a class of morphisms $\mathcal{S}$ and a class of objects $\mathcal{D}$ such that $\mathcal{D}^\perp = \mathcal{S}$ and $\mathcal{S}^\perp = \mathcal{D}$.

The operation $\perp$ satisfies $\perp\perp\perp \;=\; \perp$, implying that $(\mathcal{M}^{\perp\perp}, \mathcal{M}^\perp)$ and $(\mathcal{O}^\perp, \mathcal{O}^{\perp\perp})$ are orthogonal pairs for every class of morphisms $\mathcal{M}$, resp. of objects $\mathcal{O}$, in $\mathcal{C}$.

Localizations of $\mathcal{C}$ are described by means of idempotent monads ([1]), and every idempotent monad (E, η) determines an orthogonal pair $(\mathcal{S}, \mathcal{D})$ by

$$\mathcal{S} = \{f : A \to B \mid Ef : EA \cong EB\},$$

$$\mathcal{D} = \{X \mid \eta_X : X \cong EX\}.$$

The construction in [12] "recovers" for many orthogonal pairs $(\mathcal{S}, \mathcal{D})$ its associated idempotent monad (E, η), provided it exists. In this situation, we refer to morphisms in $\mathcal{S}$ as *E-equivalences* and to objects in $\mathcal{D}$ as *E-local* objects.

Let E' be a localizing functor on $\mathcal{C}'$, a subcategory of $\mathcal{C}$, and let $(\mathcal{S}', \mathcal{D}')$ denote the orthogonal pair associated with E'. Possible extensions of E' over $\mathcal{C}$ are identified by orthogonal pairs $(\mathcal{S}, \mathcal{D})$ of $\mathcal{C}$ extending $(\mathcal{S}', \mathcal{D}')$, i.e. such that $\mathcal{S}' \subseteq \mathcal{S}$ and $\mathcal{D}' \subseteq \mathcal{D}$. Orthogonal pairs extending $(\mathcal{S}', \mathcal{D}')$ are partially ordered by the following relation: $(\mathcal{S}_1, \mathcal{D}_1) \geq (\mathcal{S}_2, \mathcal{D}_2)$ if $\mathcal{D}_1 \supseteq \mathcal{D}_2$ (equivalently, if $\mathcal{S}_1 \subseteq \mathcal{S}_2$).

Remark 1.3. Suppose that E_1, E_2 are localizing functors whose associated orthogonal pairs satisfy $(\mathcal{S}_1, \mathcal{D}_1) \geq (\mathcal{S}_2, \mathcal{D}_2)$. Then $E_2 : \mathcal{D}_1 \to \mathcal{D}_2$ is left adjoint to the inclusion $\mathcal{D}_2 \to \mathcal{D}_1$. In particular, there is a natural transformation $E_1 \to E_2$ which commutes with the localizing transformations $\eta_1 : \mathrm{Id} \to E_1$, $\eta_2 : \mathrm{Id} \to E_2$.

Remark 1.4. If $(\mathcal{S}, \mathcal{D})$ extends $(\mathcal{S}', \mathcal{D}')$ over $\mathcal{C}$, then

$$\left((\mathcal{S}')^{\perp\perp}, (\mathcal{S}')^{\perp}\right) \geq (\mathcal{S}, \mathcal{D}) \geq \left((\mathcal{D}')^{\perp}, (\mathcal{D}')^{\perp\perp}\right).$$

Accordingly, we call $\left((\mathcal{S}')^{\perp\perp}, (\mathcal{S}')^{\perp}\right)$ the *maximal extension* of $(\mathcal{S}', \mathcal{D}')$ over $\mathcal{C}$ and $\left((\mathcal{D}')^{\perp}, (\mathcal{D}')^{\perp\perp}\right)$ the *minimal extension*.

2. *P-local* groups, modules and spaces

This section contains the algebraic foundations needed to describe a certain idempotent monad in the pointed homotopy category of connected CW-complexes. The key concept is that of a group G acting P-locally on an abelian group A (Definition 2.4). The motivation to consider such actions comes from Theorem 2.1 below.

When dealing with spaces and maps, we work in the based setting throughout. Thus $[X, Y]$ denotes the set of pointed homotopy classes of maps from X to Y. By P we always denote a (fixed) set of primes, and by P' its complement in the set of all primes. We write $n \in P'$ if n belongs to the multiplicative closure of the primes in P'. A

group G will be called P-local ([16], [22], [27]) if the n-th power map $x \mapsto x^n$ is bijective on G for every $n \in P'$.

Theorem 2.1. *For a connected CW-complex X, the following assertions are equivalent:*

(a) *On the loop space ΩX the n-th power map $\omega \mapsto \omega^n$ is a self homotopy equivalence for every $n \in P'$.*

(b) *The group $[W, \Omega X]$ is P-local for every space W.*

(c) *The groups $\pi_1(X)$ and $\pi_k(X) \rtimes \pi_1(X)$, $k \geq 2$, are P-local, where $\rtimes$ denotes the semidirect product with respect to the standard action.*

Proof. This is a consequence of the natural group isomorphism

$$\pi_k(X) \rtimes \pi_1(X) \cong [S^{k-1} \cup \mathrm{pt}, \Omega X], \tag{2.1}$$

where pt denotes a disjoint base point; see [18], [19]. □

Definition 2.2. We call *P-local* those connected CW-complexes X satisfying the equivalent conditions of Theorem 2.1.

Let $\mathrm{map}_*(A, X)$ denote the space of pointed maps from A to X. Using condition (2.1.b) and the exponential law ([8]), one infers

Proposition 2.3. *If X is P-local, then the base point component of $\mathrm{map}_*(A, X)$ is also P-local, for every finite CW-complex A.* □

Note that, if a CW-complex X is nilpotent, then the groups $\pi_k(X) \rtimes \pi_1(X)$ are nilpotent. Therefore, a nilpotent CW-complex X is P-local in our sense if and only if the groups $\pi_k(X)$ are P-local for $k \geq 1$, because P-localization is exact in the category of nilpotent groups. Thus our terminology is consistent with the classical one ([16]). Next we characterize the action of the fundamental group of a P-local space on the higher homotopy groups in a way which is more readily accessible to methods from Homological Algebra.

Definition 2.4. Let G be a group and A a (left) $\mathbb{Z}G$-module. Denote by $\omega : \mathbb{Z}G \to \mathrm{End}(A)$ the associated ring homomorphism. We say that A is a *P-local* $\mathbb{Z}G$-module if $\omega(1 + x + x^2 + \cdots + x^{n-1}) \in \mathrm{Aut}(A)$ for all $x \in G$, $n \in P'$.

We also say that the action of G on A is *P-local*, or that G *acts P-locally* on A. This notion is easily generalized to the case when A is a group, not necessarily abelian, on which G acts; see [11], [12], [14], [19]. We shall often use the notation

$$\rho_{n,x} = 1 + x + x^2 + \cdots + x^{n-1}. \tag{2.2}$$

Remark 2.5. Given a group G and a $\mathbb{Z}G$-module A, the identity

$$(a, x)^n = (\rho_{n,x}a, x^n), \qquad n \geq 1, \tag{2.3}$$

in the group $A \rtimes G$ shows that $A \rtimes G$ is *P-local* if and only if G is a *P-local* group and A is a *P-local* $\mathbb{Z}G$-module.

In the light of Remark 2.5, condition (c) in Theorem 2.1 can be reformulated as

(2.1.c') $\pi_1(X)$ *is a P-local group and each* $\pi_k(X)$, $k \geq 2$, *is a P-local* $\mathbb{Z}[\pi_1(X)]$ *-module.*

Our next step is to identify *P-local* $\mathbb{Z}[\pi_1(X)]$- modules as modules over a certain universal ring associated to $\mathbb{Z}[\pi_1(X)]$, in which the elements $1 + x + \cdots + x^{n-1}$, $x \in \pi_1(X)$, $n \in P'$, are units. We need to recall some general facts of ring theory. Let R be a ring with 1, and $S \subseteq R$ be an arbitrary subset. As explained in [13], there exists a ring homomorphism $\alpha : R \to R_S$, unique up to isomorphism, such that

  (i) $\alpha(s)$ is invertible in R_S for all $s \in S$, and
  (ii) if $f : R \to T$ is any ring homomorphism for which $f(s)$ is invertible in T for all $s \in S$, then there is a unique ring homomorphism $f' : R_S \to T$ with $f'\alpha = f$.

In particular, (ii) shows that α is a *ring epimorphism,* meaning that for every diagram of rings

$$R \xrightarrow{\alpha} R_S \underset{v}{\overset{u}{\rightrightarrows}} T$$

where $u\alpha = v\alpha$, it follows that $u = v$. Therefore ([26]), the category of R_S-modules embeds as a full subcategory in the category of R-modules via α, and this embedding has a left adjoint α^* given by

$$\alpha^*(A) = R_S \otimes_R A. \tag{2.4}$$

We apply these considerations to the case $R = \mathbb{Z}G$, G a given group, and S the multiplicative closure of the elements $\rho_{n,x}$, $x \in G$, $n \in P'$, cf. (2.2). Thus we obtain a universal ring homomorphism

$$\alpha : \mathbb{Z}G \to (\mathbb{Z}G)_S \tag{2.5}$$

inverting the elements of S.

Therefore, a $\mathbb{Z}G$-module A is P-local if and only if the ring homomorphism $\omega : \mathbb{Z}G \to \mathrm{End}\,(A)$ factors uniquely through (2.5); i.e. P-local $\mathbb{Z}G$-modules are precisely $(\mathbb{Z}G)_S$-modules.

If the group G is commutative, then $(\mathbb{Z}G)_S = S^{-1}(\mathbb{Z}G)$ is the usual ring of fractions ([2]). In particular, of course, if $G = \{1\}$, then $(\mathbb{Z}G)_S = \mathbb{Z}_P$, the ring of integers localized at P.

We now collect several elementary facts concerning $(\mathbb{Z}G)_S$-modules which will be needed in the sequel.

Lemma 2.6. *Let G be a group, and*

$$0 \to A' \to A \to A'' \to 0$$

a short exact sequence of $\mathbb{Z}G$-modules. Then, if any two of A', A, A'' are P-local, so is the third.

Proof. For each $n \in P'$ and $x \in G$, the operation $\rho_{n,x}$ induces the commuting diagram below.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \rho_{n,x}} & & \downarrow{\scriptstyle \rho_{n,x}} & & \downarrow{\scriptstyle \rho_{n,x}} & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0.
\end{array}
$$

Now our claim follows from the five-lemma. $\qquad\square$

Lemma 2.7. *Let G be a group, A a P-local $\mathbb{Z}G$-module, and B a nilpotent $\mathbb{Z}G$-module. Then the diagonal action of G on each of the following abelian groups is P-local:*

$$A \otimes_{\mathbb{Z}} B, \qquad \mathrm{Tor}^{\mathbb{Z}}(A, B), \qquad \mathrm{Hom}_{\mathbb{Z}}(B, A), \qquad \mathrm{Ext}_{\mathbb{Z}}(B, A).$$

Proof. Argue by induction on the nilpotency class of the action of G on B, using Lemma 2.6. If this action is trivial, the $\mathbb{Z}G$-module structure on each of the abelian groups listed is induced by that of A, i.e. it factors as

$$\mathbb{Z}G \xrightarrow{\omega} \mathrm{End}\,(A) \to \mathrm{End}\,(C),$$

where C is any of $A \otimes_{\mathbb{Z}} B$, $\mathrm{Tor}^{\mathbb{Z}}(A, B)$, $\mathrm{Hom}_{\mathbb{Z}}(B, A)$, $\mathrm{Ext}_{\mathbb{Z}}(B, A)$. Hence, it factors through $\alpha : \mathbb{Z}G \to (\mathbb{Z}G)_S$; cf. (2.5). $\qquad\square$

Lemma 2.8. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be a central group extension and A a P-local $\mathbb{Z}G$-module. Then the induced action of Q on $H_k(N; A)$ and $H^k(N; A)$ is P-local for all k.*

Proof. Since the conjugation action of G on N is trivial, the action of Q on $H_k(N; A)$ is induced by the action of G on A, i.e. the following diagram commutes.

$$
\begin{array}{ccc}
\mathbb{Z}G & \xrightarrow{\ \omega\ } & \mathrm{End}_{\,\mathbb{Z}N}(A) \\
\downarrow & & \downarrow \\
\mathbb{Z}Q & \longrightarrow & \mathrm{End}\,(H_k(N; A))
\end{array}
$$

Now use Lemma 2.9 below. The same argument applies to cohomology. $\qquad\square$

Lemma 2.9. *Let $\varphi : G \to Q$ be a group homomorphism.*

(i) If B is a P-local $\mathbb{Z}Q$-module, then B is also P-local as a $\mathbb{Z}G$-module.

(ii) If φ is an epimorphism then the converse in (i) also holds.

Proof. This is straightforward. $\qquad\square$

In particular, every P-local $\mathbb{Z}G$-module is a $\mathbb{Z}_P$-module. If the action of G is trivial, then the two notions coincide.

Lemma 2.10. *Every nilpotent action of a P-local nilpotent group G on a $\mathbb{Z}_P$-module A is P-local.*

Proof. This follows from Remark 2.5; cf. [16]. $\qquad\square$

Lemma 2.11. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be a group extension in which N is P-local and Q is P-torsion. Then G is P-local.*

Proof. See [3, Theorem 11.5].   □

Corollary 2.12. *Every action of a P-torsion group G on a $\mathbb{Z}_P$-module A is P-local.*

Proof. A P-torsion group G is always P-local. Moreover, $A \rtimes G$ is P-local by Lemma 2.11. Thus our assertion follows from Remark 2.5.   □

Corollary 2.13. *If the connected CW-complex X has a finite P-group as its fundamental group, and $\mathbb{Z}_P$-modules as its higher homotopy groups, then X is a P-local space.*   □

Lemma 2.14. *Suppose A is a P-local $\mathbb{Z}G$-module. Then every P'-torsion element of G acts as the identity on A.*

Proof. If x is P'-torsion, then $x^n = 1$ for some $n \in P'$, and hence we have the following identity in the ring $\mathbb{Z}G$:

$$0 = x^n - 1 = (x - 1)(1 + x + x^2 + \cdots + x^{n-1}). \tag{2.6}$$

Since $1 + x + x^2 + \cdots + x^{n-1}$ acts as an isomorphism of A, it follows that $x - 1$ acts as 0, i.e. $xa = a$ for every $a \in A$.   □

Corollary 2.15. *If G is a P'-torsion group, then all P-local $\mathbb{Z}G$-modules are trivial $\mathbb{Z}G$-modules.*   □

We conclude this section by initiating a comparison of the rings $\mathbb{Z}_P G$ and $(\mathbb{Z}G)_S$, for a P-local group G. The lemma below follows directly from the definitions.

Lemma 2.16. *Let G be any group. Then $(\mathbb{Z}G)_S$ is canonically isomorphic to $\mathbb{Z}_P G$ if and only if the elements of the form $1 + x + \cdots + x^{n-1}$, $x \in G$, $n \in P'$, are units in $\mathbb{Z}_P G$.*   □

On the other hand, we have

Lemma 2.17. *Let $f \colon \mathbb{Z}G \to R$ be a ring homomorphism. View R as a left $\mathbb{Z}G$-module via f. Then R is a P-local $\mathbb{Z}G$-module if and only if $f(1 + x + \cdots + x^{n-1})$ is a unit in R for all $x \in G$, $n \in P'$.*

Proof. This follows from considering the ring homomorphisms

$$\mathbb{Z}G \xrightarrow{f} R \xrightarrow{\mu} \mathrm{End}\,(R)$$

where $(\mu(r))(s) = rs$, and observing that the inverse image of $\mathrm{Aut}\,(R)$ under μ is precisely the group of units of R. $\qquad\square$

Theorem 2.18. *Let G be a P-torsion group. Then*
  (a) $1 + x + \cdots + x^{n-1}$ *is a unit in $\mathbb{Z}_P G$ for all $x \in G$, $n \in P'$.*
  (b) $(\mathbb{Z}G)_S \cong \mathbb{Z}_P G$.

Proof. Given $x \in G$, let $C = \langle x \rangle$ be the cyclic group of G generated by x. By Corollary 2.12, $\mathbb{Z}_P C$ is a P-local $\mathbb{Z}C$-module via the inclusion $\mathbb{Z}C \to \mathbb{Z}_P C$. Now use Lemma 2.17 to infer (a) and Lemma 2.16 to infer (b). $\qquad\square$

Alternatively, a proof of Theorem 2.18 can be based on Galois theory, using the ring isomorphism $\phi \colon \mathbb{Q}C \to \prod_{d \mid m} \mathbb{Q}(\omega^{\frac{m}{d}})$, cf. [25], where C is any cyclic subgroup of G, m the order of C, and $\omega = e^{\frac{2\pi}{m} i}$. We owe this argument to S. Sehgal.

3. The P-localization functor $X \to X_P$

Theorem 2.1 suggests to regard P-local CW-complexes as classifying spaces of A_∞-associative H-groups which are P-local in the sense that their P'-power maps are self homotopy equivalences. Due to the special nature of such spaces, we investigate them here in more detail.

P-local CW-complexes form a class $\mathcal{D}_P$ which fits into an orthogonal pair $(\mathcal{S}_P, \mathcal{D}_P)$ in **Ho** (the pointed homotopy category of connected CW-complexes). Moreover, $\mathcal{D}_P$ is the orthogonal complement of a countable collection of maps; see Theorem 3.1. Maps in $\mathcal{S}_P$ induce a P-equivalence of fundamental groups and isomorphisms in (co-)homology with certain twisted P-local coefficients; see Theorem 3.2. Next, we construct an idempotent

functor $(\ )_P$ associated to the orthogonal pair $(\mathcal{S}_P, \mathcal{D}_P)$ in the sense of Section 1. This functor indeed extends P-localization of nilpotent spaces, as is shown in Section 4.

In the category of groups, the class of P-local groups d_P is the orthogonal complement of the set

$$t_P = \{\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}, \quad n \in P'\}. \tag{3.1}$$

Setting $s_P = (d_P)^{\perp}$, we obtain an orthogonal pair (s_P, d_P) in the category of groups. We refer to homomorphisms in s_P as P-*equivalences*. It is well-known ([12], [22]) that there is an idempotent monad $((\ )_P, \ell)$ associated with the orthogonal pair (s_P, d_P), referred to as P-*localization* of groups. It is also known ([14], [24]) that $((\ )_P, \ell)$ induces on nilpotent groups P-localization in the classical sense ([16]). Moreover, $((\ )_P, \ell)$ is readily seen to be the *maximal* extension of P-localization of nilpotent groups, cf. Remark 1.4 above.

For $k \geq 2$, we denote

$$S_{\tau}^k = S^1 \wedge (S^{k-1} \cup \mathrm{pt}), \tag{3.2}$$

where pt denotes a disjoint base point. Thus S_{τ}^k is a twisted Moore space whose homotopy type is that of $S^1 \vee S^k$, for $k \geq 2$, but their respective suspension-induced co-H-structures are not isomorphic, cf. (3.4) below. For convenience, we define $S_{\tau}^1 = S^1$. Let $\rho_n : S^1 \to S^1$ be the standard map of degree n, and consider the set

$$\mathcal{T}_P = \{\rho_n^k : S_{\tau}^k \to S_{\tau}^k, \quad k \geq 1, \ n \in P'\}, \tag{3.3}$$

where $\rho_n^1 = \rho_n$ and $\rho_n^k = \rho_n \wedge \mathrm{Id}$ for $k \geq 2$. Then we have

Theorem 3.1. *With the above notation,* $(\mathcal{T}_P)^{\perp} = \mathcal{D}_P$ *in* **Ho***.*

Proof. By (2.1), there are natural group homomorphisms

$$[S_{\tau}^k, X] \cong [S^{k-1} \cup \mathrm{pt}, \Omega X] \cong \pi_k(X) \rtimes \pi_1(X) \tag{3.4}$$

for $k \geq 2$ and every connected space X. Moreover, the induced function

$$(\rho_n^k)^* : [S_{\tau}^k, X] \to [S_{\tau}^k, X]$$

is precisely the n-th power map for $k \geq 1$. Thus our claim follows from Theorem 2.1. $\square$

We conclude that the class $\mathcal{D}_P$ is saturated. Maps in $\mathcal{S}_P = (\mathcal{D}_P)^\perp$ are called *P-equivalences.* Our next goal is to give an explicit description of these. For a group G and a set of primes P, we shall write

$$P[G] = \left(\mathbb{Z}[G_P]\right)_S \tag{3.5}$$

where S is the multiplicative closure of the elements $1 + x + x^2 + \cdots + x^{n-1}$, $x \in G_P$, $n \in P'$, in the ring $\mathbb{Z}[G_P]$; cf. (2.5). Hence, $P[G]$-modules are precisely abelian groups provided with a P-local action of G_P. For a space X, we denote, again for simplicity,

$$P[X] = P[\pi_1(X)]. \tag{3.6}$$

With this notation, we have the following characterization of P-equivalences.

Theorem 3.2. *Let $f : X \to Y$ be a map between connected spaces. Then f is a P-equivalence if and only if $f_* : \pi_1(X) \to \pi_1(Y)$ is a P-equivalence of groups and*

$$(a) \qquad f^* : H^k(Y; A) \cong H^k(X; A)$$

for all k and every (left) $P[Y]$-module A or, equivalently,

$$(b) \qquad f_* : H_k(X; P[Y]) \cong H_k(Y; P[Y])$$

for all k.

Proof. Our argument is analogous to the one sketched in [1, p. 84]. If $f : X \to Y$ is a P-equivalence, then, for each P-local group G, f is orthogonal to $K(G, 1)$, and this provides a bijection

$$f^* : \mathrm{Hom}\,(\pi_1(Y), G) \cong \mathrm{Hom}\,(\pi_1(X), G).$$

This proves the first assertion. The second assertion is proved using a twisted Eilenberg-MacLane space ([15]): Given a $P[Y]$-module A and an integer $n \geq 1$, set $G = \pi_1(Y)_P$ and let G act by homeomorphisms on a $K(A, n)$. Consider the space

$$L_G(A, n) = EG \times_G K(A, n), \tag{3.7}$$

where EG is the universal cover of $K(G, 1)$. If $n \geq 2$, then $L_G(A, n)$ has precisely two nonvanishing homotopy groups: G in dimension 1 and A in dimension n. Moreover, the induced action of G on A is the given one. If $n = 1$, then

$L_G(A, n) = K(A \rtimes G, 1)$. In any case, by Theorem 2.1 and Remark 2.5, $L_G(A, n)$ is P-local and hence orthogonal to f. This gives a bijection

$$f^* : [Y, L_G(A, n)] \cong [X, L_G(A, n)],$$

which restricts to an isomorphism ([15])

$$f^* : H^n(Y; A) \cong H^n(X; A),$$

as required. The case $n = 0$ is trivial. This implies (a).

Now assume that $f_* : \pi_1(X) \to \pi_1(Y)$ is a P-equivalence and that (a) holds. Then, for a map $g : X \to Z$ with Z P-local, there is a unique homomorphism $\varphi : \pi_1(Y) \to \pi_1(Z)$ rendering commutative the diagram

$$
\begin{array}{ccc}
\pi_1(X) & \xrightarrow{f_*} & \pi_1(Y) \\
g_* \downarrow & \swarrow_{\varphi} & \\
\pi_1(Z), & &
\end{array}
\tag{3.8}
$$

because f_* is a P-equivalence and $\pi_1(Z)$ is P-local. Now φ can be realized by a map from the two-skeleton of Y to Z, whose restriction to the one-skeleton is uniquely determined, up to homotopy, by φ. Thus, the obstructions to existence and uniqueness, up to homotopy, of a map $g' : Y \to Z$ such that $g'f \simeq g$ lie in the cohomology groups

$$H^k\big(Y, X; \pi_i(Z)\big), \quad i \geq 2, \quad k \in \{i, i+1\}. \tag{3.9}$$

But these vanish by assumption, because the groups $\pi_i(Z)$, $i \geq 2$, are $P[Y]$-modules via φ, cf. Lemma 2.9. Thus $(b) \implies (a)$ follows from Lemma 5.3. Finally, $(a) \implies (b)$ follows from Corollary 7.3. $\qquad\square$

Now we proceed to prove the existence of an idempotent monad $\big((\ )_P, \ell\big)$ associated with the orthogonal pair $(\mathcal{S}_P, \mathcal{D}_P)$. The argument is a direct application of the technique developed in [12].

Theorem 3.3. *For every connected space X there is a map $\ell : X \to X_P$ in $\mathcal{S}_P$ with X_P in $\mathcal{D}_P$.*

Proof. For each map $\rho_n^k : S_\tau^k \to S_\tau^k$, $k \geq 1$, $n \in P'$, construct the homotopy push-out (double mapping cylinder)

$$
\begin{array}{ccc}
S_\tau^k & \xrightarrow{\ \rho_n^k\ } & S_\tau^k \\
\rho_n^k \downarrow & & \downarrow t_{n,k}^1 \\
S_\tau^k & \xrightarrow[\ t_{n,k}^2\ ]{} & Z_{n,k}
\end{array}
\tag{3.10}
$$

and observe that the maps $t_{n,k}^i$ are P-equivalences, because the ρ_n^k are such. Define, by means of (3.10), maps

$$u_{n,k} : Z_{n,k} \to S_\tau^k \tag{3.11}$$

such that $u_{n,k} t_{n,k}^i = \mathrm{Id}$ for all $k \geq 1$, $n \in P'$, $i = 1, 2$. Then the maps $u_{n,k}$ are P-equivalences as well. We next construct a direct system

$$X = X_0 \xrightarrow{s_0} X_1 \xrightarrow{s_1} X_2 \xrightarrow{s_2} \ldots \tag{3.12}$$

in which each s_i is a P-equivalence. Assume X_i has been constructed for $i \geq 0$. If X_i is P-local, set $X_P = X_i$. Otherwise, for each $n \in P'$ and each element $z \in \pi_k(X_i) \rtimes \pi_1(X_i)$ not having an n-th root, choose a map $\varphi_z : S_\tau^k \to X_i$ representing z. For each pair x, y of elements of $\pi_k(X_i) \rtimes \pi_1(X_i)$ such that $x^n = y^n$, $n \in P'$, choose a map $\psi_{x,y} : Z_{n,k} \to X_i$ using (3.10), in such a way that $\psi_{x,y} t_{n,k}^1$ represents x and $\psi_{x,y} t_{n,k}^2$ represents y. Define a homotopy push-out

$$
\begin{array}{ccc}
\bigvee_{n,k}\big((\vee_z S_\tau^k) \vee (\vee_{x,y} Z_{n,k})\big) & \longrightarrow & \bigvee_{n,k}\big((\vee_z S_\tau^k) \vee (\vee_{x,y} S_\tau^k)\big) \\
\downarrow & & \downarrow \\
X_i & \xrightarrow{\ \ s_i\ \ } & X_{i+1}
\end{array}
\tag{3.13}
$$

in which the upper arrow is a coproduct of copies of maps ρ_n^k and $u_{n,k}$ (which is a P-equivalence) and the left-hand vertical arrow is defined by means of the maps φ_z and $\psi_{x,y}$ as chosen above. Therefore, s_i is a P-equivalence. Let X_P be the homotopy direct limit of (3.12), and $\ell : X \to X_P$ the composite map. It is plain that ℓ is a P-equivalence. To check that X_P is P-local, it suffices to show that X_P is orthogonal to each ρ_n^k. But this is ensured by the construction of (3.13), together with the compactness of the spaces S_τ^k and $Z_{n,k}$. $\qquad\square$

Proposition 3.4.   *For every connected space* X, *the induced homomorphism* $\ell_* : \pi_1(X) \to \pi_1(X_P)$ *is* P-*localization in the category of groups.*

Proof. This is an immediate consequence of Theorem 2.1 and Theorem 3.2. $\qquad\square$

4. Behaviour of the functor $(\quad)_P$ on nilpotent spaces

Let $(\mathcal{S}'_P, \mathcal{D}'_P)$ be the orthogonal pair associated with P-localization of nilpotent spaces ([7], [16]). In this section we prove:

Theorem 4.1. *The orthogonal pair $(\mathcal{S}_P, \mathcal{D}_P)$ discussed in Section 3 extends $(\mathcal{S}'_P, \mathcal{D}'_P)$, that is, $\mathcal{S}'_P \subseteq \mathcal{S}_P$ and $\mathcal{D}'_P \subseteq \mathcal{D}_P$.*

Consequently, for every nilpotent CW-complex X the map $\ell : X \to X_P$ given by Theorem 3.3 is equal to the P-localization of X in the classical sense.

Proof of Theorem 4.1. The fact that the class $\mathcal{D}'_P$ is contained in $\mathcal{D}_P$ has been shown in Section 2. We next check that $\mathcal{S}'_P$ is contained in $\mathcal{S}_P$. Let $f : X \to Y$ be in $\mathcal{S}'_P$, i.e. the spaces X, Y are nilpotent and

$$f_* : H_k(X; \mathbb{Z}_P) \cong H_k(Y; \mathbb{Z}_P) \qquad \text{for all} \quad k. \tag{4.1}$$

Then, as pointed out at the beginning of Section 3, $f_* : \pi_1(X) \to \pi_1(Y)$ is certainly a P-equivalence of groups. Hence, it suffices to prove that

$$f_* : H_k(X; A) \cong H_k(Y; A) \qquad \text{for all} \quad k \text{ and every } P[Y]\text{-module } A; \tag{4.2}$$

see Theorem 3.2. Fix one such module A. Plainly, it suffices to prove (4.2) for maps $f : X \to Y$ which P-localize in the classical sense. In that case, we have a homotopy commutative diagram of universal covers, in which the vertical arrows are P-localizations of nilpotent spaces ([16]):

$$
\begin{array}{ccccc}
\widetilde{X} & \longrightarrow & X & \longrightarrow & K(\pi_1(X), 1) \\
\downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} \\
\widetilde{Y} & \longrightarrow & Y & \longrightarrow & K(\pi_1(Y), 1).
\end{array}
\tag{4.3}
$$

Since the spaces $\widetilde{X}, \widetilde{Y}$ are simply-connected, we have

$$(\widetilde{f})_* : H_k(\widetilde{X}; A) \cong H_k(\widetilde{Y}; A) \qquad \text{for all} \quad k, \tag{4.4}$$

because A is in particular a $\mathbb{Z}_P$-module. Now we look at the induced homomorphisms

$$(\bar{f})_* : H_r\big(\pi_1(X); H_s(\widetilde{X}; A)\big) \to H_r\big(\pi_1(Y); H_s(\widetilde{Y}; A)\big), \tag{4.5}$$

which are isomorphisms for all r, s, by Lemma 4.2 and Theorem 4.3 below, together with (4.4). Therefore, the morphism of spectral sequences with coefficient module A associated with (4.3) is indeed an isomorphism, which implies (4.2). This completes the argument. $\qquad \square$

Lemma 4.2. *Let G be a group acting by homeomorphisms on a connected CW-complex X. Assume that the induced action of G on $H_k(X;\mathbb{Z})$ is nilpotent for all k. Then, for every P-local $\mathbb{Z}G$-module A, the diagonal action of G on $H_k(X;A)$ is P-local for all k (here A is viewed as a trivial coefficient module in X).*

Proof. The case $k = 0$ is trivial. For $k \geq 1$, we have a short exact sequence of $\mathbb{Z}G$-modules (with diagonal action)

$$0 \to H_k(G;\mathbb{Z}) \otimes_{\mathbb{Z}} A \to H_k(G;A) \to \mathrm{Tor}^{\mathbb{Z}}\big(H_{k-1}(G;\mathbb{Z}), A\big) \to 0.$$

Now Lemma 2.6 and Lemma 2.7 give the desired result.   □

This applies, of course, to $G = \pi_1(Y)$, $X = \widetilde{Y}$ in (4.5). The action of $\pi_1(Y)$ on the groups $H_k(\widetilde{Y};\mathbb{Z})$ is nilpotent because the space Y is nilpotent ([16, p. 70]).

Theorem 4.3. *Let G be a nilpotent group and $\ell : G \to G_P$ its P-localization. Then, for every P-local $\mathbb{Z}[G_P]$-module A, the homomorphisms*

$$\ell_* : H_k(G;A) \to H_k(G_P;A) \qquad and$$

$$\ell^* : H^k(G_P;A) \to H^k(G;A)$$

are isomorphisms for all k.

Proof. We shall give the proof only for homology, since it is completely analogous in the case of cohomology. The following argument is partially due to A. Reynol ([21]).

   We argue inductively on the nilpotency class c of G. Thus assume first that G is commutative, so that we can consider the group extensions

$$0 \longrightarrow \mathrm{Ker}\,\ell \longrightarrow G \overset{\ell}{\longrightarrow} \mathrm{im}\,\ell \longrightarrow 0 \tag{4.6}$$

$$0 \longrightarrow \mathrm{im}\,\ell \overset{j}{\longrightarrow} G_P \longrightarrow \mathrm{coker}\,\ell \longrightarrow 0 \tag{4.7}$$

and the corresponding Lyndon-Hochschild-Serre spectral sequences with coefficient module A. Note first that the action of $\mathrm{Ker}\,\ell$ on A is trivial and, hence, from the fact that $\mathrm{Ker}\,\ell$ is P'-torsion ([16]) and the Universal Coefficient Theorem it follows that

$$H_s(\mathrm{Ker}\,\ell;A) = 0 \qquad \text{for all } s \geq 1$$

and

$$H_0(\mathrm{Ker}\,\ell; A) \cong A.$$

Therefore, the spectral sequence associated with (4.6) collapses and gives isomorphisms

$$\ell_* : H_k(G; A) \cong H_k(\mathrm{im}\,\ell; A) \qquad \text{for all} \quad k. \tag{4.8}$$

Now look at the spectral sequence associated with (4.7). The action of $\mathrm{coker}\,\ell$ on $H_s(\mathrm{im}\,\ell; A)$ is P-local by Lemma 2.8 and hence *trivial* by Corollary 2.15, because $\mathrm{coker}\,\ell$ is certainly P'-torsion. This tells us, again by the Universal Coefficient Theorem, that

$$H_r\big(\mathrm{coker}\,\ell; H_s(\mathrm{im}\,\ell; A)\big) = 0 \qquad \text{for} \quad r \geq 1$$

and

$$H_0\big(\mathrm{coker}\,\ell; H_s(\mathrm{im}\,\ell; A)\big) \cong H_s(\mathrm{im}\,\ell; A),$$

so that we obtain isomorphisms

$$j_* : H_k(\mathrm{im}\,\ell; A) \cong H_k(G_P; A) \qquad \text{for all } k. \tag{4.9}$$

Finally, (4.8) and (4.9) together complete the proof in the case $c = 1$. The inductive argument is carried out without any additional difficulty, by using Lemma 2.8 and the diagram of central extensions

$$
\begin{array}{ccccc}
\Gamma^{c-1}G & \rightarrowtail & G & \twoheadrightarrow & G/\Gamma^{c-1}G \\
\ell \downarrow & & \ell \downarrow & & \ell \downarrow \\
(\Gamma^{c-1}G)_P & \rightarrowtail & G_P & \twoheadrightarrow & \big(G/\Gamma^{c-1}G\big)_P\,,
\end{array}
$$

where $\Gamma^k G$, $k \geq 0$, denotes terms in the lower central series of G. $\qquad\square$

Remark 4.4. Theorem 4.3 actually extends Theorem 4.14 in [16, p. 40], since every nilpotent action of a nilpotent P-local group on a $\mathbb{Z}_P$-module is P-local (Lemma 2.10).

We also record a useful corollary of Theorem 4.3, applications of which have been given in [21].

Corollary 4.5. *Let G be a nilpotent group and $\ell : G \to G_P$ its P-localization. Assume that G_P acts on a $\mathbb{Z}_P$-module A through $\omega : G_P \to Aut(A)$ in such a way that $\operatorname{im}\omega$ is a torsion group of finite exponent. Then*

$$\ell_* : H_k(G; A) \cong H_k(G_P; A)$$

and

$$\ell^* : H^k(G_P; A) \cong H^k(G; A), \qquad for\ all\ \ k.$$

Proof. Set $Q = \operatorname{im}\omega$. Then Q is a finite P-group because it is an epimorphic image of a P-local group. Hence, by Corollary 2.12, the action of Q on A is P-local and, by Lemma 2.9, so is the action of G_P on A. Now Theorem 4.3 applies. $\qquad\square$

Note that the hypothesis concerning the action ω is essential. For otherwise we have the explicit counterexample $G = \mathbb{Z}$ and $A = \mathbb{Q}[\mathbb{Q}]$; see [10].

5. Localizing with respect to homology with twisted coefficients

In this section and the two next ones we consider our P-localization functor from the homology point of view. We show that it is a particular output of a certain rich source of idempotent monads in **Ho**. In fact, Bousfield's localization with respect to homology ([4]) can be paralleled for homology with twisted coefficients, provided we work in the appropriate categories. The resulting localizing functors can be "spliced" to produce localizing functors in **Ho**, some of which extend the familiar P-localization of nilpotent spaces. In this section, we explain the procedure of localizing with respect to homology with twisted coefficients. Splicing will be described in Section 6.

For a group G, denote by **Ho**(G) the category whose objects are pairs (X, φ), where X is a connected space and $\varphi : \pi_1(X) \to G$ is a group homomorphism. Morphisms from (X, φ) to (Y, ψ) are (pointed homotopy classes of) maps $f : X \to Y$ such that $\varphi = \psi f_*$, where $f_* : \pi_1(X) \to \pi_1(Y)$ is the homomorphism induced by f.

Theorem 5.1. *Let G be a group and A a (right) $\mathbb{Z}G$-module. Then there is an idempotent monad (E, η) in the category **Ho**(G) whose class of E-equivalences consists precisely of all $H_*(\ ; A)$-equivalences.*

We base our proof of Theorem 5.1 on the existence result for localizing functors given in [12]. For the convenience of the reader, it is reproduced as Theorem 5.2 below, in a form adapted to our present purposes.

Theorem 5.2. *Suppose that a class* $\mathcal{S}$ *of morphisms in* $\mathbf{Ho}(G)$ *satisfies conditions (C1) to (C6) below. Then* $\mathcal{S}$ *is saturated and* $\mathbf{Ho}(G)$ *admits a localizing functor whose class of equivalences is* $\mathcal{S}$.

>   (C1) $\mathcal{S}$ *contains all isomorphisms in* $\mathbf{Ho}(G)$.
>   (C2) *If the composite* gf *of two morphisms is defined and any two of* f, g, gf *are in* $\mathcal{S}$, *then the third is also in* $\mathcal{S}$.
>   (C3) *The coproduct of a set of morphisms in* $\mathcal{S}$ *is in* $\mathcal{S}$.
>   (C4) *There is a set* $\mathcal{S}_0 \subseteq \mathcal{S}$ *such that* $(\mathcal{S}_0)^{\perp} = \mathcal{S}^{\perp}$.
>   (C5) *For every diagram* $C \xleftarrow{f} A \xrightarrow{s} B$ *with* $s \in \mathcal{S}$ *there is a weak push-out*

$$
\begin{array}{ccc}
A & \xrightarrow{\;s\;} & B \\
{\scriptstyle f}\downarrow & & \downarrow \\
C & \xrightarrow{\;t\;} & Z
\end{array}
$$

>   *with* $t \in \mathcal{S}$.
>   (C6) *For every ordinal* c *and every directed system* $F : c \to \mathbf{Ho}(G)$ *with* $s_i : F(0) \to F(i)$ *in* $\mathcal{S}$ *for all* $i < c$, *there is a weak colimit* T *such that the morphism* $s_c : F(0) \to T$ *is in* $\mathcal{S}$. $\qquad\square$

Now we proceed to verify these hypotheses in our case.

Proof of Theorem 5.1. Let $\mathcal{S}$ be the class of $H_*(\; ; A)$-equivalences in $\mathbf{Ho}(G)$. Conditions (C1), (C2) and (C3) are obviously satisfied. Let c be the smallest infinite ordinal whose cardinality is bigger than the cardinality of A. The collection of spaces X with at most c cells forms a proper set of homotopy types. Choose a representing space in each of these homotopy types, and let $\mathcal{S}_0$ consist of all morphisms $f : (X, \varphi) \to (Y, \psi)$ in $\mathcal{S}$ where X and Y are such representing spaces. Then Lemma 11.3 in [4] can be adapted to also incorporate twisted coefficients. This proves that $(\mathcal{S}_0)^{\perp} = \mathcal{S}^{\perp}$. Hence, condition (C4) is satisfied. In (C5) and (C6) we can simply take the corresponding homotopy colimits, together with the obvious group homomorphisms. $\qquad\square$

In general, the construction of $\eta_X : X \to EX$ for a given space X will require a homotopy direct limit of length up to the ordinal c; see [4], [11].

The next lemmas are the key to interrelate the functors given by Theorem 5.1. Note that, if (X, φ) is an object of $\mathbf{Ho}(G)$, then each ring homomorphism $\alpha : \mathbb{Z}G \to R$ allows us to regard R as a twisted coefficient module in X.

Lemma 5.3. *Let G be a group and $f : (X, \varphi) \to (Y, \psi)$ a morphism in $\mathbf{Ho}(G)$. Assume given a ring homomorphism $\alpha : \mathbb{Z}G \to R$. If f is an $H_*(\ ; R)$-equivalence, then*

> (i) *f is an $H_*(\ ; A)$-equivalence for every right R-module A; and*
> (ii) *f is an $H^*(\ ; B)$-equivalence for every left R-module B.*

Proof. We can assume, without loss of generality, that X is a subcomplex of Y and that f is the corresponding inclusion. Let $q : \widetilde{Y} \to Y$ be the universal cover of Y and set $\widehat{X} = q^{-1}(X)$, $\pi = \pi_1(Y)$. Let $C_* = C_*(\widetilde{Y}, \widehat{X})$ denote the associated cellular chain complex. Then the hypothesis

$$0 = H_k(Y, X; R) = H_k(R \otimes_{\mathbb{Z}\pi} C_*) \qquad \text{for all} \quad k,$$

tells us precisely that $R \otimes_{\mathbb{Z}\pi} C_*$ is a free resolution of 0 as an R-module. Consequently,

$$H_k(Y, X; A) \cong H_k(A \otimes_R R \otimes_{\mathbb{Z}\pi} C_*)$$

$$= \mathrm{Tor}_k^R(A, 0) = 0$$

and

$$H^k(Y, X; B) \cong H^k \mathrm{Hom}_{\mathbb{Z}\pi}(C_*, \mathrm{Hom}_R(R, B))$$

$$\cong H^k \mathrm{Hom}_R(R \otimes_{\mathbb{Z}\pi} C_*, B)$$

$$= \mathrm{Ext}_R^k(0, B) = 0, \qquad \text{for all} \quad k.$$

$\square$

Let $e : \mathbb{Z} \to \mathbb{Z}_P$ denote the inclusion, and e_n denote the composite

$$\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{e} \mathbb{Z}_P. \tag{5.1}$$

Given a $\mathbb{Z}[\mathbb{Z}_P]$-module A, we shall denote by $e_n^* A$ the $\mathbb{Z}[\mathbb{Z}]$-module defined via (5.1).

With the notation of Section 3, we can consider in $\mathbf{Ho}(\mathbb{Z}_P)$ the morphisms

$$\rho_n^k : (S_\tau^k, e_n) \longrightarrow (S_\tau^k, e), \qquad k \geq 1, \ n \in P'. \tag{5.2}$$

Lemma 5.4.

  (i) ρ_n^k is an $H_*(\ ;P[\mathbb{Z}])$-equivalence for all $k \geq 1$, $n \in P'$.

  (ii) $H_1(S_\tau^k; A) = 0$ for all $k \geq 1$, where A is any one of the following coefficient modules $(n \in P')$:

$$\mathbb{Z}[\mathbb{Z}_P], \quad e_n^*\mathbb{Z}[\mathbb{Z}_P], \quad \mathbb{Z}_P[\mathbb{Z}_P], \quad e_n^*\mathbb{Z}_P[\mathbb{Z}_P], \quad P[\mathbb{Z}], \quad e_n^*P[\mathbb{Z}].$$

Proof. Let x be a generator of $\pi_1(S_\tau^k)$. The cellular chain complex of the universal cover of S_τ^k consists of a copy of $\mathbb{Z}[\mathbb{Z}]$ in dimensions $0, 1$ and k. Therefore, the homomorphisms

$$(\rho_n^k)_* : H_i(S_\tau^k; e_n^*P[\mathbb{Z}]) \longrightarrow H_i(S_\tau^k; P[\mathbb{Z}]) \tag{5.3}$$

are induced by the chain map

$$
\begin{array}{ccccccccccc}
\ldots 0 & \longrightarrow & P[\mathbb{Z}] & \xrightarrow{\partial_k} & 0 & \longrightarrow & \ldots & \longrightarrow & P[\mathbb{Z}] & \xrightarrow{\partial_1} & P[\mathbb{Z}] & \longrightarrow & 0 \\
& & \downarrow{\varphi_k} & & \downarrow & & & & \downarrow{\varphi_1} & & \downarrow{=} & & \downarrow \\
\ldots 0 & \longrightarrow & P[\mathbb{Z}] & \xrightarrow{\partial_k'} & 0 & \longrightarrow & \ldots & \longrightarrow & P[\mathbb{Z}] & \xrightarrow{\partial_1'} & P[\mathbb{Z}] & \longrightarrow & 0
\end{array}
$$

where $\partial_1 = 1 - x^n$, $\partial_1' = 1 - x$, $\partial_k = \partial_k' = 0$ (even if $k = 2$), and

$$\varphi_1 = \varphi_k = 1 + x + x^2 + \cdots + x^{n-1}.$$

Therefore, the above chain map is a chain isomorphism and (5.3) is an isomorphism for all i, as claimed. To prove (ii), consider the chain map above with coefficient module A. First of all, ∂_1 and ∂_1' are monomorphisms with any of the first four coefficient systems listed, because $\mathbb{Z}[\mathbb{Z}_P]$ and $\mathbb{Z}_P[\mathbb{Z}_P]$ contain no zero divisors. For the remaining two systems, the claim follows because $P[\mathbb{Z}]$ is flat as a $\mathbb{Z}[\mathbb{Z}_P]$-module. $\square$

6. Splicing localization functors

In this section we show how localization functors on different categories **Ho**(G) can be spliced to yield localization functors on **Ho**. The main abstract result is Theorem 6.2. In conjunction with the ordering of orthogonal pairs, as in Remark 1.4, it serves us as a pivotal point to access and compare various idempotent functors on **Ho**.

The basic ingredient in Theorem 6.2 is a pair of *matching functors*. To explain this, suppose given

  (i) a localization functor L on the category of groups, with associated orthogonal pair $(\mathcal{S}, \mathcal{D})$;

(ii) a functor $F : \mathbb{Z}\mathcal{D} \to \mathcal{R}$ together with a natural transformation $T : \mathrm{Id} \to F$. Here $\mathcal{R}$ denotes the category of rings, and $\mathbb{Z}\mathcal{D}$ denotes the category of integral group rings of groups in $\mathcal{D}$ (and ring homomorphisms induced by group homomorphisms).

To a connected CW-complex X, we apply the following process: Associate to X the object (X, φ_X) in $\mathbf{Ho}\big(L\pi_1(X)\big)$, where $\varphi_X : \pi_1(X) \to L\pi_1(X)$ is the localizing homomorphism. By Theorem 5.1, the object (X, φ_X) has an $H_*(\ ; F\mathbb{Z}[L\pi_1(X)])$-localization in the category $\mathbf{Ho}\big(L\pi_1(X)\big)$. Denote it by (X_L^F, ψ_X), $\psi_X : \pi_1(X_L^F) \to L\pi_1(X)$. Now forget the twist in this object, so as to obtain again a space X_L^F in $\mathbf{Ho}$. Schematically,

$$
\begin{array}{ccc}
\mathbf{Ho}\big(L\pi_1(X)\big): & (X, \varphi_X) \longrightarrow (X_L^F, \psi_X) & \\
& \uparrow \qquad\qquad \downarrow & \qquad (6.1) \\
\mathbf{Ho}: & X \xrightarrow{\ \eta_X\ } X_L^F &
\end{array}
$$

Definition 6.1. We say that the functors F and L *match* if the induced homomorphism $(\eta_X)_* : \pi_1(X) \to \pi_1(X_L^F)$ is an L-equivalence for every space X.

Theorem 6.2 ("Splicing theorem"). *For matching functors F and L the association $X \to X_L^F$ determines an idempotent monad on $\mathbf{Ho}$ with natural transformation η.*

Proof. First we check functoriality. Thus start with a map $f : X \to Y$ and let $f_* : \pi_1(X) \to \pi_1(Y)$ be the induced homomorphism of fundamental groups. Then the morphism in $\mathbf{Ho}\big(L\pi_1(Y)\big)$

$$
\big(X, (Lf_*)\varphi_X\big) \xrightarrow{\ \eta_X\ } \big(X_L^F, (Lf_*)\psi_X\big)
$$

is an $H_*(\ ; F\mathbb{Z}[L\pi_1(Y)])$-equivalence, by Lemma 5.3. Hence, there is a unique morphism f' in $\mathbf{Ho}\big(L\pi_1(Y)\big)$ rendering commutative the diagram

$$
\begin{array}{ccc}
\big(X, (Lf_*)\varphi_X\big) & \xrightarrow{\ \eta_X\ } & \big(X_L^F, (Lf_*)\psi_X\big) \\
f \downarrow & & \downarrow f' \qquad\qquad (6.2) \\
(Y, \varphi_Y) & \xrightarrow{\ \eta_Y\ } & (Y_L^F, \psi_Y),
\end{array}
$$

which gives a commutative diagram in $\mathbf{Ho}$

$$
\begin{array}{ccc}
X & \xrightarrow{\ \eta_X\ } & X_L^F \\
f \downarrow & & \downarrow f' \qquad\qquad (6.3) \\
Y & \xrightarrow{\ \eta_Y\ } & Y_L^F.
\end{array}
$$

Moreover, if $f'' : X_L^F \to Y_L^F$ also renders (6.3) commutative, then

$$\psi_Y(f'')_*(\eta_X)_* = \psi_Y(\eta_Y)_* f_* = \psi_Y(f')_*(\eta_X)_* = (Lf_*)\psi_X(\eta_X)_*,$$

and, since $(\eta_X)_*$ is assumed to be an L-equivalence, it follows that $\psi_Y(f'')_* = (Lf_*)\psi_X$. Thus f'' is also a morphism in $\mathbf{Ho}\big(L\pi_1(Y)\big)$ and hence $f'' \simeq f'$. Functoriality follows. Next we check idempotence. Since $L\pi_1(X_L^F) \cong L\pi_1(X)$ by assumption, the homomorphism $\varphi_{X_L^F}$ coincides with ψ_X. Therefore, the object $\big(X_L^F, \varphi_{X_L^F}\big)$ is already $H_*(\ ; F\mathbb{Z}[L\pi_1(X)])$-local, and it follows that the map $\eta_{X_L^F} = (\eta_X)' : X_L^F \to \big(X_L^F\big)_L^F$ is a homotopy equivalence. $\qquad\square$

We say that an idempotent monad (E, η) in $\mathbf{Ho}$ is *spliceable* if it can be obtained from a pair of matching functors F, L by means of (6.1).

Theorem 6.3. *The P-localization monad $((\ )_P, \ell)$ described in Section 3 is spliceable.*

Proof. Choose L to be P-localization of groups and set $F\mathbb{Z}G = (\mathbb{Z}G)_S$; cf. (2.5). Thus, for a CW-complex X, $F\mathbb{Z}[L\pi_1(X)] = P[X]$; cf. (3.6). Next we show that the functors F and L match, and that the resulting monad is precisely $((\ )_P, \ell)$. Given a space X, consider the morphism

$$\ell : (X, \varphi_X) \to (X_P, \mathrm{Id}) \tag{6.4}$$

in $\mathbf{Ho}(\pi_1(X)_P)$. We check that (6.4) is $H_*(\ ; P[X])$-localization, so that $X_L^F \simeq X_P$ and $\eta_X = \ell$, as claimed.

First we see that (X_P, Id) is an $H_*(\ ; P[X])$-local object. Thus assume given an $H_*(\ ; P[X])$-equivalence $s : (A, u) \to (B, v)$ and a morphism $f : (A, u) \to (X_P, \mathrm{Id})$. Since the groups $\pi_i(X_P)$, $i \geq 2$, are $P[X]$-modules, the cohomology groups $H^k\big(B, A; \pi_i(X_P)\big)$ vanish for $i \geq 2$, by Lemma 5.3. Now existence and uniqueness of a morphism $g : (B, v) \to (X_P, \mathrm{Id})$ satisfying $gs = f$ follow by obstruction theory, as in (3.9).

Secondly, we check that (6.4) is an $H_*(\ ; P[X])$-equivalence. Let (C, ω) be an $H_*(\ ; P[X])$-local object in $\mathbf{Ho}(\pi_1(X)_P)$, and assume given a morphism $f : (X, \varphi_X) \to (C, \omega)$. By Lemma 5.3 and Lemma 5.4, the morphisms $\rho_n^k : S_\tau^k \to S_\tau^k$ introduced in (3.3) are $H_*(\ ; P[X])$-equivalences via any homomorphism $\mathbb{Z} \to \pi_1(X)_P$. It follows that C is a P-local space, by Theorem 3.1. Since $\ell : X \to X_P$ P-localizes in $\mathbf{Ho}$,

existence and uniqueness of a morphism $g : (X_P, \mathrm{Id}) \to (C, \omega)$ satisfying $g\ell \simeq f$ is straightforward. $\hfill \square$

Example 6.4. The class $\mathcal{D}$ of connected spaces whose higher homotopy groups are P-local satisfies $\mathcal{D} = \mathcal{T}^\perp$ in **Ho**, where

$$\mathcal{T} = \{\rho_n : S^k \to S^k, \ k \geq 2, \ n \in P'\}, \tag{6.5}$$

ρ_n denoting the standard map of degree n. Using the same arguments as in Theorem 3.2, one sees that the class $\mathcal{S} = \mathcal{D}^\perp$ consists of those maps $f : X \to Y$ inducing an isomorphism of fundamental groups and being $H^*(\ ; A)$-equivalences for every $\mathbb{Z}_P[\pi_1(Y)]$-module A. Paralleling the arguments of Section 3, an idempotent monad (E, η) associated with the orthogonal pair $(\mathcal{S}, \mathcal{D})$ can be obtained. We claim that this monad is spliceable. In the setting of Theorem 6.2, choose L to be the identity and $F\mathbb{Z}G = \mathbb{Z}_P \otimes \mathbb{Z}G = \mathbb{Z}_P G$. These functors match, and the resulting monad is (E, η). The proof is completely analogous to the proof of Theorem 6.3.

The orthogonal pair generated by $\{\rho_n : S^k \to S^k, \ k \geq 1, \ n \in P'\}$ does not admit a localization functor; see [10].

Example 6.5. Given an arbitrary ring R, viewed as a trivial coefficient module, the $H_*(\ ; R)$-localization functor of [1], [4] is spliceable: Choose L to be $LG = \{1\}$ for every group G, and $F\mathbb{Z}G = R$.

Example 6.6. For a ring R, let L be the HR-localization functor in the category of groups ([4]), and let F be any functor such that $F\mathbb{Z} = R$. Then L, F match because every $H_*(\ ; F\mathbb{Z}G)$-equivalence is also an $H_*(\ ; R)$-equivalence, which induces an L-equivalence of fundamental groups; cf. [4]. If $F\mathbb{Z}G = R$ for every G, then the resulting functor is again $H_*(\ ; R)$-localization.

We close this section by displaying the ordering relations between some of the spliceable localizing functors explicitly described above; see Remark 1.3. Thus, there is a natural transformation $E \to E'$ if the class of E-equivalences is contained in the class of E'-equivalences.

We denote by $(\ )_P$ both P-localization of groups and spaces, and by E_P both $H\mathbb{Z}_P$-localization of groups and $H_*(\ ; \mathbb{Z}_P)$-localization of spaces. In the following

diagram, we represent each functor by a matching pair L, F defining it. The arrows are natural transformations.

$$(A) \quad L = \mathrm{Id}, \; F\mathbb{Z}G = \mathbb{Z}_P G$$

$$(B) \quad L = (\;)_P, \; F\mathbb{Z}G = (\mathbb{Z}G)_S \qquad\qquad (C) \quad L = E_P, \; F\mathbb{Z}G = \mathbb{Z}_P G$$

$$(D) \quad L = E_P, \; F\mathbb{Z}G = (\mathbb{Z}G)_S$$

$$\downarrow$$

$$(E) \quad L = E_P, \; F\mathbb{Z}G = \mathbb{Z}_P.$$

Here (B) is the P-localization functor, and (E) is the $H_*(\;;\mathbb{Z}_P)$-localization functor.

Proposition 6.7. $H_*(\;;\mathbb{Z}_P)$-localization is the minimal extension of P-localization of nilpotent spaces.

Proof. The orthogonal complement of all P-local nilpotent spaces is contained in the class of all $H_*(\;;\mathbb{Z}_P)$- equivalences, because the spaces $K(\mathbb{Z}_P, n)$, $n \geq 1$, are nilpotent and P-local. On the other hand, $H^*(\;;\mathbb{Z}_P)$-equivalences coincide with $H_*(\;;\mathbb{Z}_P)$-equivalences and their localization extends P-localization of nilpotent spaces. Now our claim follows from Proposition 2.3 in [12]. □

We conjecture that the functor $(\;)_P$ is the maximal extension of P-localization of nilpotent spaces. We know that the functor (C) in the diagram above does *not* extend P-localization of nilpotent spaces (nor does any functor above it), because an $H_*(\;;\mathbb{Z}_P)$-equivalence between nilpotent spaces need not be an $H_*(\;;A)$-equivalence for A an arbitrary coefficient module whose underlying abelian group is P-local ([10]).

7. The class of equivalences of a spliceable localization functor

Let L and F be a matching pair of functors in the sense of Section 6, and let (E, η) be the associated idempotent monad in **Ho**, cf. Theorem 6.2.

Lemma 7.1. *If G is an L-local group, then $K(G,1)_L^F \simeq K(G,1)$.*

Proof. It suffices to show that the object $\big(K(G,1), \mathrm{Id}\big)$ is $H_*(\ ; F\mathbb{Z}G)$-local in $\mathbf{Ho}(G)$. But $\big(K(G,1), \mathrm{Id}\big)$ is indeed orthogonal to every morphism in $\mathbf{Ho}(G)$. $\square$

Theorem 7.2. *A map $f : X \to Y$ in $\mathbf{Ho}$ is an E-equivalence if and only if $f_* : \pi_1(X) \to \pi_1(Y)$ is an L-equivalence and f is an $H_*(\ ; F\mathbb{Z}[L\pi_1(Y)])$-equivalence.*

Proof. If f is an E-equivalence, then, for every L-local group G, $K(G,1)$ is E-local by Lemma 7.1 and hence orthogonal to f. This proves the first claim. Now, in the diagram (6.2), $f' : \big(X_L^F, (Lf_*)\psi_X\big) \to \big(Y_L^F, \psi_Y\big)$ is an isomorphism by assumption and the horizontal arrows are $H_*(\ ; F\mathbb{Z}[L\pi_1(Y)])$-equivalences, so that f is also an $H_*(\ ; F\mathbb{Z}[L\pi_1(Y)])$-equivalence. Conversely, if $f_* : \pi_1(X) \to \pi_1(Y)$ is an L-equivalence, then the localization diagram in $\mathbf{Ho}$

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\eta_X \downarrow & & \downarrow \eta_Y \\
X_L^F & \xrightarrow[\ f'\ ]{} & Y_L^F
\end{array}
$$

corresponds to a localization diagram in $\mathbf{Ho}\big(L\pi_1(Y)\big)$. But then f' is a homotopy equivalence because f is an $H_*(\ ; F\mathbb{Z}[L\pi_1(Y)])$-equivalence. $\square$

For the spliceable functor $(\ )_P$, we have an independent characterization of its equivalences in terms of cohomology (see Theorem 3.2). Both descriptions together yield

Corollary 7.3. *Let $f : X \to Y$ be a map in $\mathbf{Ho}$ inducing a P-equivalence of fundamental groups. Then the following are equivalent:*

(i) f is an $H_*(\ ; P[Y])$-equivalence.

(ii) f is an $H_*(\ ; A)$-equivalence for every right $P[Y]$-module A.

(iii) f is an $H^*(\ ; B)$-equivalence for every left $P[Y]$-module B. $\square$

8. Some special features of P-localization

In this section we consider the connections between the functor $(\ )_P$ discussed in Section 3, the $H_*(\ ;\mathbb{Z}_P)$-localization functor E_P, and the $\mathbb{Z}_P$-completion functor $(\mathbb{Z}_P)_\infty$ of Bousfield-Kan ([7]). All spaces X are assumed to be connected.

From [7, II, 2.8] and the results of Section 6, we know that for every space X there is a homotopy commutative diagram

$$
\begin{array}{ccccc}
 & & X & & \\
 & \ell \swarrow & \eta \downarrow & \phi \searrow & \\
X_P & \xrightarrow{\ \beta\ } & E_P X & \xrightarrow{\ \gamma\ } & (\mathbb{Z}_P)_\infty X
\end{array}
\tag{8.1}
$$

where ℓ, η are the respective localization maps, and ϕ is the $\mathbb{Z}_P$-completion map. Moreover, γ is a homotopy equivalence if and only if the space X is $\mathbb{Z}_P$-good ([7]), i.e. if and only if ϕ is an $H_*(\ ;\mathbb{Z}_P)$-equivalence. The study of $\mathbb{Z}_P$-good spaces continues to be a matter of current interest ([6]).

If the space X is nilpotent, then X is $\mathbb{Z}_P$-good and the maps β, γ in (8.1) are homotopy equivalences. We provide the following generalization:

Proposition 8.1. *If the space* X_P *is nilpotent, then* X *is* $\mathbb{Z}_P$-*good and the maps* β, γ *in (8.1) are homotopy equivalences.*

Proof. Every nilpotent P-local space is $H_*(\ ;\mathbb{Z}_P)$-local. Thus, under our assumption, $\ell : X \to X_P$ is an $H_*(\ ;\mathbb{Z}_P)$-equivalence into an $H_*(\ ;\mathbb{Z}_P)$-local space and hence $\ell \simeq \eta$. Secondly, ℓ induces a homotopy equivalence

$$(\mathbb{Z}_P)_\infty X \simeq (\mathbb{Z}_P)_\infty X_P,$$

and plainly $(\mathbb{Z}_P)_\infty X_P \simeq X_P$, because X_P is nilpotent P-local. It follows that $\gamma\beta$ is a homotopy equivalence, and hence so is γ. $\qquad\square$

The nilpotency of X_P can be checked in practice in some cases. For example, it is guaranteed whenever $\pi_1(X)_P = \{1\}$. Note that spaces such that $\pi_1(X)_P = \{1\}$ satisfy also $H_1(X;\mathbb{Z}_P) = 0$ and hence they are indeed known to be $\mathbb{Z}_P$-good ([7, VII, 3.2]).

Example 8.2. Let $P = \{3\}$ and Σ_3 denote the symmetric group on three elements. Then $(\Sigma_3)_3 = \{1\}$ and, therefore,

$$K(\Sigma_3, 1)_3 \simeq E_3 K(\Sigma_3, 1) \simeq (\mathbb{Z}_3)_\infty K(\Sigma_3, 1), \tag{8.2}$$

which is a simply-connected space with nontrivial homotopy [7, VII, 4.4].

This last example turns out to be a particular case of a more general fact (Theorem 8.4 below). We are indebted to G. Mislin for some key ideas in the next proposition.

Proposition 8.3. *Let X be a space whose homotopy groups $\pi_i(X)$ are finite for $i \geq 1$, and let P consist of a single prime p. Then*

(i) *The homotopy groups $\pi_i\big((\mathbb{Z}_P)_\infty X\big)$ are finite p-groups for $i \geq 1$.*

(ii) *The space $(\mathbb{Z}_P)_\infty X$ is nilpotent.*

(iii) *The $\mathbb{Z}_P$-completion map $\phi : X \to (\mathbb{Z}_P)_\infty X$ is an $H_*(\ ; A)$-equivalence for every (twisted) coefficient module A whose underlying abelian group is a $\mathbb{Z}_P$-module.*

(iv) *The kernel N of $\phi_* : \pi_1(X) \to \pi_1\big((\mathbb{Z}_P)_\infty X\big)$ satisfies $N_P = \{1\}$.*

Proof. Statement (i) is proved in [7, VII, 4.3]. Statement (ii) follows because every action of a finite p-group on another finite p-group is nilpotent. Now let F be the homotopy fibre of ϕ. The homotopy groups $\pi_i(F)$ are finite for $i \geq 1$, and, hence, so are the homology groups $H_i(F; \mathbb{Z})$, $i \geq 1$. It follows that any action of $\pi_1\big((\mathbb{Z}_P)_\infty X\big)$ on the groups $H_i(F; \mathbb{Z}_P)$ must be nilpotent. Hence, we can apply the spectral sequence comparison theorem ([17]) to the diagram

$$
\begin{array}{ccccc}
F & \longrightarrow & X & \xrightarrow{\ \phi\ } & (\mathbb{Z}_P)_\infty X \\
\downarrow & & {\scriptstyle \phi}\downarrow & & {\scriptstyle =}\downarrow \\
\mathrm{pt} & \longrightarrow & (\mathbb{Z}_P)_\infty X & \xrightarrow{\ =\ } & (\mathbb{Z}_P)_\infty X \ .
\end{array}
\tag{8.3}
$$

Since X is $\mathbb{Z}_P$-good ([7, VII, 4.3]), ϕ is an $H_*(\ ; \mathbb{Z}_P)$-equivalence, implying that $H_i(F; \mathbb{Z}_P) = 0$, for $i \geq 1$. Now, given a $\mathbb{Z}_P$-module A with an action of $\pi_1\big((\mathbb{Z}_P)_\infty X\big)$, the groups $H_i(F; A)$, $i \geq 1$, vanish because the action of $\pi_1(F)$ on A via ϕ is trivial. It follows from the top fibration in (8.3) that

$$\phi_* : H_i(X; A) \cong H_i\big((\mathbb{Z}_P)_\infty X; A\big) \qquad \text{for all } i,$$

thus proving (iii). To prove (iv), observe that $N = \mathrm{Ker}\,(\phi_*)$ is an epimorphic image of $\pi_1(F)$ and, since P-localization preserves epimorphisms ([22]), it suffices to show that $\pi_1(F)_P = \{1\}$. By Corollary 2.3 in [9],

$$\pi_1(F)_P \cong \pi_1\big((\mathbb{Z}_P)_\infty F\big).$$

and $\pi_1\big((\mathbb{Z}_P)_\infty F\big)$ vanishes because $H_1(F; \mathbb{Z}_P) = 0$. $\square$

Theorem 8.4. *Let X be a space whose homotopy groups $\pi_i(X)$ are finite for $i \geq 1$, and P consist of a single prime p. Then X_P is nilpotent and the maps β, γ in (8.1) are homotopy equivalences.*

Proof. From (iii) and (iv) in Proposition 8.3 it follows that the map $\phi : X \to (\mathbb{Z}_P)_\infty X$ is a P-equivalence, and from (i) and (ii) that $(\mathbb{Z}_P)_\infty X$ is nilpotent and P-local. Hence $\phi \simeq \ell$ and X_P is nilpotent. Finally, $\phi \simeq \eta$ because X is $\mathbb{Z}_P$-good. $\square$

For a space X and a perfect normal subgroup N of $\pi_1(X)$ (i.e. $H_1(N; \mathbb{Z}) = 0$), Quillen's plus-construction yields a homology equivalence with arbitrary twisted coefficients $q : X \to X^+$ such that the kernel of $q_* : \pi_1(X) \to \pi_1(X^+)$ is N, cf. [20].

Proposition 8.5. *Let P be a set of primes and let X be a space whose fundamental group has a perfect normal subgroup N belonging to the kernel of the P-localizing homomorphism $\pi_1(X) \to \pi_1(X)_P$. Then the Quillen map $q : X \to X^+$ associated to N induces a homotopy equivalence $X_P \simeq (X^+)_P$.*

Proof. Under these hypotheses, $q_* : \pi_1(X) \to \pi_1(X^+)$ is a P-equivalence of groups and hence $q : X \to X^+$ is a P-equivalence of spaces. $\square$

The hypotheses of Proposition 8.5 are in particular satisfied if P consists of a single prime p and $\pi_1(X)$ is a finite perfect group. This provides an alternative way to construct the space X_P in that case: First, add 2-cells and 3-cells to X, so as to kill $\pi_1(X)$ preserving homology, and then P-localize the resulting simply-connected space X^+.

We conclude by explaining the effect of the functor $(\ )_P$ on a wedge of circles. Thus let $X = \bigvee_{i \in I} S^1$, where the index set I contains at least two elements. Then $(\mathbb{Z}_P)_\infty X$

is an aspherical space with uncountable fundamental group ([7, IV, 5.3]), while $E_P X$ is not known in general. However, it is known that the homomorphism

$$\gamma_* : \pi_1(E_P X) \to \pi_1\big((\mathbb{Z}_P)_\infty X\big)$$

is strictly surjective ([5]). We next prove that X_P is aspherical. In this context, note that P-localization sends countable groups to countable groups. Therefore, $\pi_1(X_P)$ is also countable, if I is countable.

We first state an obvious necessary and sufficient condition for a space $K(G, 1)_P$ to be aspherical; cf. Theorem 3.2 and Lemma 5.3.

Proposition 8.6. *Given a group G and a set of primes P, the space $K(G, 1)_P$ is a $K(G_P, 1)$ if and only if the P-localizing homomorphism $\ell : G \to G_P$ induces isomorphisms*

$$\ell_* : H_k(G; A) \cong H_k(G_P; A)$$

for all k and every $P[G]$-module A.   □

Known examples of groups G satisfying the condition stated include:

(i) Nilpotent groups (Theorem 4.3).
(ii) Finite P-nilpotent groups ([9]).
(iii) The fundamental group of the Klein bottle.
(iv) All free groups (Theorem 8.7 below).

Theorem 8.7. *If F is a free group and P a set of primes, then the P-localizing homomorphism $\ell : F \to F_P$ induces isomorphisms*

$$\ell_* : H_k(F; A) \cong H_k(F_P; A)$$

for all k and every $P[F]$-module A.

Proof. We use the construction of F_P given by Baumslag ([3]). Thus

$$F_P = \varinjlim \Phi(i)$$

for a certain direct system Φ in the category of groups, indexed by a (large) ordinal α, with $\Phi(0) = F$, and in which each map $s_i : \Phi(i) \to \Phi(i+1)$ is injective. For a

given index i, either $\Phi(i+1) = \Phi(i)$ and s_i is the identity, or there exists a subgroup $\mathbb{Z} \subseteq U \subseteq \mathbb{Z}_P$ and a push-out diagram

$$
\begin{array}{ccc}
U & \overset{j}{\hookrightarrow} & \mathbb{Z}_P \\
\downarrow & & \downarrow \\
\Phi(i) & \underset{s_i}{\rightarrow} & \Phi(i+1)
\end{array}
\qquad (8.4)
$$

where j is the inclusion and all arrows are monomorphisms. Now assume given an abelian group A provided with a P-local action of F_P. Since homology commutes with direct limits, it suffices to show that every s_i is an $H_*(\ ;A)$-equivalence. But the inclusion j P-localizes and hence it is an $H_*(\ ;A)$-equivalence, by Theorem 4.3. It follows, by the Mayer-Vietoris sequence associated with (8.4), that s_i is also an $H_*(\ ;A)$-equivalence, as desired. $\qquad \square$

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