Singly generated radicals associated with varieties of groups

Carles Casacuberta, José L. Rodríguez*, and Dirk Scevenels

Abstract

To every variety of groups $\mathcal{W}$ one can associate an idempotent radical $\mathcal{P}_{\mathcal{W}}$ by iterating the verbal subgroup. The basic example is the perfect radical, which is the intersection of the transfinite derived series. We prove that each such radical $\mathcal{P}_{\mathcal{W}}$ is generated by a single locally free group F, in the sense that, for every group G, the subgroup $\mathcal{P}_{\mathcal{W}}G$ is generated by images of homomorphisms $F \to G$.

Our motivation comes from algebraic topology. In fact we show that every variety of groups $\mathcal{W}$ determines a localization functor in the homotopy category, which kills the radical $\mathcal{P}_{\mathcal{W}}$ of the fundamental group while preserving homology with certain coefficients.

0 Introduction

Radicals have been broadly studied in abelian categories; see e.g. [15, VI.1]. However, they have received less attention in the category of groups, where the fundamentals of radical theory were first investigated by Kurosh; see [13, 1.3]. We say that a functor R is a radical if RG is a normal subgroup of G and $R(G/RG) = 1$ for all groups G.

Radicals are useful in localization theory. Given a family Φ of group homomorphisms $\varphi_\alpha\colon A_\alpha \to B_\alpha$, a group K is called Φ-local if the induced map of sets $\mathrm{Hom}(\varphi_\alpha, K)\colon \mathrm{Hom}(B_\alpha, K) \to \mathrm{Hom}(A_\alpha, K)$ is bijective for all α. Under mild assumptions (e.g., if Φ is a set, or also if each φ_α is surjective), every

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group G admits a Φ-localization $G \to LG$, which is initial among homomorphisms from G into Φ-local groups. The main examples are localizations at sets of primes and homological localizations [2].

As we explain in Section 2, there is a bijective correspondence between radicals and surjective localizations in the category of groups, and there is also a bijective correspondence between idempotent radicals (that is, such that $RRG = RG$ for all G) and localizations with respect to homomorphisms whose target is the trivial group (such localizations will be called reductions). In particular, the projection onto an arbitrary variety of groups determines a radical R, and to this radical we can associate by standard methods an idempotent radical and hence a reduction. The basic example is $RG = [G, G]$, the commutator subgroup, whose associated idempotent radical $R^{\infty}G$ is the perfect radical.

Our motivation comes from [1], where a universal locally free group $\mathcal{F}$ was constructed with the property that localization with respect to the homomorphism $\mathcal{F} \to 1$ has the effect of dividing out the perfect radical. In Theorem 3.3 below we generalize the construction of Example 5.3 in [1] so that it applies to any other idempotent radical associated with a variety.

Furthermore, every radical R associated with a variety gives rise to a localization functor in the homotopy category of CW-complexes. When applied to a space X, this functor kills the subgroup $R\pi_1(X)$ of the fundamental group and preserves homology with certain coefficients. Quillen's plus-construction [12] is the special case corresponding to the perfect radical.

1 Radicals in group theory

We shall work in the category $\mathcal{G}$ of groups. A *radical* R is a subfunctor of the identity (i.e., a functor assigning to each group G a subgroup RG in such a way that every homomorphism $G \to K$ induces $RG \to RK$ by restriction), with the property that RG is normal in G and $R(G/RG) = 1$ for all groups G. A radical R is said to be *idempotent* if $RRG = RG$ for all groups G. These notions are standard in abelian categories (see e.g. [15, Ch. VI]), although the terminology varies slightly depending on the authors; cf. [7], [8], [9], [13].

Example 1.1 The best-known example of a (nonidempotent) radical is the commutator subgroup $RG = [G, G]$. Two idempotent examples are the perfect radical (i.e., the largest perfect subgroup, where a group G is called *perfect* if $[G, G] = G$), and the torsion radical (i.e., the smallest normal subgroup $\tau(G)$ such that $G/\tau(G)$ is torsion-free). In fact, for each set of primes J there is a J-torsion radical, and there is also a largest J-perfect subgroup, where a group G is called J-perfect if the first mod p homology group $H_1(G; \mathbf{Z}/p)$ is zero for $p \in J$.

Given any radical R, the class of groups G such that $RG = G$ is closed under quotients and free products, and the class of groups G such that $RG = 1$ is closed under subgroups and cartesian products. The proof is the same as the one given in [15, VI.1.2] for abelian categories. It follows that, if two groups G and K satisfy $RG = G$ and $RK = 1$, then $\mathrm{Hom}(G, K)$ is trivial.

Proposition 1.2 *Let R be any radical in the category of groups. Then every group G contains a unique subgroup $R^\infty G$ maximal with the property that $RR^\infty G = R^\infty G$. Moreover, $R^\infty G$ is normal in G and it is contained in RG.*

Proof. Let $R^\infty G$ be the product of all subgroups H of G such that $RH = H$. Then $R^\infty G$ is a quotient of the free product of all such subgroups H, and this family of subgroups is closed under conjugation. Hence, $RR^\infty G = R^\infty G$ and $R^\infty G$ is normal. Finally, since R is a subfunctor of the identity, we have $RR^\infty G \subseteq RG$, that is, $R^\infty G \subseteq RG$. $\sharp$

It follows, similarly as in [15, VI.1.6], that R^∞ is also a radical and it is in fact the largest idempotent radical which is a subfunctor of R. There is another standard way of constructing R^∞ from R by transfinite induction. Namely, if α is a successor ordinal, define $R^\alpha = RR^{\alpha-1}$, with $R^0 = R$, and if α is a limit ordinal, then let R^α be the intersection of R^β for all $\beta < \alpha$. Finally, take $R^\infty G$ to be $R^\alpha G$ for the smallest α such that $R^{\alpha+1}G = R^\alpha G$.

Proposition 1.3 *Every family of groups $\mathcal{C}$ determines a radical R, by defining RG to be the intersection of the kernels of all epimorphisms $f\colon G \to C$ where C is in $\mathcal{C}$.*

We omit the details since this is a classical construction. Note that every radical R arises this way, by taking $\mathcal{C}$ to be the class of groups G such that $RG = 1$.

There is another important source of radicals. Namely, the same argument as in [9, 2.2] proves the following.

Proposition 1.4 *Let $\mathcal{G}$ be the category of groups and $\mathcal{D}$ any category. Suppose given an adjoint pair of functors $L\colon \mathcal{G} \to \mathcal{D}$ and $E\colon \mathcal{D} \to \mathcal{G}$, where L is the left adjoint. Then we obtain a radical by defining RG to be the kernel of the unit map $G \to ELG$, for each group G.*

2 Localizations

A full subcategory $\mathcal{D}$ of the category of groups is called *reflective* if the inclusion $E\colon \mathcal{D} \to \mathcal{G}$ has a left adjoint L. In this case, for every group G we have a group homomorphism $G \to LG$ which is initial among homomorphisms from G into groups in $\mathcal{D}$. Then the groups in $\mathcal{D}$ are called *L-local* and the homomorphisms $G \to K$ inducing isomorphisms $LG \cong LK$ are called *L-equivalences*. The functor L will be called a *localization* or a *reflection* onto the subcategory $\mathcal{D}$. It will be called an *epireflection* if, for all groups G, the localization homomorphism $G \to LG$ is surjective.

If L is any reflection in the category of groups, and RG denotes the kernel of the localization homomorphism $G \to LG$, then it follows from Proposition 1.4 that R is a radical. In fact, there is a bijective correspondence between radicals and epireflections in the category of groups.

Given any family of group homomorphisms $\varphi_\alpha\colon A_\alpha \to B_\alpha$, we can consider the subcategory $\mathcal{D}$ of all groups D which are *orthogonal* to φ_α for all α, that is, such that the induced map

$$\mathrm{Hom}(\varphi_\alpha, D)\colon \mathrm{Hom}(B_\alpha, D) \to \mathrm{Hom}(A_\alpha, D)$$

is bijective for all α. If the family $\{\varphi_\alpha\}$ is a set, then we can form the free product φ of all homomorphisms in the family, and it follows by standard methods that the orthogonal subcategory $\mathcal{D}$ is reflective. For further details and applications to homotopy theory, see [1], [4], [5], [14]. If φ is a surjective homomorphism, then the φ-localization functor L_φ is an epireflection. In the special case when φ is of the form $A \to 1$ for some group A, the orthogonal groups are called *A-reduced*. Thus, a group G is A-reduced if and only if $\mathrm{Hom}(A, G)$ is trivial. In this case, the φ-localization of a group G will be called *A-reduction* and denoted by $G//A$. It is the largest quotient of G admitting no nontrivial homomorphisms from A.

If the given class of homomorphisms $\{\varphi_\alpha\}$ is proper (i.e., not a set), then we cannot infer in general that the orthogonal subcategory $\mathcal{D}$ is reflective.

However, there is a special situation where it can be proved that the subcategory orthogonal to a (possibly proper) class of homomorphisms is reflective.

Proposition 2.1 *The subcategory $\mathcal{D}$ orthogonal to any class of group epimorphisms is reflective.*

Proof. Let TG be the intersection of all kernels of epimorphisms from G onto groups in $\mathcal{D}$. By Proposition 1.3, T is a radical, and hence $G \to G/TG$ is a reflection. Thus, it suffices to check that a group G belongs to $\mathcal{D}$ if and only if $TG = 1$. If G is in $\mathcal{D}$, then the identity homomorphism $G \to G$ is an epimorphism onto a group in $\mathcal{D}$ and hence TG is indeed trivial. To prove the converse, suppose that $TG = 1$ and let $\gamma\colon G \to \widehat{G}$ be the inverse limit of all epimorphisms from G onto groups in $\mathcal{D}$. Our assumption ensures that γ is injective. Since $\mathcal{D}$ is closed under subgroups and inverse limits, it follows that G is in $\mathcal{D}$. $\sharp$

In the special case where the targets of all homomorphisms φ_α are trivial, the associated localization will be called a *reduction*. This terminology is consistent with our previous use of the same word. The same arguments as in [14, Theorem 2.7] lead to the following characterization.

Theorem 2.2 *Let L be a localization in the category of groups and let $\mathcal{D}$ be the class of L-local groups. Then L is an epireflection if and only if $\mathcal{D}$ is closed under subgroups, and L is a reduction if and only if $\mathcal{D}$ is closed under subgroups and formation of extensions.*

Finally, we prove that the bijection between epireflections and radicals makes reductions correspond with idempotent radicals.

Theorem 2.3 *There is a bijective correspondence between idempotent radicals and reduction functors in the category of groups.*

Proof. We show that, given an idempotent radical R, the functor $LG = G/RG$ is a reduction. Consider the family of all homomorphisms $A \to 1$ where $LA = 1$, and let L' denote the corresponding reduction. From the fact that all such homomorphisms are L-equivalences it follows that there is a natural transformation $L' \to L$. Conversely, let G be L'-local. Then we have $LRG = RG/RRG = 1$ and this implies that the homomorphism $RG \to 1$ is an L'-equivalence. Therefore $\mathrm{Hom}(RG, G)$ is trivial, from which it follows that $RG = 1$ and G is L-local. Thus we have proved that $L' = L$, as desired.

The kernel R of any reduction with respect to a class $\varphi_\alpha\colon A_\alpha \to 1$ can be described as a possibly transfinite direct limit, as in [4, Theorem 3.2], where the first step is the subgroup generated by the images of all homomorphisms from A_α for all α. From this description one sees that R is in fact an idempotent radical. $\sharp$

3 Varieties of groups

Let $\mathcal{W}$ be the variety of groups defined by a set of words W. That is, W is a set of elements of the free group F_∞ on a countably infinite set of generators $\{x_1, x_2, x_3 \ldots\}$, and $\mathcal{W}$ is the family of groups G with the property that every homomorphism $f\colon F_\infty \to G$ satisfies $f(w) = 1$ for all $w \in W$; see [11]. For an arbitrary group G, the verbal subgroup WG of G is the subgroup generated by all the images of words in W under homomorphisms $F_\infty \to G$.

Proposition 3.1 *A group G is in the variety $\mathcal{W}$ if and only if G is orthogonal to the natural homomorphism $\varphi\colon F_\infty \to F_\infty/WF_\infty$.*

Proof. A group G is orthogonal to φ if and only if every homomorphism $f\colon F_\infty \to G$ satisfies $f(WF_\infty) = 1$. But this condition is equivalent to $WG = 1$, and this means that G is in the variety $\mathcal{W}$. $\sharp$

Let φ be as in Proposition 3.1. Then the φ-localization functor L_φ is the projection onto the variety $\mathcal{W}$, sending each group G onto G/WG. Thus, the verbal subgroup is a radical, which we denote by the same letter W. It is not idempotent in general; indeed, WWG need not be equal to WG.

The groups G such that $WG = G$ will be called *$\mathcal{W}$-perfect*. That is, G is $\mathcal{W}$-perfect if and only if $L_\varphi G = 1$. This notion specializes to ordinary perfect groups when WG is the commutator subgroup of G.

It follows from Proposition 1.2 that every group G has a largest $\mathcal{W}$-perfect subgroup. We call it the $\mathcal{W}$-perfect radical of G, and denote it by $\mathcal{P}_\mathcal{W}G$. This radical $\mathcal{P}_\mathcal{W}$ is idempotent. Thus, we can consider the reduction functor assigning to each group G the quotient $G/\mathcal{P}_\mathcal{W}G$; cf. Theorem 2.3. We say that a group F *generates* the radical $\mathcal{P}_\mathcal{W}$ if $G/\!/F = G/\mathcal{P}_\mathcal{W}G$ for all groups G. Our main result in this section (Theorem 3.3) states that each radical $\mathcal{P}_\mathcal{W}$ is generated by some locally free group.

In the case where W consists of the word x^m alone, where m is a non-negative integer, the $\mathcal{W}$-perfect radical of a group G is the largest subgroup

H such that $H = H^m$, where H^m denotes the subgroup generated by all m-powers of elements of H. We call this radical the *Burnside radical* of exponent m. The Burnside radical of exponent 0 is the trivial subgroup and the Burnside radical of exponent 1 is the whole group. Note that the Burnside radical of exponent m coincides with the radical generated by $\mathbf{Z}[1/m]$ on commutative groups, but not on other groups in general.

Recall from [11, 12.12] that every word w is equivalent to a power word x^m together with a commutator word c, in the sense that w is a law in a group G if and only if the words x^m and c are both laws in G. (A commutator word is any element of $[F_\infty, F_\infty]$.) Thus, given a variety $\mathcal{W}$, we can assume without loss of generality that $\mathcal{W}$ is defined by a set of words of the form

$$W = \{x^m, c_1, c_2, c_3, \ldots\},$$

where each c_i is a commutator word and m is a nonnegative integer, called the exponent of the variety.

If a variety $\mathcal{W}$ is defined by commutator words only, then the $\mathcal{W}$-perfect radical is contained in the ordinary perfect radical. Indeed, the inclusion $WG \subseteq [G,G]$ yields an epimorphism $G/WG \to G/[G,G]$ and hence all $\mathcal{W}$-perfect groups are perfect. However, the inclusion of the $\mathcal{W}$-perfect radical into the perfect radical can be proper, as the following example shows.

Example 3.2 Let $\mathcal{W}$ be the variety defined by the word $c = [x,y]^m$, where m is any integer greater than 2. Then there exist perfect groups which are not $\mathcal{W}$-perfect; it suffices to pick any perfect group G such that $G \neq G^m$.

We next prove that, for every variety $\mathcal{W}$, the reduction $G \to G/\mathcal{P}_{\mathcal{W}}G$ coincides with F-reduction (that is, localization with respect to $F \to 1$), for some locally free group F. To this aim, we shall generalize the construction described in Example 5.3 of [1].

Theorem 3.3 *Let $\mathcal{W}$ be any variety of groups. Then there exists a locally free, $\mathcal{W}$-perfect group F such that, for all groups G, the radical $\mathcal{P}_{\mathcal{W}}G$ is generated by images of homomorphisms $F \to G$.*

Proof. Let $W = \{w_1, w_2, \ldots\}$ be a set of words defining the variety $\mathcal{W}$. In order to simplify the notation, we will assume, as we may (by reordering the words in W and inserting the trivial word as many times as needed), that

w_j is a word on a subset of the generators $x_1, \ldots, x_j$ of the free group F_∞. Thus we write $w_j = w_j(x_1, \ldots, x_j)$.

We shall construct a countable, locally free group $F_{\mathbf{n}}$ for each sequence $\mathbf{n} = (n_1, n_2, n_3, \ldots)$ of positive integers, and define F to be the free product of the groups $F_{\mathbf{n}}$ for all increasing sequences $\mathbf{n}$. The group $F_{\mathbf{n}}$ is defined as the colimit of a directed system $(F_{\mathbf{n},r}, \varphi_r)$ of free groups and homomorphisms. For $r = 0$, the group $F_{\mathbf{n},0}$ is infinite cyclic with a generator x_0. For $r \geq 1$, the group $F_{\mathbf{n},r}$ is the free group on the symbols

$$x_r(\delta_1, \ldots, \delta_r; \varepsilon_1, \ldots, \varepsilon_r; i_1, \ldots, i_r),$$

where $1 \leq \varepsilon_k \leq \delta_k \leq n_k$ and $1 \leq i_k \leq n_k$, for $k = 1, \ldots, r$. The homomorphism $\varphi_r \colon F_{\mathbf{n},r} \to F_{\mathbf{n},r+1}$ is determined by letting

$$\varphi_r\left(x_r(\delta_1, \ldots, \delta_r; \varepsilon_1, \ldots, \varepsilon_r; i_1, \ldots, i_r)\right)$$

be the product

$$\prod_{i_{r+1}=1}^{n_{r+1}} \bar{w}_1(i_{r+1})\, \bar{w}_2(i_{r+1}) \cdots \bar{w}_{n_{r+1}}(i_{r+1})$$

in which $\bar{w}_j(i_{r+1})$ denotes the value of the word w_j on the symbols

$$x_{r+1}(\delta_1, \ldots, \delta_r, j; \varepsilon_1, \ldots, \varepsilon_r, \varepsilon_{r+1}; i_1, \ldots, i_r, i_{r+1}),$$

where ε_{r+1} runs from 1 to j.

The image of φ_r is contained in the verbal subgroup of $F_{\mathbf{n},r+1}$. Hence, $F_{\mathbf{n}} = WF_{\mathbf{n}}$ for each sequence $\mathbf{n}$, so that $F = WF$ as well; i.e., F is $\mathcal{W}$-perfect. Since every epimorphic image of a $\mathcal{W}$-perfect group is $\mathcal{W}$-perfect, it follows that the image of every homomorphism $F \to G$ is contained in the $\mathcal{W}$-perfect radical of G. Thus the argument will be complete if we show that, for every element $x \in \mathcal{P}_{\mathcal{W}}G$, there is an increasing sequence $\mathbf{n}$ and a homomorphism $F_{\mathbf{n}} \to G$ whose image contains the element x. To see this, pick the minimum n_1 such that x can be written as a product of n_1 values (or less) of some of the words $w_1, \ldots, w_{n_1}$ (possibly repeated and in any order), and choose one such decomposition of x to continue the process. Consider all the elements of G which appear in the chosen expression of x as arguments in the words w_j; pick the minimum n_2 which is greater than or equal to the lengths of the expressions of these elements as products of values of words in W, and greater than or equal to the subindices of the words involved. Replace n_2 with $1 + n_1$ if necessary, in order that the sequence $\mathbf{n}$ be increasing. By continuing this way, one obtains a sequence $\mathbf{n} = (n_1, n_2, \ldots)$ of positive integers and a homomorphism $\psi\colon F_{\mathbf{n}} \to G$ sending x_0 to x. (In order to illustrate how ψ is defined, suppose e.g. that $x = w_3(a,b,c)\, w_2(d,e)$, so that $n_1 = 3$. Then ψ sends $x_1(3;1;1) \mapsto a$, $x_1(3;2;1) \mapsto b$, $x_1(3;3;1) \mapsto c$, $x_1(2;1;2) \mapsto d$, $x_1(2;2;2) \mapsto e$, and it sends all the other generators $x_1(\delta;\varepsilon;i)$ of $F_{\mathbf{n},1}$ to 1. Then one proceeds similarly by choosing decompositions of a, b, c, d, e as products of values of words in W, and so on.) $\sharp$

4 Applications to homotopy theory

For a map $f\colon A \to B$ between CW-complexes, a space X is called *f-local* if the induced map of function spaces

$$\mathrm{map}(f, X)\colon \mathrm{map}(B, X) \to \mathrm{map}(A, X)$$

is a weak homotopy equivalence. Each map f determines a localization functor L_f in the homotopy category of CW-complexes; see [6]. Thus, for every CW-complex X there is a map $X \to L_f X$ which is homotopy initial among maps from X into f-local spaces.

Let $\mathcal{W}$ be any variety of groups. Let F be the locally free $\mathcal{W}$-perfect group constructed in the proof of Theorem 3.3 (or any other locally free group generating the same radical). Since F is a direct limit of free groups, its classifying space $K(F,1)$ is a homotopy colimit of wedges of circles and hence it is two-dimensional. It then follows from [5, Theorem 2.1] that localization with respect to $f\colon K(F,1) \to *$ is *π_1-compatible*; that is,

$$\pi_1(L_f X) \cong L_\varphi \pi_1(X) \qquad \text{for all spaces } X,$$

where φ denotes the homomorphism $F \to 1$ induced by f on the fundamental group; thus, the localization functor L_φ has the efect of dividing out the $\mathcal{W}$-perfect radical. Therefore, the functor L_f assigns to each space X a space for which the $\mathcal{W}$-perfect radical of the fundamental group is trivial. The following theorem ensures that such localizations are not trivial themselves, since they preserve homology with certain coefficients. The steering example is Quillen's plus-construction; cf. [1].

Recall once more that, by [11, 12.12], every variety $\mathcal{W}$ can be defined by a power word together with commutator words.

Theorem 4.1 *Let $\mathcal{W}$ be any variety of groups of exponent $m \geq 0$. Let F be any locally free group generating the $\mathcal{W}$-perfect radical. Consider the map $f\colon K(F,1) \to *$. Then, for each space X, the natural map $X \to L_f X$ kills the $\mathcal{W}$-perfect radical from the fundamental group of X, and it induces an isomorphism in homology with coefficients in $\mathbf{Z}/m$.*

Proof. By assumption, we have $F = WF$; that is, every element of F can be written as a product of commutators and power words of the form a^m with $a \in F$. If $m = 0$, then F is perfect and, since it is is locally free, it is in fact acyclic. Therefore, the map $f\colon K(F,1) \to *$ is an integral homology equivalence and hence all maps $X \to Y$ inducing a homotopy equivalence $L_f X \simeq L_f Y$ are integral homology equivalences. In particular, the natural map $X \to L_f X$ is an integral homology equivalence for all spaces X. If $m \geq 2$, then the abelianization of F is a group A such that $A = mA$; that is, A is p-divisible for all primes p dividing m. Hence, $H_1(F; \mathbf{Z}/m) = 0$ and, since F is locally free, F is mod m acyclic. In other words, the map f is a mod m homology equivalence. It then follows, as above, that the natural map $X \to L_f X$ is a mod m homology equivalence for all X. $\sharp$

Plus-constructions for homology with mod m coefficients have long been known; see [3, VII.6] or [10]. These occur in our framework, up to homotopy, by choosing the variety defined by the words x^m and $[x,y]$; what they kill is the J-perfect radical of the fundamental group, where J is the set of prime divisors of m. The word x^m alone yields a localization which kills the Burnside radical of exponent m from the fundamental group, while preserving homology with mod m coefficients. This localization does not alter, for example, spaces whose fundamental group is a finite perfect group of exponent m.

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