Contemporary Mathematics

Homological localizations preserve 1-connectivity

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Abstract. Every generalized homology theory E yields a localization functor $\mathbf{L}_E$ that sends the E-equivalences to homotopy equivalences. We prove that if X is any 1-connected space, then $\mathbf{L}_E X$ is also 1-connected, for every generalized homology theory E. This is deduced from a result by Hopkins and Smith stating that if $K(\mathbb{Z}, 2)$ is E-acyclic then E is trivial.

Introduction

A number of results in the literature suggest that idempotent functors in the homotopy category of spaces preserve 1-connectivity, although no proof of this fact has so far been given. One of the earliest examples is localization with respect to ordinary homology, which in fact preserves n-connectivity for all n; see [**1**].

In the same article [**1**], Bousfield proved the existence of localization with respect to any generalized homology theory E; that is, a functor $\mathbf{L}_E$ which assigns to every space X a space $\mathbf{L}_E X$ together with a natural map $X \to \mathbf{L}_E X$ which is terminal in the homotopy category among E-equivalences with source X. (An E-equivalence is a map $X \to Y$ inducing isomorphisms $E_n(X) \cong E_n(Y)$ for all n.)

In [**9**], Mislin showed that K-theory localization does not preserve n-connectivity in general, since for example $\pi_3(\mathbf{L}_K S^{2p+2}; \mathbb{Z}/p) \neq 0$ for every odd prime p. However, Mislin also proved in [**9**] that the K-localization of every 1-connected space is 1-connected. Further evidence of the fact that 1-connectivity could be preserved by arbitrary idempotent functors in the homotopy category was given by Neisendorfer in [**10**] and by Tai in his detailed study of the problem in [**11**].

It is therefore natural to address the question of whether or not localizations with respect to generalized homology theories preserve 1-connectivity. Such localizations were thoroughly discussed by Bousfield in [**2**], where a description was given of their effect on abelian Eilenberg–Mac Lane spaces. The main tool was an arithmetic square, already exploited by Mislin in [**9**], allowing one to determine the E-localization of a space (with some restrictions on the fundamental group) from its $E\mathbb{Z}/p$-localizations and rational coherence data.

1991 *Mathematics Subject Classification.* Primary 55P60, 55N20; Secondary 20K21.

The first-named author was supported by DGES grant PB97-0202.

The second-named author was supported by the Centre de Recerca Matemàtica, Barcelona, and Swiss NSF grant 81LA-51213.

Our main result is that $\mathbf{L}_E X$ is 1-connected if X is 1-connected, for any generalized homology theory E. This follows by combining the methods of Bousfield in [**2**] with a result proved by Hopkins and Smith in [**8**], according to which a $K(\mathbb{Z},2)$ is never E-acyclic if E is nontrivial. We note, however, that $K(\mathbb{Z},3)$ is $K\mathbb{Z}/p$-acyclic for all p, by [**9**, Corollary 2.3]. It is known that, if L is any homotopy idempotent functor, then $LK(\mathbb{Z},n)$ is necessarily a $K(A,n)$ where A is either zero or a commutative ring with 1, for all n; see [**5**]. If $L = \mathbf{L}_E$ for some nontrivial homology theory E, then the possibility that $A = 0$ has been discarded for $n = 2$ in [**8**], and this opens the way to substantial improvements of earlier results or to new results as in this article.

Acknowledgements The plausibility of the main result in this article was communicated to us by A. K. Bousfield, to whom we are indebted. We first learned a proof of the non-acyclicity of $K(\mathbb{Z},2)$, due to M. J. Hopkins and J. H. Smith, from a very helpful letter written by E. Devinatz [**6**]. We also thank W. Chachólski for informing us and for several conversations on this subject.

1. Torsion homology theories

Throughout the paper we denote by E a spectrum or the associated homology theory. For an abelian group R, the corresponding spectrum with coefficients in R is defined as $ER = E \wedge SR$ where SR is the Moore spectrum of type $(R,0)$. The only cases of interest in this article are $R = \mathbb{Z}/p$ and R a subring of $\mathbb{Q}$. A spectrum E is called *torsion* if $E\mathbb{Q}$ is contractible. The ordinary Eilenberg–Mac Lane spectrum with coefficients in R is denoted by HR. We denote by $\mathbb{Z}_p^\wedge$ the p-adics, by $\mathbb{Z}(p^\infty)$ the Prüfer group $\bigcup_{n=1}^\infty \mathbb{Z}/p^n$ and, for a set of primes P, we denote by $\mathbb{Z}_P$ the integers localized at P.

In this first section we concentrate on mod p homology theories, where p is any prime. Using the Atiyah–Hirzebruch spectral sequence, one sees that if E is any homology theory, then every $H\mathbb{Z}/p$-equivalence is an $E\mathbb{Z}/p$-equivalence; details are given in [**9**, § 1]. Hence, all $E\mathbb{Z}/p$-local spaces are $H\mathbb{Z}/p$-local and there is a natural transformation of functors $\mu\colon \mathbf{L}_{H\mathbb{Z}/p} \to \mathbf{L}_{E\mathbb{Z}/p}$.

We next prove that, if X is connected, then the induced homomorphism

$$\mu_*\colon \pi_1(\mathbf{L}_{H\mathbb{Z}/p}X) \to \pi_1(\mathbf{L}_{E\mathbb{Z}/p}X)$$

is surjective. This result is essentially contained in the proof of Proposition 7.1 in [**2**], as we next recall for the sake of completeness. The argument is based on Bousfield's version of the Whitehead theorem (cf. [**2**, Theorem 5.2]), stating that if R is $\mathbb{Z}/p$ or a subring of $\mathbb{Q}$, and $f\colon X \to Y$ is a map inducing isomorphisms $H_i(X;R) \cong H_i(Y;R)$ for $i < n$ and an epimorphism $H_n(X;R) \twoheadrightarrow H_n(Y;R)$, where $n \geq 1$, then f also induces isomorphisms $\pi_i(\mathbf{L}_{HR}X) \cong \pi_i(\mathbf{L}_{HR}Y)$ for $i < n$ and an epimorphism $\pi_n(\mathbf{L}_{HR}X) \twoheadrightarrow \pi_n(\mathbf{L}_{HR}Y)$.

Theorem 1.1. *Let E be any homology theory and p any prime. Then, for every connected space X, the natural homomorphism $\mu_*\colon \pi_1(\mathbf{L}_{H\mathbb{Z}/p}X) \to \pi_1(\mathbf{L}_{E\mathbb{Z}/p}X)$ is surjective.*

Proof. The claim is obvious if $E\mathbb{Z}/p$ is trivial. If $E\mathbb{Z}/p$ is not trivial, then $K(\mathbb{Z}/p,1)$ is not $E\mathbb{Z}/p$-acyclic, as shown in [**2**, Proposition 2.2]. Since the natural map $\mu\colon \mathbf{L}_{H\mathbb{Z}/p}X \to \mathbf{L}_{E\mathbb{Z}/p}X$ is an $E\mathbb{Z}/p$-equivalence, we obtain an isomorphism

$$\mu_*\colon H_1(\mathbf{L}_{H\mathbb{Z}/p}X;\mathbb{Z}/p) \cong H_1(\mathbf{L}_{E\mathbb{Z}/p}X;\mathbb{Z}/p) \qquad (1.1)$$

using [**2**, Proposition 2.1] or [**5**, Theorem 1.3], according to which $K(\mathbb{Z}/p,1)$ is $E\mathbb{Z}/p$-local. By the generalized Whitehead theorem stated above, μ induces then an epimorphism $\pi_1(\mathbf{L}_{H\mathbb{Z}/p}X) \twoheadrightarrow \pi_1(\mathbf{L}_{E\mathbb{Z}/p}X)$, since $\mathbf{L}_{E\mathbb{Z}/p}X$ is $H\mathbb{Z}/p$-local. □

COROLLARY 1.2. *If E is any torsion homology theory and X is 1-connected, then $\mathbf{L}_E X$ is also 1-connected.*

PROOF. As in [**2**], we denote by $\mathcal{P}E$ the set of primes p such that $\pi_*(E)$ is not uniquely p-divisible. By [**2**, Proposition 7.1], for each torsion homology theory E and every 1-connected space X, we have a homotopy equivalence

$$\mathbf{L}_E X \simeq \prod_{p\in\mathcal{P}E} \mathbf{L}_{E\mathbb{Z}/p}X.$$

Now recall from [**1**] that $\mathbf{L}_{H\mathbb{Z}/p}X$ is 1-connected if X is 1-connected. Therefore, Theorem 1.1 tells us that $\mathbf{L}_E X$ is 1-connected. □

Before discussing non-torsion homology theories, we need to study the second homotopy group $\pi_2(\mathbf{L}_{E\mathbb{Z}/p}X)$ when X is 1-connected. The following result is the main input in our discussion.

THEOREM 1.3. *Let E be a homology theory and p any prime. Suppose that $E\mathbb{Z}/p$ is nontrivial. Then either $K(\mathbb{Z}/p,2)$ or $K(\mathbb{Z}_p^\wedge,2)$ is $E\mathbb{Z}/p$-local.*

PROOF. The classification of acyclicity patterns for Eilenberg–Mac Lane spaces given by Bousfield in [**2**, § 4] implies that $\mathbf{L}_{E\mathbb{Z}/p}K(\mathbb{Z},n) = K(A,n)$ for each $n \geq 1$, where the group A can be $\mathbb{Z}_p^\wedge$, or $\mathbb{Z}/p^i$ for some $i \geq 1$, or zero. In [**8**], it is shown that if a reduced homology theory vanishes on $K(\mathbb{Z},2)$, then it is trivial. (Thus, nontrivial mod p homology theories of type IV-1 as defined in [**2**, § 4] do not exist.) Therefore, if $\mathbf{L}_{E\mathbb{Z}/p}$ is nontrivial, then the localization $\mathbf{L}_{E\mathbb{Z}/p}K(\mathbb{Z},2)$ is necessarily $K(\mathbb{Z}_p^\wedge,2)$ or $K(\mathbb{Z}/p^i,2)$ for some $i \geq 1$. In the latter case, $K(\mathbb{Z}/p,2)$ cannot be $E\mathbb{Z}/p$-acyclic, as one sees by induction using the fibre sequences

$$K(\mathbb{Z}/p,2) \to K(\mathbb{Z}/p^i,2) \to K(\mathbb{Z}/p^{i-1},2).$$

Hence, $K(\mathbb{Z}/p,2)$ is $E\mathbb{Z}/p$-local, by [**2**, Proposition 2.1] or [**5**, Lemma 1.4]. □

If $K(\mathbb{Z}/p,2)$ is $E\mathbb{Z}/p$-local and X is 1-connected, then, using the fact that $\mu\colon \mathbf{L}_{H\mathbb{Z}/p}X \to \mathbf{L}_{E\mathbb{Z}/p}X$ is an $E\mathbb{Z}/p$-equivalence, we obtain as in (1.1) an isomorphism

$$\mu_*\colon H_2(\mathbf{L}_{H\mathbb{Z}/p}X;\mathbb{Z}/p) \cong H_2(\mathbf{L}_{E\mathbb{Z}/p}X;\mathbb{Z}/p). \tag{1.2}$$

Thus, the homomorphism $\pi_2(\mathbf{L}_{H\mathbb{Z}/p}X) \to \pi_2(\mathbf{L}_{E\mathbb{Z}/p}X)$ induced by μ is surjective, by the generalized Whitehead theorem.

Now suppose that $K(\mathbb{Z}_p^\wedge,2)$ is $E\mathbb{Z}/p$-local and X is 1-connected. Similarly as in the previous case, since μ is an $E\mathbb{Z}/p$-equivalence, we have an isomorphism

$$\mu^*\colon \mathrm{Hom}(\pi_2(\mathbf{L}_{E\mathbb{Z}/p}X),\mathbb{Z}_p^\wedge) \cong \mathrm{Hom}(\pi_2(\mathbf{L}_{H\mathbb{Z}/p}X),\mathbb{Z}_p^\wedge). \tag{1.3}$$

In order to use this information, we recall the following concept from [**4**, VI.3] and [**7**]. An abelian group A is called Ext-*p-complete* if the natural homomorphism $A \to \mathrm{Ext}(\mathbb{Z}(p^\infty),A)$ derived from the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[1/p] \to \mathbb{Z}(p^\infty) \to 0$$

is an isomorphism. Equivalently, an abelian group A is Ext-p-complete if and only if both $\mathrm{Hom}(\mathbb{Z}[1/p],A) = 0$ and $\mathrm{Ext}(\mathbb{Z}[1/p],A) = 0$. As explained in [**4**, VI.4], Ext-p-complete abelian groups are uniquely q-divisible for primes $q \neq p$, and they admit a canonical $\mathbb{Z}_p^\wedge$-module structure.

An Ext-p-complete abelian group A is called *adjusted* if the quotient A/TA of A by its torsion subgroup TA is p-divisible (hence divisible). Thus, A is adjusted if and only if A does not admit any torsion-free Ext-p-complete quotients other than zero. Since $TA \otimes \mathbb{Z}(p^\infty) = 0$, it also follows that an Ext-p-complete abelian group A is adjusted if and only if $A \otimes \mathbb{Z}(p^\infty) = 0$.

THEOREM 1.4. *Let E be a homology theory and p a prime. Suppose that $E\mathbb{Z}/p$ is nontrivial. Then, for every 1-connected space X, the cokernel of the natural homomorphism $\mu_*\colon \pi_2(\mathbf{L}_{H\mathbb{Z}/p}X) \to \pi_2(\mathbf{L}_{E\mathbb{Z}/p}X)$ is an adjusted* Ext-*p-complete abelian group, which is zero if $K(\mathbb{Z}/p, 2)$ is $E\mathbb{Z}/p$-local.*

PROOF. The spaces $\mathbf{L}_{H\mathbb{Z}/p}X$ and $\mathbf{L}_{E\mathbb{Z}/p}X$ are $H\mathbb{Z}/p$-local. The abelian groups $\pi_2(\mathbf{L}_{H\mathbb{Z}/p}X)$ and $\pi_2(\mathbf{L}_{E\mathbb{Z}/p}X)$ are thus Ext-p-complete, by [**1**, Theorem 5.5]. Hence, $\operatorname{Coker}\mu_*$ is Ext-p-complete, since the cokernel of any homomorphism between Ext-p-complete abelian groups is Ext-p-complete. If $K(\mathbb{Z}/p, 2)$ is $E\mathbb{Z}/p$-local, then we already proved, by means of (1.2), that $\operatorname{Coker}\mu_*$ is zero. Thus, we assume that $K(\mathbb{Z}_p^\wedge, 2)$ is $E\mathbb{Z}/p$-local. In this case, the isomorphism displayed in (1.3) shows that $\operatorname{Hom}(\operatorname{Coker}\mu_*, \mathbb{Z}_p^\wedge) = 0$. For an abelian group A, if $\operatorname{Hom}(A, \mathbb{Z}_p^\wedge) = 0$ then we have $\operatorname{Hom}(A \otimes \mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty)) = 0$ by adjunction. Since $A \otimes \mathbb{Z}(p^\infty)$ is a p-torsion divisible abelian group, we may infer that $A \otimes \mathbb{Z}(p^\infty) = 0$ and this implies that A/TA is p-divisible, as we needed. (In fact, an Ext-p-complete abelian group A is adjusted if and only if the condition $\operatorname{Hom}(A, \mathbb{Z}_p^\wedge) = 0$ holds. This has also been pointed out in [**3**, Lemma 7.7].) □

2. Non-torsion homology theories

In this section we deal with non-torsion homology theories. In this case, there is an arithmetic square allowing one to compute E-localizations of 1-connected spaces by combining mod p data and rational data. Specifically, the following diagram is a homotopy pull-back square if X is 1-connected (and also under less restrictive conditions; see [**2**, Proposition 7.2]). Recall that $\mathcal{P}E$ denotes the set of primes p such that $\pi_*(E)$ is not uniquely p-divisible.

$$\begin{array}{ccc}
\mathbf{L}_E X & \longrightarrow & \prod_{p\in\mathcal{P}E} \mathbf{L}_{E\mathbb{Z}/p}X \\
\downarrow & & \downarrow \\
\mathbf{L}_{H\mathbb{Q}} X & \longrightarrow & \mathbf{L}_{H\mathbb{Q}}\left(\prod_{p\in\mathcal{P}E} \mathbf{L}_{E\mathbb{Z}/p}X\right).
\end{array}$$

We also need the following remark.

LEMMA 2.1. *Suppose given a set of primes P and an adjusted* Ext-*p-complete abelian group A_p for all $p \in P$. The rationalization $\prod_{p\in P} A_p \to \left(\prod_{p\in P} A_p\right) \otimes \mathbb{Q}$ is then an epimorphism.*

PROOF. Fix any prime $q \in P$. Then we have $A_q \otimes \mathbb{Z}(q^\infty) = 0$ since A_q is adjusted, and $\left(\prod_{p\neq q} A_p\right) \otimes \mathbb{Z}(q^\infty) = 0$ as well, since $\prod_{p\neq q} A_p$ is uniquely q-divisible.

Therefore, $\left(\prod_{p\in P} A_p\right)\otimes\mathbb{Z}(q^\infty) = 0$. This shows that $\left(\prod_{p\in P} A_p\right)\otimes\mathbb{Q}/\mathbb{Z} = 0$, which proves our claim. □

Our main result is the following.

Theorem 2.2. *Let E be any homology theory and let X be 1-connected. Then $\mathbf{L}_E X$ is also 1-connected.*

Proof. By Corollary 1.2, we may assume that E is not torsion. Our strategy is to compare the arithmetic squares for E and ordinary homology $H\mathbb{Z}_{\mathcal{P}E}$. The natural maps $\mu\colon \mathbf{L}_{H\mathbb{Z}/p}X \to \mathbf{L}_{E\mathbb{Z}/p}X$ yield a commutative diagram

$$\begin{array}{ccccc}
Y & \longrightarrow & F' & \longrightarrow & F \\
\downarrow & & \downarrow & & \downarrow \\
F'' & \longrightarrow & \prod_{p\in\mathcal{P}E} \mathbf{L}_{H\mathbb{Z}/p}X & \xrightarrow{\ \mu\ } & \prod_{p\in\mathcal{P}E} \mathbf{L}_{E\mathbb{Z}/p}X \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{L}_{H\mathbb{Q}}F'' & \longrightarrow & \mathbf{L}_{H\mathbb{Q}}\left(\prod_{p\in\mathcal{P}E} \mathbf{L}_{H\mathbb{Z}/p}X\right) & \longrightarrow & \mathbf{L}_{H\mathbb{Q}}\left(\prod_{p\in\mathcal{P}E} \mathbf{L}_{E\mathbb{Z}/p}X\right)
\end{array}$$

where each row and each column is a fibre sequence. The four spaces in the lower right square are 1-connected by Corollary 1.2. Therefore, all the fibres except perhaps Y are connected. The group $\pi_1(F'')$ is the product of the cokernels of the homomorphisms $\mu_*\colon \pi_2(\mathbf{L}_{H\mathbb{Z}/p}X) \to \pi_2(\mathbf{L}_{E\mathbb{Z}/p}X)$, so it is a product of adjusted Ext-p-complete groups, by Theorem 1.4. Hence, Lemma 2.1 tells us that the induced homomorphism $\pi_1(F'') \to \pi_1(\mathbf{L}_{H\mathbb{Q}}F'')$ is surjective. This implies that Y is connected as well, so the homomorphism $\pi_1(F') \to \pi_1(F)$ is surjective.

From the arithmetic square for E we see that $\mathbf{L}_E X$ is 1-connected if and only if the boundary homomorphism $\pi_2(\mathbf{L}_{H\mathbb{Q}}X) \to \pi_1(F)$ is surjective. Consider now the fibre sequence $F' \to \mathbf{L}_{H\mathbb{Z}_{\mathcal{P}E}}X \to \mathbf{L}_{H\mathbb{Q}}X$ appearing in the arithmetic square for $H\mathbb{Z}_{\mathcal{P}E}$. Since we know that $\mathbf{L}_{H\mathbb{Z}_{\mathcal{P}E}}X$ is 1-connected, the homomorphism $\pi_2(\mathbf{L}_{H\mathbb{Q}}X) \to \pi_1(F')$ is surjective. The composite $\pi_2(\mathbf{L}_{H\mathbb{Q}}X) \to \pi_1(F') \to \pi_1(F)$ is thus also surjective, as we needed. □

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